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On a Family of q-Weighted Bergman Spaces and Applications

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Abstract. In this paper, we introduce a q-weighted Bergman spaces $\{\mathcal{A}_{\alpha,n,q}\}_{n\in\mathbb{N}}$. For $n = 0$ an uncertainty inequality of the Heisenberg-type for the space $\mathcal{A}_{\alpha,q}$ is given by considering the operators $\nabla_{\alpha,q} := \nabla_{\alpha,q}$ and $L_{\alpha,q} := L_{\alpha,q}$. Also, we study on this space the q-Toeplitz operators, the q-Hankel operators. At the end, we study the theory of extremal function and reproducing kernel of Hilbert space and we use it to establish the extremal function associated to an bounded linear operator $T : \mathcal{A}_{\alpha,q} \to H$, for any Hilbert space H.

1. Introduction

Many studies has happened in the last decade, characterizing the action of operators on Bergman and weighted Bergman spaces. This line of inquiry has attracted interest owing to its intimate links with complex analysis, functional analysis, and operator theory [6]. Many techniques have been investigate in different types of operators. For example, Hankel operators have been thoroughly analyzed using function-theoretic and operator-theoretic approaches ([1], [12]); composition operators have been studied through dynamical and analytic techniques ([14]); and multiplier operators have been explored in the context of reproducing kernel Hilbert spaces and boundedness criteria. These developments have significantly enriched the theory and opened new directions for further investigation.

The main results of this paper is to deal with operators acting on a general q-weighted Bergman spaces $\{\mathcal{A}_{\alpha,n,q}\}_{n\in\mathbb{N}}$. We prove some properties concerning q-Toeplitz operators and q-Hankel operators; we establish a more general Heisenberg-type uncertainty principle given in [16] for the space $\mathcal{A}_{\alpha,q}$ by considering the operators $\nabla_{\alpha,q} := \nabla_{\alpha,q}$ and $L_{\alpha,q} := L_{\alpha,q}$; we study the theory of extremal function and reproducing kernel of Hilbert space, to establish the extremal function associated to a bounded linear operator T. Noting that, there exist many similar uncertainty

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principles, in physics [2], [4], [10], and mathematics [3], [19], that are based on position, momentum, energy, time, and so on.

The weighted Bergman space is one of the complex analysis tools used in harmonic analysis [7]. Let $\mathbb{C}$ be the complex plane, $\mathbb{D} = \left\{z \in \mathbb{C} : |z| < 1\right\}$ the open unit disk and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. For any $\alpha > 0$,

$$d\nu_\alpha(z) := \frac{1}{\pi}\alpha(1-|z|^2)^{\alpha-1}dxdy$$

is the weighted Lebesgue measure on $\mathbb{D}$. The weighted Bergman space $\mathcal{A}_\alpha$ is the space $H(\mathbb{D})\bigcap L^2(\mathbb{D}, d\nu_\alpha)$. Noting that, it is an Hilbert when space equipped with the inner product

$$\langle f, g\rangle_{\mathcal{A}_\alpha} := \int_{\mathrm{D}} f(z)\overline{g(z)}d\nu_\alpha(z),$$

and the norm $\|f\|_{\mathcal{A}_\alpha} = \|f\|_{L^2_{\alpha,q}(\mathbb{D})}$, see [8, 16, 20] for more details on the theory of Bergman spaces. The contents of the paper are as follows. Section 2 reviewers from [16] the q-analogue of the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$ and we will introduce the q-analogue of q-weighted Bergman spaces $\{\mathcal{A}_{\alpha,n,q}\}_{n\in\mathbb{N}}$. In Sect.3, we will study the q-derivative operator $\nabla_{\alpha,q}$ and its adjoint operator $L_{\alpha,q}$ on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$, we will prove some properties concerning q-Toeplitz operators and q-Hankel operators and we will establish at the end of this section a general uncertainty inequality of Heisenberg type for the space $\mathcal{A}_{\alpha,q}$. In Sect.4, we will give an application of the theory of extremal function and reproducing kernel of Hilbert space by establishing the extremal function associated to a bounded linear operator T.

2. Preliminaries

In all the sequel, assume that $0 < q < 1$ and $\alpha > 0$. The reader can refer to [9] and [13] for more details for the definitions and notations of the basic hypergeometric series, the Jackson's q-derivative and q-integrals, q-Gamma and q-Beta functions. The reference [16] is devoted to the q-weighted Bergman space on the disk.

The standard Watson's notation for the q-shifted factorials are defined for any complex number a by

$$(a;q)_0 := 1,\ (a;q)_n := \prod_{k=0}^{n}(1-aq^{k-1}),\quad n = 1,2,\ldots,\ (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^{k-1}),$$

and $[a]_q$ is standing for the number associated to a,

$$[a]_q := \frac{1-q^a}{1-q},\ [a]_q! := \frac{(q;q)_n}{(1-q)^n},\ n \in \mathbb{N}.$$

For any complex z, $(a;q)_z$ is defined by

$$(a;q)_z := \frac{(a;q)_\infty}{(aq^z;q)_\infty}, \tag{1}$$

and the q-binomial theorem [9] is given by

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{2}$$

The q-analogue of the classical Euler Gamma and Beta functions defined by Jackson in [11] are

$$\Gamma_q(a) := \frac{(q;q)_\infty}{(q^a;q)_\infty}(1-q)^{1-a}, \ \Re(a) > 0.$$

$$\beta_q(a,b) := \int_0^1 t^{a-1}(qt;q)_{b-1} d_q t = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}, \ \Re(a), \Re(b) > 0. \tag{3}$$

The q-analogue exponential functions $e_q(z)$ and $E_q(z)$ [9] are given by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{(1-q)z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty},$$

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(1-q)z^n}{(q;q)_n} = (-z;q)_\infty.$$

The q-derivative [9] on a subset of $\mathbb{C}$ is defined by

$$D_{q,z} f(z) := \frac{f(z) - f(qz)}{(1-q)z}, \quad z \neq 0. \tag{4}$$

In all the sequel, we need the following spaces:

- $H(\mathbb{D})$ the space of all analytic functions on the unit open disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$.
- $L^2_{\alpha,q}(\mathbb{D}) := L^2_q(\mathbb{D}, d\nu_{\alpha,q})$ the space of measurable functions f on the unit disk $\mathbb{D}$ satisfying

$$\| f \|^2_{L^2_{\alpha,q}(\mathbb{D})} := \frac{[\alpha]_q}{2\pi} \int_0^1 \Big(\int_0^{2\pi} | f(re^{i\theta}) |^2 \, d\theta \Big) (qr^2;q)_{\alpha-1} d_q(r^2) := \int_{\mathbb{D}} | f(z) |^2 \, d\nu_{\alpha,q}(z)$$

is finite, where $d\nu_{\alpha,q}$ [5] the measure defined on the unit disk $\mathbb{D}$ for $\alpha > 0$ by

$$d\nu_{\alpha,q}(z) := \frac{[\alpha]_q}{2\pi}(qr^2;q)_{\alpha-1} d_q(r^2) d\theta; \ z = re^{i\theta},$$

and $d\theta$ is the usual Lebesgue measure on $[0, 2\pi[$ and the integral with respect to $d_q(r^2)$ is related to the q-Jackson's integral over $[0,1]$ defined by:

$$\int_0^1 f(t) d_q t := (1-q) \sum_{n=0}^{\infty} f(q^n) q^n.$$

- $\mathcal{A}_{\alpha,q} := \mathcal{A}_{\alpha,q}(\mathbb{D})$ the q-weighted Bergman space of all functions in $H(\mathbb{D}) \bigcap L^2_{\alpha,q}(\mathbb{D})$. It is a Hilbert space when equipped with the inner product

$$\langle f, g \rangle_{\mathcal{A}_{\alpha,q}} = \int_{\mathbb{D}} f(z)\overline{g(z)} d\nu_{\alpha,q}(z).$$

and the norm

$$\|f\|_{\mathcal{A}_{\alpha,q}} = \Big(\int_{\mathbb{D}} |f(z)|^2 d\nu_{\alpha,q}(z) \Big)^{1/2}.$$

- $\mathcal{A}_{\alpha,n,q} := \mathcal{A}_{\alpha,n,q}(\mathbb{D})$, the Hilbert space of functions on $H(\mathbb{D})$, such that

$$\begin{aligned} \|f\|^2_{\mathcal{A}_{\alpha,n,q}} &:= |f(0)|^2 + \int_{\mathbb{D}} |\ N_q^n f(z)\ |^2\ d\nu_{\alpha,q},\ \ n = 1, 2, \ldots \\ \|f\|^2_{\mathcal{A}_{\alpha,0,q}} &:= \|f\|^2_{\mathcal{A}_{\alpha,q}}, \end{aligned}$$

N_q is the q-multiplication operator on $\mathcal{A}_{\alpha,q}$ given by $N_q := zD_{q,z}$.

Moreover, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ then

$$\|f\|^2_{\mathcal{A}_{\alpha,n,q}} = |a_0|^2 + \sum_{k=1}^{\infty} [k]_q^{2n} C_k(\alpha; q)|a_k|^2,$$

where

$$C_n(\alpha; q) := \frac{(q; q)_n}{(q^{\alpha+1}; q)_n}.$$

3. Uncertainty inequality on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$

Consider the q-operator $\nabla_{\alpha,q}$ and $L_{\alpha,q}$ are the operators on $\mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q})$ defined by

$$\nabla_{\alpha,q} := q^{-\alpha-1} D_{q,z},\ \ N_q := zD_{q,z}\ L_{\alpha,q} := z^2 D_{q,z} + [\alpha+1]_q q^{-\alpha-1} z. \tag{5}$$

So, we have the following q-commutation relation

Lemma 3.1. $[\nabla_{\alpha,q}, L_{\alpha,q}]_q := \nabla_{\alpha,q} L_{\alpha,q} - L_{\alpha,q}\nabla_{\alpha,q} = q^{-\alpha-1}\Lambda_q\Big([\alpha+1]_q I + (1+q^{-1})q^{-\alpha-1}N_q\Big)$, *where I is the identity operator and Λ_q is the q-shift operator given by $\Lambda_q f(z) = f(qz)$.*

We derive the following results

Proposition 3.1. *Let $f, g \in \mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, we have*

(i) $\langle f, g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D}, d\nu_{\alpha,q})} = \sum_{n=0}^{\infty} a_n \overline{b}_n \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} = \sum_{n=0}^{\infty} a_n \overline{b}_n\ C_n(\alpha; q)$.

(ii) $||f||^2_{\mathcal{A}_{\alpha,q}(\mathrm{D}, d\nu_{\alpha,q})} = \sum_{n=0}^{\infty} |a_n|^2 \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} = \sum_{n=0}^{\infty} |a_n|^2\ C_n(\alpha; q)$.

(iii) *The set* $\left\{\xi^{\alpha}_{n,q}(z) := \frac{z^n}{\sqrt{C_n(\alpha; q)}}\right\}_{n\geq 0}$ *, forms a Hilbert's basis for the space $\mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q})$.*

Proof. Given $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, the result follows by using dominate convergence theorem and relation (4.6) in [5] we have

$$\langle f, g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D}, d\nu_{\alpha,q})} = \sum_{m,n=0}^{\infty} a_m \overline{b}_n \int_{\mathbb{D}} z^m\ \overline{z^n}\ d\nu_{\alpha,q}(z) = \sum_{n=0}^{\infty} a_m \overline{b}_n \frac{(q; q)_n}{(q^{\alpha+1}; q)_n}. \tag{6}$$

The last assertion follows directly from Proposition 4.1 in [5]. □

Theorem 3.1. *The function $\mathcal{K}_{\alpha,q}$ given for $w,z\in\mathbb{D}$, by*

$$\mathcal{K}_{\alpha,q}(z,w)=\mathcal{K}_{\alpha,q}(z\overline{w})=\frac{1}{(z\overline{w};q)_{\alpha+1}}, \tag{7}$$

is a reproducing kernel for the q-weighted Bergman space $\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q})$.
That is

(i) *for all $w\in\mathbb{D}$, $z\longmapsto\mathcal{K}_{\alpha,q}(z,w)$ belong to $\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q})$.*
(ii) *for all $w,z\in\mathbb{D}$ and $f\in\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q})$, we have $\langle f,\mathcal{K}_{\alpha,q}(.,w)\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}=f(w)$.*
(iii) *For all $f\in\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q})$ and $z\in\mathbb{C}$, $\mid f(z)\mid\leq\Big[e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)\Big]^{1/2}\parallel f\parallel_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}$.*
(iv) *Let $w\in\mathbb{D}$. The function $u(z)=\mathcal{K}_{\alpha,q}(z\overline{w})$ is the unique analytic solution on $\mathbb{D}$ of the initial problem*

$$z\nabla_{\alpha,q}u(z)=wL_{\alpha,q}u(z),\ \ u(0)=1.$$

Proof. To prove the first assertion (i), we use Proposition 3.1 (iii) the function $\xi^{\alpha}_{n,q}(z)$ constitute an orthonormal basis of $\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q})$. Therefore for any $z,w\in\mathbb{D}$, $\mathcal{K}_{\alpha,q}$ can be computed by evaluating the following sum

$$\mathcal{K}_{\alpha,q}(z,w)=\sum_{n=0}^{\infty}\xi^{\alpha}_{n,q}(z)\overline{\xi^{\alpha}_{n,q}(w)}=\sum_{n=0}^{\infty}\frac{1}{C_n(\alpha;q)}z^n\overline{w}^n.$$

Hence by (1) combined with (2) we deduce easily

$$\mathcal{K}_{\alpha,q}(z,w)=\sum_{n=0}^{\infty}\frac{(q^{\alpha+1};q)_n}{(q;q)_n}(z\overline{w})^n=\frac{(q^{\alpha+1}z\overline{w};q)_\infty}{(z\overline{w};q)_\infty}=\frac{1}{(z\overline{w};q)_{\alpha+1}}.$$

To prove (ii), we use the same as in Proposition 4.2 in [5]. The last assertion follows by using (4). □

The domain of the operator $\nabla_{\alpha,q}$ denoted by $\mathrm{Dom}(\nabla_{\alpha,q})$ is defined by

$$\mathrm{Dom}(\nabla_{\alpha,q}):=\Big\{f\in\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q});\ \nabla_{\alpha,q}f\in\mathcal{A}_{\alpha,q}(\mathbf{D},d\nu_{\alpha,q})\Big\},$$

and same for $\mathrm{Dom}_q(N_q)$ and $\mathrm{Dom}_q(L_{\alpha,q})$.

Lemma 3.2. *The operators $\nabla_{\alpha,q}$, N_q and $L_{\alpha,q}$ satisfies the following*

(i) *$Dom(\nabla_{\alpha,q})=Dom(L_{\alpha,q})=Dom(N_q)=\mathcal{A}_{\alpha,1,q}$.*
(ii) *For any f,g in $\mathcal{A}_{\alpha,1,q}$ we have: $\langle\nabla_{\alpha,q}f,g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}=\langle f,L_{\alpha,q}g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}$.*
(iii) *For any f in $\mathcal{A}_{\alpha,1,q}$ we have*

$$\parallel L_{\alpha,q}f\parallel^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}=\parallel\nabla_{\alpha,q}f\parallel^2_{\mathcal{A}_{\alpha,q}}+q^{-\alpha-1}[\alpha+1]_q\parallel\Lambda_{q^{1/2}}f\parallel^2_{\mathcal{A}_{\alpha,q}}+q^{-\alpha-1}(1+q^{-1})\langle N_q\Lambda_{q^{1/2}}f,\Lambda_{q^{1/2}}f\rangle_{\mathcal{A}_{\alpha,q}}.$$

Proof. Let $f\in\mathcal{A}_{\alpha,1,q}$, with $f(z)=\sum_{k=0}^{\infty}a_kz^k$. Then using relation (4), we have respectively

$$\nabla_{\alpha,q}f(z)=\sum_{k=1}^{\infty}q^{-\alpha-1}[k]_qa_kz^{k-1}=\sum_{k=0}^{\infty}q^{-\alpha-1}[k+1]_qa_{k+1}z^k \tag{8}$$

and

$$L_{\alpha,q}f(z)=\sum_{k=0}^{\infty}([k]_q+q^{-\alpha-1}[\alpha+1]_q)a_kz^{k+1}=\sum_{k=1}^{\infty}([k-1]_q+q^{-\alpha-1}[\alpha+1]_q)a_{k-1}z^k. \quad (9)$$

Thus from the previous relation, we get

$$\| \nabla_{\alpha,q}f \|^2_{\mathcal{A}_{\alpha,q}}=\langle \nabla_{\alpha,q}f,\nabla_{\alpha,q}f\rangle_{\mathcal{A}_{\alpha,q}}=\langle f,L_{\alpha,q}\nabla_{\alpha,q}f\rangle_{\mathcal{A}_{\alpha,q}}=\sum_{k=1}^{\infty}q^{-\alpha-1}[k]_q\Big([k-1]_q+q^{-\alpha-1}[\alpha+1]_q\Big)|a_k|^2C_k(\alpha;q), \quad (10)$$

$$\| L_{\alpha,q}f \|^2_{\mathcal{A}_{\alpha,q}}=\langle L_{\alpha,q}f,L_{\alpha,q}f\rangle_{\mathcal{A}_{\alpha,q}}=\langle f,\nabla_{\alpha,q}L_{\alpha,q}f\rangle_{\mathcal{A}_{\alpha,q}}=\sum_{k=1}^{\infty}q^{-\alpha-1}[k+1]_q\Big([k]_q+q^{-\alpha-1}[\alpha+1]_q\Big)|a_k|^2C_k(\alpha;q), \quad (11)$$

and

$$\| N_qf \|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}=\langle N_qf,N_qf\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}=\sum_{k=1}^{\infty}[k]_q^2|a_k|^2C_k(\alpha;q). \quad (12)$$

Therefore, from Proposition 3.1, (10), (11) and (12) we deduce easily

$$\begin{aligned}
\|f\|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}-|f(0)|^2 &\leq \|\nabla_{\alpha,q}f\|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}\leq(1+q^{-\alpha-1}[\alpha+1]_q)\|f\|^2_{\mathcal{A}_{\alpha,1,q}(\mathrm{D},d\nu_{\alpha,q})}\\
\|f\|^2_{\mathcal{A}_{\alpha,1,q}(\mathrm{D},d\nu_{\alpha,q})} &\leq \|L_{\alpha,q}f\|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}\leq[2]_q(1+q^{-\alpha-1}[\alpha+1]_q)\|f\|^2_{\mathcal{A}_{\alpha,1,q}(\mathrm{D},d\nu_{\alpha,q})}\\
\|f\|^2_{\mathcal{A}_{\alpha,1,q}(\mathrm{D},d\nu_{\alpha,q})}-|f(0)|^2 &\leq \|N_qf\|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}\leq\|f\|^2_{\mathcal{A}_{\alpha,1,q}(\mathrm{D},d\nu_{\alpha,q})}.
\end{aligned}$$

So, $\mathrm{Dom}(\nabla_{\alpha,q})=\mathrm{Dom}(L_{\alpha,q})=\mathrm{Dom}(N_q)=\mathcal{A}_{\alpha,1,q}(\mathbf{D},d\nu_{\alpha,q})$.

To prove (ii), let f,g in $\mathcal{A}_{\alpha,1,q}(\mathbf{D},d\nu_{\alpha,q})$ with $f(z)=\sum_{k=0}^{\infty}a_kz^k$ and $g(z)=\sum_{k=0}^{\infty}b_kz^k$. From Proposition 3.1, (8) and (9) we have

$$\begin{aligned}
\langle\nabla_{\alpha,q}f,g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} &= \sum_{k=0}^{\infty}q^{-\alpha-1}[k+1]_qa_{k+1}\overline{b}_kC_k(\alpha;q)\\
&= \sum_{k=0}^{\infty}a_{k+1}\overline{b}_k\frac{(q;q)_{k+1}}{(1-q)q^{\alpha+1}(q^{\alpha+1};q)_k}\\
&= \sum_{k=1}^{\infty}a_k\overline{b}_{k-1}\frac{(q;q)_k}{(1-q)q^{\alpha+1}(q^{\alpha+1};q)_{k-1}},
\end{aligned}$$

on the other hand

$$\begin{aligned}
\langle f,L_{\alpha,q}g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} &= \sum_{k=0}^{\infty}([k-1]_q+q^{-\alpha-1}[\alpha+1]_q)[k+1]_qa_k\overline{b}_{k-1}C_k(\alpha;q)\\
&= \sum_{k=0}^{\infty}\frac{1-q^{\alpha+k}}{1-q}a_k\overline{b}_{k-1}C_k(\alpha;q)\\
&= \sum_{k=1}^{\infty}[k+\alpha]_qa_k\overline{b}_{k-1}\frac{(q;q)_k}{(q^{\alpha+1};q)_k}\\
&= \sum_{k=1}^{\infty}a_k\overline{b}_{k-1}\frac{(q;q)_k}{q^{\alpha+1}(1-q)(q^{\alpha+1};q)_{k-1}}=\langle\nabla_{\alpha,q}f,g\rangle_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}.
\end{aligned}$$

Finally, to prove (iii), using $[k+1]_q = [k]_q + q^k$ we deduce easily that

$$\begin{aligned}[k+1]_q\Big([k]_q &+ q^{-\alpha-1}[\alpha+1]_q\Big) = \Big([k]_q + q^k\Big)\Big([k-1]_q + q^{k-1} + q^{-\alpha-1}[\alpha+1]_q\Big) \\ &= [k]_q\Big([k-1]_q + q^{-\alpha-1}[\alpha+1]_q\Big) + q^{k-\alpha-1}[\alpha+1]_q + \Big(1+q^{-1}\Big)q^k[k]_q.\end{aligned}$$

Which leads to the result using (10), (11), (12) and the fact that $\Lambda_q N_q = N_q \Lambda_q$. □

Lemma 3.3. *$Dom(\nabla_{\alpha,q}L_{\alpha,q}) = Dom(\nabla_{\alpha,q}L_{\alpha,q}) = \mathcal{A}_{\alpha,2,q}(\mathbf{D}, d\nu_{\alpha,q})$.*

Proof. Let $f \in \mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q})$, with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then using relation (8) and (9) we obtain

$$\nabla_{\alpha,q}L_{\alpha,q}f(z) = \sum_{k=0}^{\infty} q^{-\alpha-1}[k+1]_q([k]_q + q^{-\alpha-1}[\alpha+1]_q)a_k z^k$$

and

$$L_{\alpha,q}\nabla_{\alpha,q}f(z) = \sum_{k=1}^{\infty} q^{-\alpha-1}[k]_q([k-1]_q + q^{-\alpha-1}[\alpha+1]_q)a_k z^k.$$

Therefore,

$$\| \nabla_{\alpha,q}L_{\alpha,q}f \|_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} = \sum_{k=0}^{\infty} q^{-2(\alpha-+)}[k+1]_q^2([k]_q + q^{-\alpha-1}[\alpha+1]_q)^2|a_k|^2 C_k(\alpha;q)$$

and

$$\| L_{\alpha,q}\nabla_{\alpha,q}f \|_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} = \sum_{k=1}^{\infty} q^{-2(\alpha+1)}[k]_q^2([k-1]_q + q^{-\alpha-1}[\alpha+1]_q)^2|a_k|^2 C_k(\alpha;q).$$

So, as in the previous lemma, from Proposition 3.1, we deduce easily

$$\|f\|^2_{\mathcal{A}_{\alpha,2,q}(\mathrm{D},d\nu_{\alpha,q})} - |f(0)|^2 \le \|L_{\alpha,q}\nabla_{\alpha,q}f\|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} \le (1+q^{-\alpha-1}[\alpha+1]_q)^2\|f\|^2_{\mathcal{A}_{\alpha,2,q}(\mathrm{D},d\nu_{\alpha,q})},$$

and

$$\|f\|^2_{\mathcal{A}_{\alpha,2,q}(\mathrm{D},d\nu_{\alpha,q})} \le \|\nabla_{\alpha,q}L_{\alpha,q}f\|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} \le [2]_q^2(1+q^{-\alpha-1}[\alpha+1]_q)^2\|f\|^2_{\mathcal{A}_{\alpha,2,q}(\mathrm{D},d\nu_{\alpha,q})}.$$

Thus, $\mathrm{Dom}(\nabla_{\alpha,q}L_{\alpha,q}) = \mathrm{Dom}(\nabla_{\alpha,q}L_{\alpha,q}) = \mathcal{A}_{\alpha,2,q}$. □

We can now establish an uncertainty inequality of Heisenberg-type on the space $\mathcal{A}_{\alpha,q}(\mathbf{D}, d\nu_{\alpha,q})$, by the virtue of the following lemma:

Lemma 3.4. *[6] Let X and Y be self-adjoint operators on Hilbert space H (i.e $X^* = X$ and $Y^* = Y$). Then*

$$\| (X-a)f \|_H \| (Y-b)f \|_H \ge \frac{1}{2} \mid \langle [X,Y]f, f\rangle_H \mid,$$

for all f in $Dom(XY) \cap Dom(YX)$ and $a, b \in \mathbb{R}$

Theorem 3.2. *Let $f \in \mathcal{A}_{\alpha,2,q}(\mathbf{D}, d\nu_{\alpha,q})$. For all $a, b \in \mathbb{R}$, we have*

$$\begin{aligned} &\| (\nabla_{\alpha,q} + L_{\alpha,q} - a)f \|_{\mathcal{A}_{\alpha,q}} \| (\nabla_{\alpha,q} - L_{\alpha,q} + ib)f \|_{\mathcal{A}_{\alpha,q}} \\ \geq \quad & q^{-\alpha-1}[\alpha+1]_q \| \Lambda_{q^{1/2}} f \|^2_{\mathcal{A}_{\alpha,q}} + q^{-\alpha-1}(1+q^{-1})\langle N_q \Lambda_{q^{1/2}} f, \Lambda_{q^{1/2}} f\rangle_{\mathcal{A}_{\alpha,q}}. \end{aligned}$$

Proof. Consider $X := \nabla_{\alpha,q} + L_{\alpha,q}$ and $Y := i(\nabla_{\alpha,q} - L_{\alpha,q})$. By Lemma 3.2 and Lemma 3.3, the operators X and Y verifies the following properties

(a.) $X^* = X$ and $Y^* = Y$
(b.) $\mathrm{Dom}(XY) = \mathrm{Dom}(YX) = \mathcal{A}_{\alpha,2,q}$
(c.) $[X, Y]_q = -2i[\nabla_{\alpha,q}, L_{\alpha,q}]_q$.

So, the result follows from Lemma 3.1 and Lemma 3.2. □

Proposition 3.2. *Let $a, b \in \mathbb{R}$.*

(i) *For all $f \in \mathcal{A}_{\alpha,2,q}(\mathbf{D}, d\nu_{\alpha,q})$, we have*

$$\begin{aligned} &\| (\nabla_{\alpha,q} + L_{\alpha,q} - a)f \|_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} \| (\nabla_{\alpha,q} - L_{\alpha,q} + ib)f \|_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} \\ &\qquad \geq \| L_{\alpha,q} f \|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})} - \| \nabla_{\alpha,q} f \|^2_{\mathcal{A}_{\alpha,q}(\mathrm{D},d\nu_{\alpha,q})}. \end{aligned}$$

(ii) *For all $f \in \mathcal{A}_{\alpha,1,q}$, we have*

$$\begin{aligned} &q^{\alpha+1} \| (\nabla_{\alpha,q} + L_{\alpha,q} - a)f \|_{\mathcal{A}_{\alpha,1,q}} \| (\nabla_{\alpha,q} - L_{\alpha,q} + ib)f \|_{\mathcal{A}_{\alpha,1,q}} \\ &\geq [\alpha+1]_q \| \Lambda_{q^{1/2}} f \|^2_{\mathcal{A}_{\alpha,1,q}} + (1+q^{-1})\langle N_q \Lambda_{q^{1/2}} f, \Lambda_{q^{1/2}} f\rangle_{\mathcal{A}_{\alpha,1,q}}. \end{aligned}$$

Proof. Let $a, b \in \mathbb{R}$. The first inequality (i) hold from Lemma 3.2 (iii) and the second inequality (ii) hold by applying Lemma 3.2 (i). □

4. Operators on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$

4.1. **q-Toepliz Operator on $\mathcal{A}_{\alpha,q}$.** Consider the orthogonal projection operator $P_{\alpha,q} : L^2_{\alpha,q}(\mathbb{D}) \to \mathcal{A}_{\alpha,q}$. Since $L^2_{\alpha,q}(\mathbb{D}) = \mathcal{A}_{\alpha,q} \oplus \mathcal{A}^{\perp}_{\alpha,q}$ then for any $f \in L^2_{\alpha,q}(\mathbb{D})$, we have $f = (f - f^{\perp}) + f^{\perp}$ where $f - f^{\perp} \in \mathcal{A}_{\alpha,q}$ and $f^{\perp} \in \mathcal{A}^{\perp}_{\alpha,q}$. Furthermore, for $z \in \mathbb{D}$,

$$P_{\alpha,q} f(z) = (f - f^{\perp})(z) = \langle (f - f^{\perp})(z), \mathcal{K}_{\alpha,q}(z, .)\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f(z), \mathcal{K}_{\alpha,q}(z, .)\rangle_{L^2_{\alpha,q}(\mathbb{D})},$$

where $\mathcal{K}_{\alpha,q}$ is the reproducing kernel given by (7). The following assertions then follow

Proposition 4.1. *For all $f, g \in L^2_{\alpha,q}(\mathbb{D})$, we have:*

(i) $P_{\alpha,q} \circ P_{\alpha,q} f = P_{\alpha,q} f$.
(ii) $\langle P_{\alpha,q} f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, P_{\alpha,q} g\rangle_{L^2_{\alpha,q}(\mathbb{D})}$.
(iii) *The operator $P_{\alpha,q}$ is bounded with $\| P_{\alpha,q} \| = 1$ and $\| I - P_{\alpha,q} \| \leq 1$.*

Let $\phi \in L^\infty(\mathbb{D})$. The q-multiplication operators M_ϕ are the operators defined by

$$M_\phi : L^2_{\alpha,q}(\mathbb{D}) \to L^2_{\alpha,q}(\mathbb{D}),\ M_\phi f(z) := \phi(z)f(z),\ z \in \mathbb{D}.$$

The q-Toepliz operators T_ϕ are the operators defined by

$$T_\phi : \mathcal{A}_{\alpha,q} \to \mathcal{A}_{\alpha,q},\ T_\phi f(z) := P_{\alpha,q} M_\phi(z) f(z),\ z \in \mathbb{D}.$$

Theorem 4.1. *Let $\phi \in L^\infty(\mathbb{D})$.*

(i) *The operators T_ϕ are bounded and $\| T_\phi \| \leq \| \phi \|_\infty$.*

(ii) *For all $f, g \in \mathcal{A}_{\alpha,q}$, we have*

$$\langle T_\phi f, g\rangle_{\mathcal{A}_{\alpha,q}} = \langle f, T_{\overline{\phi}} g\rangle_{\mathcal{A}_{\alpha,q}}.$$

Proof. Let $\phi \in L^\infty(\mathbb{D})$. To prove (i), let $f \in \mathcal{A}_{\alpha,q}$ then from Proposition 4.1 (iii) we have

$$\| T_\phi f \|_{\mathcal{A}_{\alpha,q}} = \| P_{\alpha,q} M_\phi f \|_{\mathcal{A}_{\alpha,q}} = \| P_{\alpha,q}(\phi f) \|_{\mathcal{A}_{\alpha,q}} \leq \| \phi f \|_{L^2_{\alpha,q}(\mathbb{D})} \leq \| \phi f \|_{L^\infty(\mathbb{D})} \| f \|_{\mathcal{A}_{\alpha,q}} .$$

Thus, $\| T_\phi \| \leq \| \phi \|$.

To prove the second assertion, we use the fact that for any $f, g \in \mathcal{A}_{\alpha,q}$, $P_{\alpha,q} f = f$ and $P_{\alpha,q} g = g$. From Proposition 4.1 (ii), we obtain

$$\langle T_\phi f, g\rangle_{\mathcal{A}_{\alpha,q}} = \langle \phi f, P_{\alpha,q} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, \overline{\phi} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle P_{\alpha,q} f, T_{\overline{\phi}} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, T_{\overline{\phi}} g\rangle_{\mathcal{A}_{\alpha,q}}.$$

□

Theorem 4.2. *Let $\phi \in L^\infty(\mathbb{D})$ has compact support, then T_ϕ is a compact operator.*

Proof. Let $\phi \in L^\infty(\mathbb{D})$ and $n, m = 0, 1, 2, ...$ From Proposition 3.1, we have

$$T_\phi \xi^\alpha_{n,q}(z) = \sum_{m=0}^{\infty} \frac{\langle T_\phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})}}{C_m(\alpha; q)} z^m.$$

So,

$$\langle T_\phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{\mathcal{A}_{\alpha,q}} = \langle \phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})}$$

Since $\phi \in L^\infty(\mathbb{D})$ with compact support, there exist a positive constant a and K such that $| \phi(z) | \leq a$ and $\phi(z) = 0$, for any $| z | > a$. Then for all $n, m \in \mathbb{N}$, we get from (3) and Proposition 3.1 (i),

$$\langle \phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \frac{1}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_{|z|\leq a} \phi(z) z^n \overline{z^m} d\nu_{\alpha,q}(z)$$

Thus, we obtain

$$\begin{aligned} \left| \langle \phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{L^2_{\alpha,q}(\mathbb{D})} \right| &\leq \frac{K}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_{|z|\leq a} |z|^{n+m} d\nu_{\alpha,q}(z) \\ &\leq \frac{2K}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_0^a r^{n+m} d\nu_{\alpha,q}(z) \\ &\leq \frac{2K[\alpha]_q a^{n+m}}{\sqrt{C_n(\alpha; q) C_m(\alpha; q)}} \int_0^1 (qr^2; q)_{\alpha-1} d_q(r^2) \end{aligned}$$

$$\leq \frac{Ka^{n+m}}{\sqrt{C_n(\alpha;q)C_m(\alpha;q)}}.$$

Hence,

$$\sum_{n,m=0}^{\infty} \frac{\left|\langle T_\phi \xi^\alpha_{n,q}, \xi^\alpha_{m,q}\rangle_{\mathcal{A}_{\alpha,q}}\right|^2}{C_n(\alpha;q)C_m(\alpha;q)} \leq 4K^2\left(\sum_{n=0}^{\infty}\frac{a^{2n}}{C_n(\alpha;q)}\right)^2 \leq 4K^2 e_q(a^2)(q^{\alpha+1};q)^2_\infty < \infty$$

Then T_ϕ is an Hilbert-Schmidt operator, and consequently it is compact. □

4.2. **q-Hankel Operator on $\mathcal{A}_{\alpha,q}$.** Let $\phi \in L^\infty(\mathbb{D})$. The q-Hankel operators H_ϕ are the operators defined by

$$H_\phi : \mathcal{A}_{\alpha,q} \to \mathcal{A}_{\alpha,q}, \;\; H_\phi := (I - P_{\alpha,q})M_\phi.$$

Theorem 4.3. *Let $\phi, \psi \in L^\infty(\mathbb{D})$.*

(i.) *The operators H_ϕ are bounded and $\| H_\phi \| \leq \| \phi \|_\infty$.*

(ii) *For all $f \in \mathcal{A}_{\alpha,q}$ and $g \in L^2_{\alpha,q}(\mathbb{D})$, we have*

$$\langle H_\phi f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} = \langle f, H^*_\phi g\rangle_{\mathcal{A}_{\alpha,q}}, \;\; H^*_\phi = P_{\alpha,q}M_{\overline{\phi}}(I - P_{\alpha,q}).$$

(iii) $T_{\phi\psi} - T_\phi T_\psi = H^*_{\overline{\phi}} H_\phi$.

Proof. Let $\phi, \psi \in L^\infty(\mathbb{D})$. To prove (i), from Proposition 4.1 (iii) for any $f \in \mathcal{A}_{\alpha,q}$

$$\| H_\phi \|_{L^2_{\alpha,q}(\mathbb{D})} = \| (I - P_{\alpha,q}) \|_{L^2_{\alpha,q}(\mathbb{D})} \leq \| \phi f \|_{L^2_{\alpha,q}(\mathbb{D})} \leq \| \phi \|_{L^\infty_{\alpha,q}(\mathbb{D})} \| \phi f \|_{L^2_{\alpha,q}(\mathbb{D})} .$$

So, $\| H_\phi \| \leq \| \phi \|_{L^\infty(\mathbb{D})}$.

(ii) Let $f \in \mathcal{A}_{\alpha,q}$ and $g \in L^2_{\alpha,q}(\mathbb{D})$. From Proposition 4.1 (ii) and the fact that $P_{\alpha,q} f = f$ we obtain

$$\begin{aligned}
\langle H_\phi f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} &= \langle \phi f, g\rangle_{L^2_{\alpha,q}(\mathbb{D})} - \langle \phi f, P_{\alpha,q} g\rangle_{L^2_{\alpha,q}(\mathbb{D})} \\
&= \langle f, \overline{\phi}(I - P_{\alpha,q})g\rangle_{L^2_{\alpha,q}(\mathbb{D})} \\
&= \langle P_{\alpha,q} f, \overline{\phi}(I - P_{\alpha,q})g\rangle_{L^2_{\alpha,q}(\mathbb{D})} \\
&= \langle f, P_{\alpha,q} M_{\overline{\phi}}(I - P_{\alpha,q})g\rangle_{\mathcal{A}_{\alpha,q}}.
\end{aligned}$$

(iii) Let $\phi, \psi \in L^\infty(\mathbb{D})$. Then

$H^*_{\overline{\phi}} H_\psi = P_{\alpha,q} M_\phi (I - P_{\alpha,q})^2 M_\psi = P_{\alpha,q} M_\phi (I - P_{\alpha,q}) M_\psi = P_{\alpha,q} M_{\phi\psi} - P_{\alpha,q} M_\phi P_{\alpha,q} M_\psi = T_{\phi\psi} - T_\phi T_\psi.$ □

5. Extremal function on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$

Let $\eta > 0$ and $T : \mathcal{A}_{\alpha,q} \to H$ be a bounded operator from $\mathcal{A}_{\alpha,q}$ into a Hilbert space H. We denote by $\langle ., .\rangle_{T,\eta,q}$ the inner product defined on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$ by

$$\langle f, g\rangle_{T,\eta,q} := \eta\langle f, g\rangle_{\mathcal{A}_{\alpha,q}} + \langle Tf, Tg\rangle_H,$$

and $\| f \|_{T,\eta,q} := \sqrt{\langle f, f\rangle_{T,\eta,q}}$.

By the virtue of the theory of reproducing kernels of Hilbert space, we study the extremal function associated to the operator T on the q-weighted Bergman space $\mathcal{A}_{\alpha,q}$.

Theorem 5.1. *Let $\eta > 0$. The space $(\mathcal{A}_{\alpha,q}, \langle .,. \rangle_{T,\eta,q})$ possesses a reproducing kernel $\mathcal{K}_{T,\eta,q}(z,w); z,w \in \mathbb{D}$ which satisfies the equation $(\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.) = \mathcal{K}_{\alpha,q}(z,.)$, where $\mathcal{K}_{\alpha,q}$ is the kernel given by (7). Moreover, the kernel $\mathcal{K}_{T,\eta,q}$ satisfies the following properties*

(i) $\| \mathcal{K}_{T,\eta,q}(z,.) \|_{\mathcal{A}_{\alpha,q}} \leq \frac{1}{\eta}\sqrt{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}$.

(ii) $\| T\mathcal{K}_{T,\eta,q}(z,.) \|_{H} \leq \sqrt{\frac{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}{2\eta}}$.

(iii) $\| T^*T\mathcal{K}_{T,\eta,q}(z,.) \|_{\mathcal{A}_{\alpha,q}} \leq \sqrt{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}$,

Proof. Let $f \in \mathcal{A}_{\alpha,q}$. Using Theorem 3.1 (iii), the map $f \mapsto f(z)$ is a continuous linear functional on $(\mathcal{A}_{\alpha,q}, \langle .,. \rangle_{T,\eta,q})$. Thus, $(\mathcal{A}_{\alpha,q}, \langle .,. \langle_{T,\eta,q})$ has a reproducing kernel denoted $\mathcal{K}_{T,\eta,q}$. Now using the fact that

$$f(z) = \eta\langle f, \mathcal{K}_{T,\eta,q}(z,.)\rangle_{\mathcal{A}_{\alpha,q}} + \langle Tf, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_H = \langle f, (\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.)\rangle_{\mathcal{A}_{\alpha,q}},$$

we deduce easily that $(\eta I + T^*T)\mathcal{K}_{T,\eta,q}(z,.) = \mathcal{K}_{\alpha,q}(z,.)$. So the previous relation implies that

$$\eta^2 \| \mathcal{K}_{T,\eta,q}(z,.) \|^2_{\mathcal{A}_{\alpha,q}} + 2\eta \| T\mathcal{K}_{T,\eta,q}(z,.) \|^2_H + \| T^*T\mathcal{K}_{T,\eta,q}(z,.) \|^2_{\mathcal{A}_{\alpha,q}} = \| \mathcal{K}_{\alpha,q}(z,.) \|^2_{\mathcal{A}_{\alpha,q}} .$$

So we obtain the properties (i), (ii) and (iii) by using relation (7). □

Since relations (10), (11) and (12), we get

Example 5.1. *For any $w, z \in \mathbb{D}$, let $H = \mathcal{A}_{\alpha,q}$.*

(a) *If $T = \nabla_{\alpha,q}$, then*

$$\mathcal{K}_{T,\eta,q}(z,w) = \frac{1}{\eta C_0(\alpha;q)} + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\Big(\eta + q^{-\alpha-1}[n]_q([n-1]_q + q^{-\alpha-1}[\alpha+1]_q)\Big)C_n(\alpha;q)}.$$

(b) *If $T = L_{\alpha,q}$, then*

$$\mathcal{K}_{T,\eta,q}(z,w) = \frac{1}{\eta C_0(\alpha;q)} + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\Big(\eta + q^{-\alpha-1}[n+1]_q([n]_q + q^{-\alpha-1}[\alpha+1]_q)\Big)C_n(\alpha;q)}.$$

(c) *If $T = N_q$, then*

$$\mathcal{K}_{T,\eta,q}(z,w) = \frac{1}{\eta C_0(\alpha;q)} + \sum_{n=1}^{\infty} \frac{(z\overline{w})^n}{\Big(\eta + [n]_q^2\Big)C_n(\alpha;q)}.$$

We can state now the main result of this section.

Theorem 5.2. *For any $h \in H$ and $\eta > 0$, there exists a unique function $f^*_{\eta,h}$, where the infimum*

$$\inf_{f\in\mathcal{A}_{\alpha,q}} \left\{ \eta \parallel f \parallel^2_{\mathcal{A}_{\alpha,q}} + \parallel h - Tf \parallel^2_H \right\} \tag{13}$$

is attained. Moreover, the extremal function $f^*_{\eta,h}$ is given by

$$f^*_{\eta,h}(z) = \langle h, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_H, \tag{14}$$

and satisfies the following $| f^*_{\eta,h}(z) | \leq \sqrt{\dfrac{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}{2\eta}} \parallel h \parallel_H$.

Proof. The existence and unicity of the extremal function $f^*_{\eta,h}$ satisfying (13) is obtained in [15,17]. In particular, $f^*_{\eta,h}$ is given by the reproducing kernel of $\mathcal{A}_{\alpha,q}$ with $\parallel . \parallel_{T,\eta,q}$ norm as $f^*_{\eta,h}(z) = \langle h, T\mathcal{K}_{T,\eta,q}(z,.)\rangle_H$. This yields the result, by using relation (14), Theorem 5.1 (ii) and the fact that

$$| f^*_{\eta,h}(z) | \leq \parallel h \parallel_H \parallel T\mathcal{K}_{T,\eta,q}(z,.) \parallel_H \leq \sqrt{\frac{e_q(|z|^2)E_q(q^{\alpha+1}|z|^2)}{2\eta}} \parallel h \parallel_H,$$

which completes the proof of the theorem. □

5.1. **Applications.** Let H be the prehilbertian space of analytic functions on the disk $\mathbb{D}$ equipped with the inner product

$$\langle f, g\rangle_H := \int_{\mathbb{D}} f(z)\overline{g(z)}|z|^2 d\nu_{\alpha,q}(z).$$

For any $f, g \in H$ with $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ we have from Proposition 3.1 and relation (6)

$$\langle f, g\rangle_H = \sum_{n\geq 0} a_n \overline{b_n} C_{n+1}(\alpha; q), \ \parallel f \parallel_H = \sum_{n\geq 0} | a_n |^2 C_{n+1}(\alpha; q).$$

The space H is a Hilbert space with Hilbert's basis $\left\{ \dfrac{z^n}{\sqrt{C_{n+1}(\alpha; q)}} \right\}_{n\geq 0}$ and reproducing kernel

$$\mathcal{S}_{\alpha,q}(z, w) = \sum_{n=0}^{\infty} \frac{(z\overline{w})^n}{C_{n+1}(\alpha; q)} = \frac{\mathcal{K}_{\alpha,q}(z\overline{w}) - 1}{z\overline{w}}. \tag{15}$$

5.1.1. *Application 1.* Let T be the q-difference operator defined on $\mathcal{A}_{\alpha,q}$ by

$$Tf(z) := \frac{1}{z}(f(z) - f(0)).$$

The operator T maps continuously from $\mathcal{A}_{\alpha,q}$ into H and $\parallel Tf \parallel_H \leq \parallel f \parallel_{\mathcal{A}_{\alpha,q}}$. So, if $f, g \in \mathcal{A}_{\alpha,q}$ with $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ we can deduce easily that

$$\langle f, g\rangle_{T,\eta} = \eta a_0 \overline{b_0} + (\eta + 1) \sum_{n=1}^{\infty} a_n \overline{b_n} C_n(\alpha; q).$$

Thus, for $z, w \in \mathbb{D}$ we have

$$\begin{aligned} \mathcal{K}_{T,\eta,q}(z,w) &= \frac{1}{\eta} + \frac{1}{\eta+1}(\mathcal{K}_{\alpha,q}(z\overline{w}) - 1), \\ T\mathcal{K}_{T,\eta,q}(z,.)(w) &= \frac{1}{\eta+1}\frac{\mathcal{K}_{\alpha,q}(z\overline{w}) - 1}{\overline{w}}, \end{aligned}$$

hence for all $h \in H$ we deduce that

$$f^*_{\eta,h}(z) = \frac{1}{\eta+1} z h(z).$$

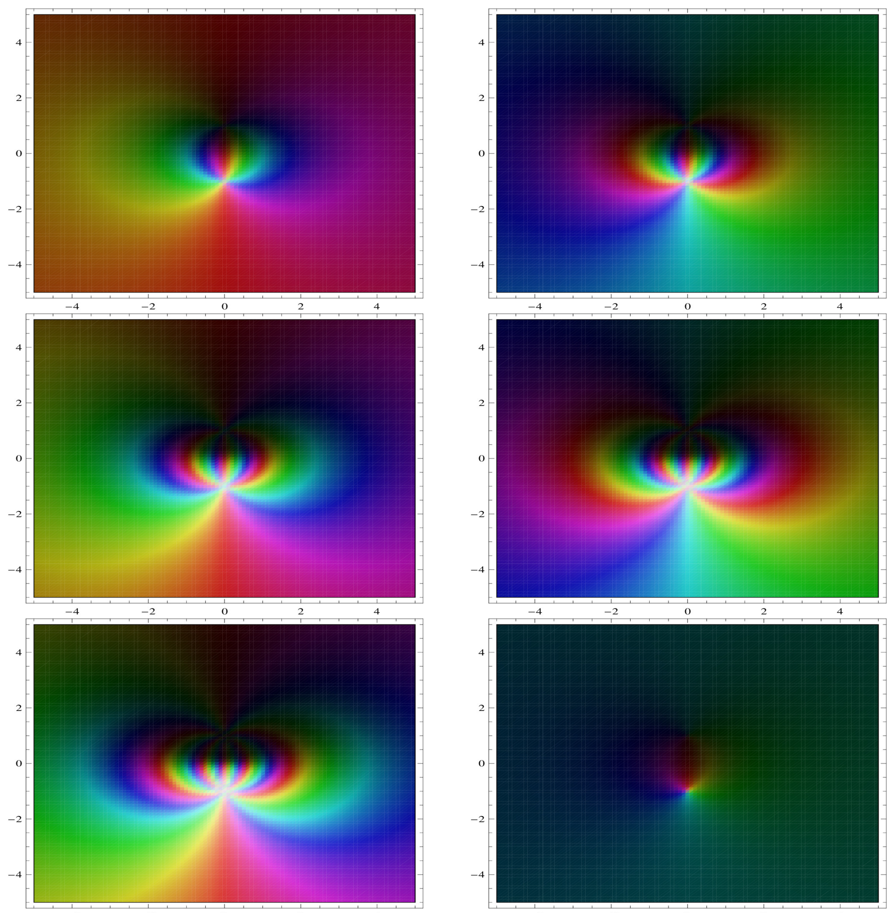

Figure 1. The following is the color function of $f^*_{\eta,h}(z)$ associated to the q–difference operator $Tf(z) := \frac{1}{z}(f(z) - f(0))$ for $\lambda = 10$, $z = x + iy$, $(x, y) \in [-5, 5] \times [-5, 5]$ and respectively $h(z) = 1, z, z^2, z^3, z^4, z^5$. The argument of a complex value is encoded by the hue of a color (red = positive real, and then counterclockwise through yellow, green, cyan, blue and purple; cyan stands for negative real). Strong colors denote points close to the origin, black = 0, weak colors denote points with large absolute value, white = ∞.

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