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New Characterization of Hardy-Fofana Spaces and Temperature Equation

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Abstract. The aim of this paper is to give a characterization of Hardy-Fofana spaces via Riesz transforms. This characterization allows us to describe the distributions belonging to these spaces as a bounded solutions of Cauchy-Riemann's general temperature equations.

1. Introduction

Let $\mathbb{R}^d$ (d is a positive integer) be the Euclidean space of dimension d equipped with the Lebesgue measure dx and the Euclidean norm. The classical Hardy space $\mathcal{H}^p(\mathbb{R}^d)$ $(0 < p < \infty)$ is defined as the space of tempered distributions f satisfying $\|\mathcal{M}f\|_p < \infty$, where the maximal function $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) = \sup_{t>0} |(f * \varphi_t)(x)|, \tag{1.1}$$

with φ in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ having non vanish integral, and $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$.

It is well known that not only this space does not depends on φ, but one can replaced Schwartz function by Poisson kernel in the definition of the maximal function (1.1).

In [1], Ablé and the second author studied Hardy-amalgam spaces $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ $(0 < p, q < \infty)$ by taking in the above maximal characterization of classical Hardy space the Wiener amalgam quasi-norm $\|\cdot\|_{p,q}$ instead of Lebesgue's.

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A locally integrable function u belongs to the amalgam space $(L^p, \ell^q)(\mathbb{R}^d)$ if

$$\|u\|_{p,q} := \left[\sum_{k\in\mathbb{Z}^d} \|u\chi_{Q_k}\|_p^q\right]^{\frac{1}{q}} < \infty,$$

where for $k \in \mathbb{Z}^d$, $Q_k = k + [0,1)^d$ and χ_{Q_k} stands for the characteristic function of Q_k.

Multiple characterizations of $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ spaces including atomic and Poisson kernel characterization, were given in [1]. We notice that the atoms in this context are exactely the one used in classical Hardy space.

Recently, Assaubay et al in [3] characterized this spaces by using first-order classical Riesz transforms and composition of first-order Riesz transformations. They also describe the distributions in $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ as the boundary values of solutions of harmonic and caloric Cauchy-Riemann systems. Here we intend to prove that similar characterizations are possible in the context of Hardy-Fofana spaces.

It is well known that for $0 < p, \alpha, q < \infty$ and $r > 0$, there exists a constant $C_{r;\alpha} > 0$ depending on r and α such that

$$C_{r;\alpha}^{-1}\|u\|_{p,q} \leq \|St_r^\alpha u\|_{q,p} \leq C_{r;\alpha}\|u\|_{p,q}, \qquad u \in (L^p, \ell^q)(\mathbb{R}^d), \tag{1.2}$$

where $(St_r^\alpha u)(x) = r^{-\frac{d}{\alpha}}u(r^{-1}x)$. It follows from the above relation that for $u \in (L^p, \ell^q)(\mathbb{R}^d)$, we have $\mathrm{St}_r^\alpha u \in (L^p, \ell^q)(\mathbb{R}^d)$ for $\alpha > 0$ and $r > 0$. Unfortunately, the family $\{\mathrm{St}_r^\alpha u\}_{r>0}$ is not bounded in $(L^p, \ell^q)(\mathbb{R}^d)$. Ibrahim Fofana considered in [7], the spaces $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ defined for $0 < p, q, \alpha \leq \infty$ by

$$(L^p, \ell^q)^\alpha(\mathbb{R}^d) = \left\{f \in (L^p, \ell^q)(\mathbb{R}^d) / \|f\|_{p,q,\alpha} < \infty\right\}$$

where

$$\|f\|_{p,q,\alpha} := \sup_{r>0} \|St_r^\alpha f\|_{p,q}\,. \tag{1.3}$$

These spaces known as Fofana's spaces are non trivial if and only if $p \leq \alpha \leq q$ (see [7]). In the rest of the paper we will always assume that this condition is fulfilled. It is proved in [6] that for $u \in (L^p, \ell^q)^\alpha(\mathbb{R}^d)$, we have $\|St_r^\alpha u\|_{p,q,\alpha} = \|u\|_{p,q,\alpha}$ and that $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ $(1 \leq p \leq \alpha \leq q)$ is the biggest norm space which is continuously embedded in $(L^p, \ell^q)(\mathbb{R}^d)$ and for which the translation St_r^α is an isometry. These spaces can also be viewed as some generalized Morrey spaces since for $p < \alpha$, the space $(L^p, \ell^\infty)^\alpha(\mathbb{R}^d)$ is exactly the classical Morrey space $L^{p,d\frac{p}{\alpha}}(\mathbb{R}^d)$.

For $0 < p \leq \alpha \leq q < \infty$, Hardy-Fofana space $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$, introduced by the authors in [4], is a subspace of Hardy-amalgam spaces consists of tempered distributions f satisfying

$$\|f\|_{\mathcal{H}^{(p,q,\alpha)}} := \|\mathcal{M}f\|_{p,q,\alpha} < \infty.$$

The purpose of this article is twofold. We first characterize these spaces via Riesz transforms and secondly, we describe the distributions belonging to these spaces as bounded solutions of certain general temperature equations of CAUCHY-RIEMANN.

This paper is organized as follow:

The next Section is devoted to the prerequisites on Hardy-Fofana spaces. In Section 3, we give the characterizations of Hardy-Fofana spaces with Riesz transforms. In the last section, we characterize distributions belonging to our spaces as bounded solutions of certain general temperature equations of CAUCHY-RIEMANN.

In this work, $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ will denote the Schwartz class of rapidly decreasing smooth functions equipped with its usual topology. The dual space of $\mathcal{S}$ is the space of tempered distributions denoted by $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$. The pairing between $\mathcal{S}'$ and $\mathcal{S}$ is denoted by $\langle \cdot, \cdot \rangle$.

We denote by $|E|$, the Lebesgue measure of a measurable subset E of $\mathbb{R}^d$. The notation $A \approx B$ means that there exist two constants $0 < C_1$ and $0 < C_2$ such that $A \leq C_1 B$ and $B \leq C_2 A$, while $A := B$ means that B is the definition of A.

2. Prerequisites for Hardy-Fofana Spaces

Fofana's spaces have among others, the following properties (see for example [6] and [7]):

(1) let $0 < p, \alpha, q \leq \infty$. The space $\left((L^p, \ell^q)^\alpha(\mathbb{R}^d), \|\cdot\|_{p,q,\alpha}\right)$ is a Banach space if $1 \leq p \leq \alpha \leq q$ and a quasi-Banach space if $0 < p < 1$;

(2) if $\alpha \in \{p, q\}$ then $(L^p, \ell^q)^\alpha(\mathbb{R}^d) = L^\alpha(\mathbb{R}^d)$ with equivalent norms;

(3) if $p < \alpha < q$ then $L^\alpha(\mathbb{R}^d) \subsetneq (L^p, \ell^q)^\alpha(\mathbb{R}^d) \subsetneq (L^p, \ell^q)(\mathbb{R}^d)$;

(4) let f and g be two measurable functions on $\mathbb{R}^d$. If $|f| \leq |g|$, then $\|f\|_{p,q,\alpha} \leq \|g\|_{p,q,\alpha}$.

For many operators including the maximal Hardy-Littlewood operator, norm inequalities are given in these spaces for $1 \leq p \leq \alpha \leq q$.

Let f be a locally integrable function and $\mathfrak{M}(f)$ be the centered Hardy-Littlewood maximal function defined by

$$\mathfrak{M}(f)(x) := \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)| dy, \ \forall \ x \in \mathbb{R}^d.$$

It is proved in [6, Proposition 4.2] that $\mathfrak{M}$ is bounded on $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$, whenever $1 < p \leq \alpha \leq q \leq \infty$. Using [8, Proposition 11.12], it is easy to extablish the following result whose proof is omitted.

Proposition 2.1. *Let $1 < p \leq \alpha \leq q < +\infty$ and $1 < u \leq +\infty$. For all sequences $\{f_n\}_{n\geq 0}$ of measurable functions, we have*

$$\left\| \left(\sum_{n\geq 0} |\mathfrak{M}(f_n)|^u \right)^{\frac{1}{u}} \right\|_{p,q,\alpha} \approx \left\| \left(\sum_{n\geq 0} |f_n|^u \right)^{\frac{1}{u}} \right\|_{p,q,\alpha},$$

with the equivalence constants not depending on the sequence $\{f_n\}_{n\geq 0}$.

As Hardy-Fofana spaces are concerned, we have among others, the following properties which can be found in [4].

Proposition 2.2. *Let $1 \le p \le \alpha \le q < \infty$.*

(1) *If $1 < p$ then the space $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ and $(L^p, L^q)^\alpha(\mathbb{R}^d)$ are equal with equivalence norms.*
(2) *The space $\mathcal{H}^{(1,q,\alpha)}(\mathbb{R}^d)$ is continuously embedded in $(L^1, \ell^q)^\alpha(\mathbb{R}^d)$.*

Notice that for $p < 1$, we have as in the classical Hardy and Hardy-amalgam spaces, that the spaces $\left(\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}^{(p,q,\alpha)}}\right)$ are quasi-Banach and for $f, g \in \mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$,

$$\|f+g\|^p_{\mathcal{H}^{(p,q,\alpha)}} \le \|f\|^p_{\mathcal{H}^{(p,q,\alpha)}} + \|g\|^p_{\mathcal{H}^{(p,q,\alpha)}}.$$

We can also define (see [5]) these spaces as subspaces of Hardy-amalgam spaces for which the familly of dilations $\left\{\mathrm{St}^\alpha_\rho\right\}_{\rho>0}$ is locally bounded.

More precisely, for a tempered distribution f, $\rho > 0$ and α two real numbers we put

$$\left\langle \mathrm{St}^\alpha_\rho f, \varphi\right\rangle := \left\langle f, \mathrm{St}^{\alpha'}_{\rho^{-1}}\varphi\right\rangle,$$

where $\frac{1}{\alpha'} + \frac{1}{\alpha} = 1$. We have (see [4]) that for $0 < p \le \alpha \le q \le \infty$,

$$\|f\|_{\mathcal{H}^{(p,q,\alpha)}} = \sup_{\rho>0} \|\mathrm{St}^\alpha_\rho f\|_{\mathcal{H}^{(p,q)}}. \tag{2.1}$$

Just as Hardy-amalgam spaces was characterized in [1] with Poisson kernel, so are Hardy-Fofana's spaces. In fact, a tempered distribution f belonging to Hardy-amalgam spaces is bounded; i.e $f * \psi \in L^\infty(\mathbb{R}^d)$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$. A convolution of such distribution with integrable functions can be defined in term of distribution. More precisely, if $f \in \mathcal{S}'(\mathbb{R}^d)$ is bounded and $u \in L^1(\mathbb{R}^d)$, then the convolution $f * u$ is defined as a tempered distribution acting on $\mathcal{S}(\mathbb{R}^d)$ by the pairing

$$\langle f * u, \varphi\rangle := \langle f * \tilde{\varphi}, \tilde{u}\rangle_{(L^\infty, L^1)} \qquad \varphi \in \mathcal{S}(\mathbb{R}^d)$$

where $\tilde{u}(x) = u(-x)$ and $\langle f * \tilde{\varphi}, \tilde{u}\rangle_{(L^\infty, L^1)}$ is the pairing between $L^\infty(\mathbb{R}^d)$ and $L^1(\mathbb{R}^d)$. But if we take as u the Poisson kernel P defined by

$$P(x) := \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{d+1}{2}}} \quad x \in \mathbb{R}^d,$$

then $f * P_t$ can be identified for all $t > 0$, to a well defined bounded function. As we can see for example in [9], there exist $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$f * P_t = (f * \varphi_t) * P_t + f * \psi_t \text{ for } t > 0.$$

It is proved in [1] that for an element $f \in \mathcal{H}^{(p,q)}(\mathbb{R}^d)$, we have

$$\|x \mapsto \sup_{t>0} \sup_{|x-y|<t} |f * P_t(y)|\|_{p,q} \approx \|\mathcal{M}f\|_{p,q} \tag{2.2}$$

where $\mathcal{M}f$ is the maximal function defined in Relation (1.1). It follows that

$$\|x \mapsto \sup_{t>0} \sup_{|x-y|<t} |f * P_t(y)|\|_{p,q,\alpha} \approx \|\mathcal{M}f\|_{p,q,\alpha}, \tag{2.3}$$

thanks to Relations (2.1) and (2.2) and the fact that St_ρ^α commute with the maximal function $\mathcal{M}$.

Lemma 2.3. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, ρ and α positive real numbers. We have*

$$\mathrm{St}_\rho^\alpha (f * \varphi_t) = \left(\mathrm{St}_\rho^\alpha f\right) * \varphi_{\rho t}, \quad t > 0.$$

Infact,

$$\begin{aligned} \rho^{\frac{-d}{\alpha}} (f * \varphi_t)(\rho^{-1}x) &= \rho^{\frac{-d}{\alpha}} \left\langle f, \rho^d \varphi_{\rho t}(x - \rho \cdot)\right\rangle \\ &= \rho^{\frac{d}{\alpha'}} \langle f, \varphi_{\rho t}(x - \rho \cdot)\rangle \\ &= \left\langle f, \mathrm{St}_{\rho^{-1}}^{\alpha'} [\varphi_{\rho t}(x - \cdot)]\right\rangle = \left(\mathrm{St}_\rho^\alpha f * \varphi_{\rho t}\right)(x). \end{aligned}$$

Lemma 2.4. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, ρ and α positive real numbers. We have*

$$\mathrm{St}_\rho^\alpha [(f * \varphi_t) * P_t] = \left(\mathrm{St}_\rho^\alpha f * \varphi_{\rho t}\right) * P_{\rho t}. \tag{2.4}$$

Relation (2.4) follows from the fact that

$$\begin{aligned} \rho^{\frac{-d}{\alpha}} \left(f * \varphi_{\rho^{-1}t}\right) * P_{\rho^{-1}t}(\rho^{-1}x) &= \rho^{\frac{-d}{\alpha}} \int_{\mathbb{R}^d} \left(f * \varphi_{\rho^{-1}t}\right)(\rho^{-1}x - y) P_{\rho^{-1}t}(y) dy \\ &= \int_{\mathbb{R}^d} \left\langle f, \rho^{\frac{d}{\alpha'}} \varphi_t(x - z - \rho \cdot)\right\rangle P_t(z) dz \\ &= \int_{\mathbb{R}^d} \left\langle \mathrm{St}_\rho^\alpha f, \varphi_t(x - z - \cdot)\right\rangle P_t(z) dz \\ &= \int_{\mathbb{R}^d} \left(\mathrm{St}_\rho^\alpha f * \varphi_t\right)(x - z) P_t(z) dz = \left(\mathrm{St}_\rho^\alpha f * \varphi_t\right) * P_t(x). \end{aligned}$$

It comes from Lemma 2.3 and 2.4 that for a bounded tempered distribution f and $u(x,t) = f * P_t(x)$,

$$\left(\mathrm{St}_\rho^\alpha u\right)(x,t) = \left[\left(\mathrm{St}_\rho^\alpha f\right) * P_t\right](x), \quad \rho > 0 \text{ and } \alpha > 0 \tag{2.5}$$

for all $t > 0$.

3. Cauchy-Riemann equations, Riesz transforms and Hardy-Fofana spaces

Let u be a harmonic function on $\mathbb{R}_+^{d+1}$; i.e, $u \in \mathcal{C}^2(\mathbb{R}_+^{d+1})$ and $\Delta u := \sum_{j=1}^{d+1} \frac{\partial^2 u}{(\partial x_j)^2} = 0$, where $x_{d+1} = t$ and $\mathbb{R}_+^{d+1} := \mathbb{R}^d \times]0, +\infty[$. We define its non tangential maximal function u^* by

$$u^*(x) := \sup_{t>0} \sup_{|x-y|<t} |u(y,t)| \quad \forall x \in \mathbb{R}^d. \tag{3.1}$$

Let f be a bounded tempered distribution, and $u(x,t) = P_t * f(x)$. As we can see in [4], $u^* \in (L^p, \ell^q)^\alpha(\mathbb{R}^d)$ whenever $f \in \mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$. We give in the next result a necessary and sufficient conditions for a harmonic function u in $\mathbb{R}^{d+1}$ to have its non tangential maximal function in $(L^p,\ell^q)^\alpha(\mathbb{R}^d)$. The proof is based on the dilation characterization of Hardy-Fofana spaces and [3, Proposition 2.1].

Proposition 3.1. *Let* $0<p\leq\alpha\leq q<+\infty$ *and* u *an harmonic function on* $\mathbb{R}^{d+1}_+$. *The maximal function* u^* *belongs to* $(L^p,\ell^q)^\alpha(\mathbb{R}^d)$ *if and only if there exists* $f\in\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ *such that*

$$u(x,t):=f*P_t(x),\ \ (x,t)\in\mathbb{R}^{d+1}_+.$$

Moreover, $\|f\|_{\mathcal{H}^{(p,q,\alpha)}}\approx\|u^*\|_{p,q,\alpha}$.

Proof. Let u be an harmonic function on $\mathbb{R}^{d+1}_+$, and u^* the associate non tangential maximal function as defined in Relation (3.1).

We suppose that there exists $f\in\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ such that $u(x,t):=f*P_t(x)$ for all $(x,t)\in\mathbb{R}^{d+1}_+$. From the Poisson characterization of Hardy-Fofana spaces (see [4, Theorem 2.3.8]), we deduce that $\|u^*\|_{p,q,\alpha}\leq C\|f\|_{\mathcal{H}^{(p,q,\alpha)}}$.

For the converse, let us suppose that $u^*\in(L^p,\ell^q)^\alpha(\mathbb{R}^d)\subset(L^p,\ell^q)(\mathbb{R}^d)$. It comes from [3, Proposition 2.1] that there exists $f\in\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ and a constant $C>0$ such that

$$u(x,t)=(f*P_t)(x),\qquad (x,t)\in\mathbb{R}^{d+1}_+ \tag{3.2}$$

and

$$\frac{1}{C}\|f\|_{\mathcal{H}^{(p,q)}}\leq\|u^*\|_{p,q}\leq C\|f\|_{\mathcal{H}^{(p,q)}}.$$

Since $\mathrm{St}^\alpha_\rho f\in\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ for all $\rho>0$, $\mathrm{St}^\alpha_\rho u$ harmonic on $\mathbb{R}^{d+1}_+$ and $\left(\mathrm{St}^\alpha_\rho u\right)(x,t)=(\mathrm{St}^\alpha_\rho f)*P_t(x)$, it comes that $(\mathrm{St}^\alpha_\rho u)^*\in(L^p,L^q)(\mathbb{R}^d)$ and

$$\frac{1}{C}\|\mathrm{St}^\alpha_\rho f\|_{\mathcal{H}^{(p,q)}}\leq\|(\mathrm{St}^\alpha_\rho u)^*\|_{p,q}\leq C\|\mathrm{St}^\alpha_\rho f\|_{\mathcal{H}^{(p,q)}}.$$

This relation being thrue for all $\rho>0$, we have

$$\frac{1}{C}\|f\|_{\mathcal{H}^{(p,q,\alpha)}}\leq\|u^*\|_{p,q,\alpha}\leq C\|f\|_{\mathcal{H}^{(p,q,\alpha)}},$$

where we use the trivial identity $(St^\alpha_\rho u)^*=St^\alpha_\rho u^*,\ \rho>0$ and $0<\alpha<\infty$. □

We say that a vector values function $F:=(u_1,u_2,...,u_{d+1})$, with $u_j:\mathbb{R}^{d+1}_+\to\mathbb{R}$, $j\in\{1,2,...,d+1\}$, satisfies the generalized Cauchy-Riemann equations (in short $F\in\mathrm{CR}(\mathbb{R}^{d+1}_+)$) if

$$\frac{\partial u_j}{\partial x_k}=\frac{\partial u_k}{\partial x_j},\ 1\leq j,k\leq d+1\qquad\text{and}\qquad\sum_{j=1}^{d+1}\frac{\partial u_j}{\partial x_j}=0, \tag{3.3}$$

where we set $x_{d+1}=t$. Also recall that for $j\in\{1,2,...d\}$, the j-th Riesz transform $\mathcal{R}_j(g)$ of a measurable function g is formally defined by

$$\mathcal{R}_j(g)(x):=\lim_{\epsilon\to0^+}\int_{|x-y|>\epsilon}K_j(x-y)g(y)dy\quad\text{a.e}\quad x\in\mathbb{R}^d.$$

where $K_j(x) := \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \frac{x_j}{|x|^{d+1}}$, $x \in \mathbb{R}^d \backslash \{0\}$.

In [2, Corollary 4.19], Ablé and Feuto demonstrated that Riesz transformations are extendable into bounded linear operators on Hardy-amalgam spaces $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ for $0 < p \leq 1$. We will keep the notations $\mathcal{R}_j$, $j = 1, \cdots, d$ for these extentions. Assaubay et al proved the following result.

Proposition 3.2 ([3], Proposition 2.3)**.** *Let $\frac{d-1}{d} < \min\{p, q\} < +\infty$. Suppose that u is harmonic function in $\mathbb{R}^{d+1}_+$. Then $u^* \in (L^p, \ell^q)(\mathbb{R}^d)$ if and only if there exists an harmonic vector $F := (u_1, ..., u_{d+1}) \in \mathrm{CR}(\mathbb{R}^{d+1}_+)$ such that $u_{d+1} := u$ and $\sup_{t>0} |||F(., t)|||_{p,q} < +\infty$.*

Furthermore, $\sup_{t>0} |||F(., t)|||_{p,q} \approx \|u^*\|_{p,q}$.

Since $u^* \in (L^p, \ell^q)(\mathbb{R}^d)$ if and only if $u = f * P_t$ for some $f \in \mathcal{H}^{(p,q)}(\mathbb{R}^d)$, they proved that one can take $u_j(x, t) = \mathcal{R}_j(f) * P_t(x)$, $j = 1, \cdots, d$.

In the case of Hardy-Fofana's spaces, we have the following.

Proposition 3.3. *Assume that $\frac{d-1}{d} < p \leq \alpha \leq q < +\infty$ and u is an harmonic function in $\mathbb{R}^{d+1}_+$. Then $u^* \in (L^p, \ell^q)^\alpha(\mathbb{R}^d)$ if and only if there exists an harmonic vector $F := (u_1, ..., u_{d+1}) \in \mathrm{CR}(\mathbb{R}^{d+1}_+)$ such that $u_{d+1} := u$ and $\sup_{t>0} |||F(., t)|||_{p,q,\alpha} < +\infty$. Furthermore*

$$\sup_{t>0} |||F(., t)|||_{p,q,\alpha} \approx \|u^*\|_{p,q,\alpha} \tag{3.4}$$

Proof. Let $\frac{d-1}{d} < p \leq \alpha \leq q < +\infty$ and u an harmonic function on $\mathbb{R}^{d+1}_+$.

We suppose that $u^* \in (L^p, \ell^q)^\alpha(\mathbb{R}^d)$. Since $(L^p, \ell^q)^\alpha(\mathbb{R}^d) \subset (L^p, \ell^q)(\mathbb{R}^d)$, Proposition 3.2 assert that there exists $f \in (L^p, \ell^q)(\mathbb{R}^d)$ so that:

- $u(x, t) = f * P_t(x)$,
- the harmonic vector $F = (u_1, \cdots, u_{d+1})$ with $u_j(x, t) = R_j(f) * P_t(x)$ for $j \in \{1, \cdots, d\}$ and $u_{d+1} = u$ belongs to $\mathrm{CR}(\mathbb{R}^{d+1}_+)$,
- $\sup_{t>0} |||F(\cdot, t)|||_{p,q} \approx \|u^*\|_{p,q}$.

Since $u^* \in (L^p, \ell^q)^\alpha (\mathbb{R}^d)$ we have that the tempered distribution f belongs to $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$, thanks to Proposition 3.1. All we have to prove now is that $x \mapsto F(x, t)$ belongs to $(L^p, \ell^q)^\alpha(\mathbb{R}^d)$ for $t > 0$ and that Relation (3.4) is satisfies.

Fix $t > 0$ and $\rho > 0$. Since $u^* \in (L^p, \ell^q)^\alpha (\mathbb{R}^d)$ and $\left(\mathrm{St}^\alpha_\rho u\right)^* = \mathrm{St}^\alpha_\rho (u^*)$, we have that for $\rho > 0$, $\| \left(\mathrm{St}^\alpha_\rho u\right)^* \|_{p,q} \leq \|u^*\|_{p,q,\alpha}$. Hence $\left(\mathrm{St}^\alpha_\rho u\right)^* \in (L^p, \ell^q) (\mathbb{R}^d)$ so that there exists $f_\rho \in \mathcal{H}^{(p,q)}(\mathbb{R}^d)$ satisfying

$$(\mathrm{St}^\alpha_\rho u)(x, t) = (f_\rho * P_t)(x),$$

with

$$F_\rho(x, t) := (\mathcal{R}_1(f_\rho) * P_t(x), \cdots, \mathcal{R}_d(f^\rho) * P_t(x), (f_\rho) * P_t)(x)) \tag{3.5}$$

belonging to $\mathrm{CR}_+(\mathbb{R}^{d+1}_+)$ and

$$\sup_{t>0} |||F_\rho(\cdot, t)|||_{p,q} \approx \|\mathrm{St}^\alpha_\rho(u^*)\|_{p,q}. \tag{3.6}$$

Moreover $(\mathrm{St}_\rho^\alpha u)(x,t) = (\mathrm{St}_\rho^\alpha f) * P_t(x)$, thanks to Relation (2.5). It follows that $f_\rho * P_t = (\mathrm{St}_\rho^\alpha f) * P_t$ for all $t > 0$ so that $f_\rho = \mathrm{St}_\rho^\alpha f$. We recall that the last equality comes from the fact that for $f \in \mathcal{H}^{(p,q)}(\mathbb{R}^d)$, $f * P_t$ tends to f in $\mathcal{S}'(\mathbb{R}^d)$ as t goes to 0. Replacing f_ρ by $\mathrm{St}_\rho^\alpha f$ in Relation (3.5) yields $F_\rho(x,t) = \big(\mathcal{R}_1(\mathrm{St}_\rho^\alpha f) * P_t(x), \cdots, \mathcal{R}_d(\mathrm{St}_\rho^\alpha f) * P_t(x), (\mathrm{St}_\rho^\alpha f) * P_t)(x)\big)$. Since the operator St_ρ^α commute with R_j we have that

$$\begin{aligned} F_\rho(\cdot,t) &= \big(\mathrm{St}_\rho^\alpha(\mathcal{R}_1 f) * P_t(\cdot), \cdots, \mathrm{St}_\rho^\alpha(\mathcal{R}_d f) * P_t(\cdot), (\mathrm{St}_\rho^\alpha f) * P_t)(\cdot)\big) \\ &= \left(\mathrm{St}_\rho^\alpha\left(u_1(\cdot,\rho^{-1}t)\right), \cdots, \mathrm{St}_\rho^\alpha\left(u_d(\cdot,\rho^{-1}t)\right), \mathrm{St}_\rho^\alpha\left(u_{d+1}(\cdot,\rho^{-1}t)\right)\right) \\ &= \mathrm{St}_\rho^\alpha\left(F(\cdot,\rho^{-1}t)\right). \end{aligned}$$

If we take this expression of F_ρ in Relation (3.6) we obtain that

$$\sup_{t>0} \||\mathrm{St}_\rho^\alpha(F(\cdot,\rho^{-1}t)|\|_{p,q} \approx \|\mathrm{St}_\rho^\alpha(u^*)\|_{p,q}.$$

But $\sup_{t>0} \||\mathrm{St}_\rho^\alpha(F(\cdot,\rho^{-1}t))|\|_{p,q} = \sup_{t>0} \||\mathrm{St}_\rho^\alpha(F(\cdot,t))|\|_{p,q}$ and the result follow from the definition of Hardy-Fofana space. □

The next result gives a characterization of $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ via Riesz transforms $\mathcal{R}_j(f * \phi)$. Since we need to use the characterization of $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ given in [3], we give the following definition.

Definition 3.4. *Let $0 < p \le \alpha \le q < \infty$. A tempered distribution f is said to be :*

- *(p,q)-restricted at infinity if there exists $\mu_0 \ge 1$ such that for $\mu \ge \mu_0$, we have*

$$f * \phi \in (L^{p\mu}, \ell^{q\mu})(\mathbb{R}^d), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

- *(p,q,α)-restricted at infinity if there exists $\mu_0 \ge 1$ such that for $\mu \ge \mu_0$, we have*

$$f * \phi \in (L^{p\mu}, \ell^{q\mu})^{\alpha\mu}(\mathbb{R}^d), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

It is easy to see that tempered distributions which are (p,q,α)-restricted for $p \le \alpha \le q$, are also (p,q)-restricted. Theorem 1.1 in [3] assert that a tempered distribution f belongs to $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ for $\frac{d-1}{d} < \min(p,q) < \infty$, if and only if it is (p,q)-restricted at infty and, for $\phi \in \mathcal{S}(\mathbb{R}^d)$ with non vanish integral,

$$\sup_{t>0}\left(\|f * \phi_t\|_{p,q} + \sum_{j=1}^d \|(\mathcal{R}_j f) * \phi_t\|_{p,q}\right) < \infty.$$

When this is the case,

$$\|f\|_{\mathcal{H}^{(p,q)}} \approx \sup_{t>0}\left(\|f * \phi_t\|_{p,q} + \sum_{j=1}^d \|(\mathcal{R}_j f) * \phi_t\|_{p,q}\right).$$

In the case of Hardy-Fofana space, we have the following result.

Theorem 3.5. *Let $\frac{d-1}{d} < p \leq \alpha \leq q < \infty$, $f \in \mathcal{S}'(\mathbb{R}^d)$.Then $f \in \mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ if and only if f is (p,q,α)-restricted at infinity and, for $\phi \in \mathcal{S}(\mathbb{R}^d)$ with non vanish integral,*

$$\sup_{t>0}\left(\|f*\phi_t\|_{p,q,\alpha}+\sum_{j=1}^{d}\left\|(\mathcal{R}_jf)*\phi_t\right\|_{p,q,\alpha}\right)<+\infty. \tag{3.7}$$

Moreover,

$$\|f\|_{\mathcal{H}^{(p,q,\alpha)}}\approx\sup_{t>0}\left(\|f*\phi_t\|_{p,q,\alpha}+\sum_{j=1}^{d}\left\|(\mathcal{R}_jf)*\phi_t\right\|_{p,q,\alpha}\right). \tag{3.8}$$

Proof. Let $\frac{d-1}{d} < p \leq \alpha \leq q < \infty$ and $f \in \mathcal{S}'(\mathbb{R}^d)$.

We suppose that f is (p,q,α)-restricted at infinity and satisfies (3.7) for non vanishing Schwartz function ϕ. There exists $\mu_0 > 1$ (large enought) such that for $\mu > \mu_0$, we have

$$f*\phi\in\left(L^{p\mu};\ell^{q\mu}\right)^{\alpha\mu}(\mathbb{R}^d),\quad \phi\in\mathcal{S}(\mathbb{R}^d). \tag{3.9}$$

It comes from the definition of Fofana spaces that

$$\mathrm{St}_{\rho}^{\alpha\mu}(f*\phi)\in\left(L^{p\mu},\ell^{\mu q}\right)(\mathbb{R}^d)\quad \phi\in\mathcal{S}(\mathbb{R}^d),\quad \rho>0.$$

Taking $\rho = 1$, we obtain that f is (p,q)-restricted at infinity. Since for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ with non vanishing integral we also have that

$$A=\sup_{t>0}\left(\sup_{\rho}\left\|\mathrm{St}_{\rho}^{\alpha}(f*\phi_t)\right\|_{p,q}+\sum_{j=1}^{d}\sup_{\rho>0}\left\|\mathrm{St}_{\rho}^{\alpha}\left((\mathcal{R}_jf)*\phi_t\right)\right\|_{p,q}\right)<\infty,$$

it follows that

$$\sup_{t>0}\left(\|f*\phi_t\|_{p,q}+\sum_{j=1}^{d}\left\|(\mathcal{R}_jf)*\phi_t\right\|_{p,q}\right)\leq A.$$

Thus $f \in \mathcal{H}^{(p,q)}(\mathbb{R}^d)$ thanks to [3, Theorem 1.1]. It remains to prove that the familly $\left\{\mathrm{St}_{\rho}^{\alpha}f\right\}_{\rho>0}$ is uniformly bounded in $\mathcal{H}^{(p,q)}(\mathbb{R}^d)$.

Fix $\rho > 0$. We have $\mathrm{St}_{\rho}^{\alpha}f \in \mathcal{H}^{(p,q)}(\mathbb{R}^d)$ so that

$$\|\mathrm{St}_{\rho}^{\alpha}f\|_{\mathcal{H}^{(p,q)}}\approx\sup_{t>0}\left(\left\|\mathrm{St}_{\rho}^{\alpha}(f)*\phi_t\right\|_{p,q}+\sum_{j=1}^{d}\left\|\mathcal{R}_j(\mathrm{St}_{\rho}^{\alpha}f)*\phi_t\right\|_{p,q}\right)$$

thanks once more to [3, Theorem 1.1]. But we have in one hand that

$\mathcal{R}_j(f)*\phi_t = \mathcal{R}_j(f*\phi_t)$, so that

$$\mathrm{St}_{\rho}^{\alpha}\left[(\mathcal{R}_jf)*\phi_t\right]=\mathrm{St}_{\rho}^{\alpha}\left[\mathcal{R}_j(f*\phi_t)\right]=\mathcal{R}_j\left[\mathrm{St}_{\rho}^{\alpha}(f*\phi_t)\right], \tag{3.10}$$

where the last equality comes from the fact that dilation comute with Riesz transforms. In the other hand we have that

$$\sup_{t>0}\|\mathrm{St}_{\rho}^{\alpha}(f*\phi_t)\|_{p,q}=\sup_{t>0}\|\mathrm{St}_{\rho}^{\alpha}(f)*\phi_t\|_{p,q}, \tag{3.11}$$

thanks to Lemma 2.3. Therefore, we have

$$\sup_{\rho>0}\sup_{t>0}\|\mathrm{St}_\rho^\alpha(f)*\phi_t\|_{p,q}=\sup_{\rho>0}\sup_{t>0}\|\mathrm{St}_\rho^\alpha(f*\phi_t)\|_{p,q}\leq A$$

and

$$\sup_{\rho>0}\sup_{t>0}\sum_{j=1}^{d}\|\mathcal{R}_j\left[\mathrm{St}_\rho^\alpha(f*\phi_t)\right]\|_{p,q}=\sup_{\rho>0}\sup_{t>0}\sum_{j=1}^{d}\|\mathrm{St}_\rho^\alpha\left[\left(\mathcal{R}_jf\right)*\phi_t\right]\|_{p,q}\leq A.$$

We deduce that $\sup_{\rho>0}\|\mathrm{St}_\rho^\alpha f\|_{\mathcal{H}^{(p,q)}}<\infty$, which prove that $f\in\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$.

For the converse, we suppose that $f\in\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$. It follows that $\mathrm{St}_\rho^\alpha f\in\mathcal{H}^{(p,q)}(\mathbb{R}^d)$ with $\|\mathrm{St}_\rho^\alpha f\|_{\mathcal{H}^{(p,q)}}\leq\|f\|_{\mathcal{H}^{(p,q,\alpha)}}<\infty$ for all $\rho>0$. It comes from [3, Theorem 1.1] that $\mathrm{St}_\rho^\alpha f$ is (p,q)-resticted at infinity and

$$\|\mathrm{St}_\rho^\alpha f\|_{\mathcal{H}^{(p,q)}}\approx\sup_{t>0}\left(\left\|\mathrm{St}_\rho^\alpha(f)*\phi_t\right\|_{p,q}+\sum_{j=1}^{d}\left\|\mathrm{St}_\rho^\alpha(\mathcal{R}_jf)*\phi_t\right\|_{p,q}\right)$$

for all $\phi\in\mathcal{S}(\mathbb{R}^d)$ with non vanish integral.

From Relations (3.11) and (3.10), and the definitions of $\|\cdot\|_{p,q,\alpha}$ and of $\|\cdot\|_{\mathcal{H}^{(p,q,\alpha)}}$, we have that

$$\|f\|_{\mathcal{H}^{(p,q,\alpha)}}\approx\sup_{t>0}\left(\|f*\phi_t\|_{p,q,\alpha}+\sum_{j=1}^{d}\left\|(\mathcal{R}_jf)*\phi_t\right\|_{p,q,\alpha}\right)<\infty.$$

Let $\phi\in\mathcal{S}(\mathbb{R}^d)$. We have $\|f*\phi\|_{p,q,\alpha}\leq C\|f\|_{\mathcal{H}^{(p,q,\alpha)}}$. For $\mu\geq1$ we have $f*\phi\in(L^{p\mu},\ell^{q\mu})^{\alpha\mu}$. In fact assuming that $\|f*\varphi\|_\infty\neq0$ we have

$$f*\phi\in(L^p,\ell^q)^\alpha\quad\text{and}\quad\|f*\phi\|_{p\mu,q\mu,\alpha\mu}\leq C\|f*\phi\|_\infty^{1-\frac{1}{\mu}}\|f*\phi\|_{p,q,\alpha}^{\frac{1}{\mu}}$$

and then f is (q,p,α)-restricted at infinity. □

4. Temperature Cauchy-Riemann Equations and Hardy-Fofana Spaces

A vector $F=(u_1,u_2,\cdots,u_{d+1})$ of functions in $\mathbb{R}_+^{d+1}$ satisfy the generalized temperature Cauchy-Riemann equations, if it satisfies the following conditions :

(1) $\sum_{j=1}^{d}\frac{\partial u_j}{\partial x_j}=i\partial_t^{1/2}u_{d+1}$
(2) $\frac{\partial u_j}{\partial x_k}=\frac{\partial u_k}{\partial x_j}$ for $j,k=1,2,\cdots,d$
(3) $\frac{\partial u_{d+1}}{\partial x_j}=-i\partial_t^{1/2}u_j, j=1,2,\cdots,d$, with

$$(\partial_t^{1/2}g)(t):=\frac{e^{i\pi/2}}{\sqrt{\pi}}\int_t^\infty\frac{g'(s)}{\sqrt{s-t}}ds,\, t>0$$

when g is a smooth enough function on $(0,\infty)$

In [3], the authors defined the space $\mathbb{H}^{p,q}(\mathbb{R}_+^{d+1})$ $(0<p,q<\infty)$ as the vector space of vector functions $F=(u_1,u_2,\cdots,u_{d+1})$ satisfying generalized temperature Cauchy-Riemann equations and such that

$$\|F\|_{\mathbb{H}^{p,q}(\mathbb{R}_+^{d+1})}:=\sup_{t>0}\||F(\cdot,t)|\|_{p,q}<\infty.$$

They also proved that under appropriate conditions on the exponents p and q, the space $\mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)$ is topologically isomorphic to $\mathcal{H}^{p,q}(\mathbb{R}^d)$. To carry out the proof of this result, they use a subspace of what they call the temperature space $\mathcal{T}(\mathbb{R}^{d+1}_+)$; that is the space of functions $u \in \mathcal{C}^2(\mathbb{R}^{d+1}_+)$, satisfying

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} \text{ in } \mathbb{R}^{d+1}_+.$$

More precisely, for $0 < p, q < \infty$, they put

$$\mathcal{T}^{p,q}(\mathbb{R}^{d+1}_+) := \left\{ u \in \mathcal{T}(\mathbb{R}^{d+1}_+) : ||u||_{\mathcal{T}^{(p,q)}} < \infty \right\}$$

where

$$||u||_{\mathcal{T}^{(p,q)}} := \sup_{t>0} ||u(.,t)||_{q,p}.$$

They proved [3, Proposition 3.2 (i)] that for $\frac{d-1}{d} < p, q < \infty$, $F = (u_1, u_2, \cdots, u_{d+1}) \in \mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)$ implies that $u := u_{d+1} \in \mathcal{T}^{p,q}(\mathbb{R}^{d+1}_+)$ and $u_j(\cdot,t) = \mathcal{R}_j(u(\cdot,t)), \quad t > 0, \quad j = 1, \cdots, d$.

We claim that for $0 < p \leq \alpha \leq q < \infty$ and $r > 0$, the space $\mathcal{T}^{p,q}(\mathbb{R}^{d+1}_+)$ is stable under the dilation St^{α}_r. This is due to the fact that for $f \in (L^p, \ell^q)(\mathbb{R}^d)$, there exists a constant $C(\alpha, r, p, q) > 0$ such that

$$C(\alpha, r, p, q)^{-1} \|f\|_{p,q} \leq \|\mathrm{St}^{\alpha}_r f\|_{p,q} \leq C(\alpha, r, p, q) \|f\|_{p,q},$$

and this dilation commute with Riesz transforms. It follows that if $F = (u_1, \cdots, u_{d+1}) \in \mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)$ then $\mathrm{St}^{\alpha}_r F \in \mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)$.

We put

$$\|F\|_{\mathbb{H}^{(p,q,\alpha)}} := \sup_{r>0} \|\mathrm{St}^{\alpha}_r F\|_{\mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)}$$

and defined the space $\mathbb{H}^{(p,q,\alpha)}(\mathbb{R}^{d+1}_+)$ as the subspace of $\mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)$ consits of F satisfying $\|F\|_{\mathbb{H}^{(p,q,\alpha)}} < \infty$. We have the following result in Hardy-Fofana spaces.

Theorem 4.1. *Let $\frac{d-1}{d} < p \leq \alpha \leq q < \infty$, and W_t the heat kernel defined by*

$$W_t(x) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{d/2}}.$$

The map $\mathcal{L}$ define on $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ by

$$\mathcal{L}(f)(x,t) := \big(((\mathcal{R}_1 f) * W_t)(x), \cdots, ((\mathcal{R}_d f) * W_t)(x), (f * W_t)(x) \big)$$

for all $x \in \mathbb{R}^d$ and $t > 0$, is a topological isomorphism from $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$ onto $\mathbb{H}^{(p,q,\alpha)}(\mathbb{R}^{d+1}_+)$.

Proof. Let $f \in \mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$. For $r > 0$ we have $\mathrm{St}^{\alpha}_r f \in \mathcal{H}^{p,q}(\mathbb{R}^d)$, thanks to the definition of $\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$. It comes from [3, Theorem 1.3] that

$$\mathcal{L}(St^{\alpha}_r f) \in \mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+), \text{ with } \|\mathcal{L}(\mathrm{St}^{\alpha}_r f)\|_{\mathbb{H}^{p,q}(\mathbb{R}^{d+1}_+)} \leq C \|\mathrm{St}^{\alpha}_r f\|_{\mathcal{H}^{p,q}(\mathbb{R}^d)} \tag{4.1}$$

for all $r > 0$. Since $\mathcal{R}_j(f * W_t) = \mathcal{R}_j(f) * W_t$ and St_r^α commute with Riesz transforms, we have that $\mathcal{L}(St_r^\alpha f) = St_r^\alpha(\mathcal{L}(f))$, $r > 0$. Thaking this remark in (4.1) we obtain that

$$\|\mathcal{L}(f)\|_{\mathbb{H}^{(p,q,\alpha)}(\mathbb{R}_+^{d+1})} \leq C\|f\|_{\mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)}.$$

Let now $F = (u_1, u_2, \cdots, u_d, u_{d+1})$ belonging to $\mathbb{H}^{(p,q,\alpha)}(\mathbb{R}_+^{d+1})$. This implies that $\mathrm{St}_r^\alpha F = (\mathrm{St}_r^\alpha u_1, \mathrm{St}_r^\alpha u_2, \cdots, \mathrm{St}_r^\alpha u_d, \mathrm{St}_r^\alpha u_{d+1}) \in \mathbb{H}^{p,q}(\mathbb{R}_+^{d+1})$ for all $r > 0$. As we can see in the proof of [3, Theorem 3.1] this implies that for all $r > 0$, there exists $f_r \in \mathcal{H}^{p,q}(\mathbb{R}^d)$ so that $\mathrm{St}_r^\alpha u_{d+1} \in \mathcal{T}^{p,q}(\mathbb{R}_+^{d+1})$ with $\mathrm{St}_r^\alpha u_{d+1}(x,t) = f_r * W_t(x)$ and

$$\|f_r\|_{\mathcal{H}^{p,q}} \leq C \sup_{t>0} \|\mathrm{St}_r^\alpha F(\cdot,t)\|_{p,q} \leq C\|F\|_{\mathbb{H}^{(p,q,\alpha)}}, \tag{4.2}$$

and $\mathrm{St}_r^\alpha u_j(\cdot,t) = \mathcal{R}_j(\mathrm{St}_r^\alpha u_{d+1}(\cdot,t))$, $t > 0$, $j = 1, \cdots, d$.

We put $f := f^1$. We have $\mathrm{St}_r^\alpha f = f_r$ for all $r > 0$. Taking this in estimate (4.2) yields

$$\|\mathrm{St}_r^\alpha f\|_{\mathcal{H}^{p,q}} \leq C\|F\|_{\mathbb{H}^{(p,q,\alpha)}}$$

wich prove that $f \in \mathcal{H}^{(p,q,\alpha)}(\mathbb{R}^d)$.

The vector

$$G(x,t) = ((\mathcal{R}_1(f) * P_t)(x), \cdots, \mathcal{R}_d(f) * P_t)(x), (f * P_t)(x)), \quad x \in \mathbb{R}^d, t > 0$$

is harmonic, satisfies the generalized Cauchy-Riemann equation, and

$$\sup_{t>0} \||G(\cdot,t)|\|_{p,q,\alpha} \leq C\|F\|_{\mathbb{H}^{(p,q,\alpha)}}.$$

□

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