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On Local and Semi-Local Convergence Analysis of A High-Order Iterative Method for Solving Nonlinear Systems Without High Derivatives

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Abstract. In this paper, we study a general high-order iterative method for solving nonlinear systems in Banach spaces without requiring higher-order derivatives. The proposed method constructs each iteration by combining evaluations of the operator and its derivative, together with an adapted correction scheme. A detailed local convergence analysis under majorant conditions is provided, establishing the convergence to the solution. We also show a semi-local convergence by introducing new majorizing sequences. The theoretical results are illustrated with examples, and confirm the theoretical predictions.

1. Introduction

Numerous problems in applied mathematics, scientific computing, and engineering are modeled by nonlinear systems of the form

$$G : D \subset B_0 \to B,$$

where D is an open and convex subset of the Banach space B_0, and B is another Banach space. The goal is to find $x^* \in D$ satisfying

$$G(x) = 0. \tag{1}$$

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Finding exact analytical solutions to such nonlinear systems is typically difficult or impossible. Consequently, iterative methods are commonly employed. Newton's method, defined by

$$x_{n+1} = x_n - G'(x_n)^{-1}G(x_n),$$

is among the most classical and efficient iterative schemes, offering quadratic convergence under suitable conditions. However, many real-world problems demand faster convergence with lower computational cost, motivating the development of high-order methods.

Following this line, we study a general high-order iterative method, which extends classical schemes by incorporating additional correction steps while using only evaluations of G and G' (without requiring higher derivatives). Inspired by the high-order framework of Behl *et al.* [9], which attains order $3(k-1)$ for systems in $\mathbb{R}^m$ by reusing a frozen inverse Jacobian and relying solely on first-order information, the present study generalizes that scheme to Banach spaces, discards the seventh-derivative assumptions underpinning their local Taylor analysis, and furnishes a unified local and semi-local convergence theory with computable error bounds, larger attraction regions, and sharpened uniqueness criteria—thereby achieving broader applicability.

Let $k \geq 3$ be a natural number and $x_0 \in D$ an initial point. Then, the method is defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n^{(1)} &= x_n - G'(x_n)^{-1}G(x_n), \\ y_n^{(2)} &= x_n - 2T^{-1}G(x_n), \\ y_n^{(3)} &= y_n^{(2)} - MG(y_n^{(2))}, \\ &\dots \\ x_{n+1} = y_n^{(k)} &= y_n^{(k-1)} - MG(y_n^{(k-1)}), \end{aligned} \tag{2}$$

where the operators T, L, and M are given by

$$\begin{aligned} T = T_n &= G'(x_n) + G'(y_n^{(1))}, \\ L = L_n &= 3F'(y_n^{(2))} - G'(x_n), \\ M = M_n &= L^{-1}TG'(x_n)^{-1}. \end{aligned}$$

The method (2) is shown in [9] to possess convergence order $3(j-1)$ for $j = 3, \dots, k$ using Taylor expansions when $B_0 = B = \mathbb{R}^m$ (m natural number), assuming the existence of at least the seventh derivative $G^{(7)}$, although the derivatives G'', $G^{(3)}$, ..., $G^{(7)}$ are not explicitly required in the iteration steps.

Unlike many classical methods that impose strong smoothness assumptions, this method is designed to work under weaker differentiability conditions. It builds upon ideas from previous studies, including Parhi and Gupta [14], Wang et al. [18], and Cordero et al. [10], while aiming for a better balance between convergence speed, computational cost, and robustness.

To illustrate the need for such an approach, consider the scalar function $g : D \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} b_1 t^7 \log t + b_1 t^8 + b_2 t^9, & t \neq 0, \\ 0, & t = 0, \end{cases}$$

where $D = [-2, 2)$, and b_1, b_2 are real constants with $b_1 \neq 0$ and $b_1 + b_2 = 0$. Although g has a zero at $t^* = 1$, the seventh derivative $g^{(7)}(t)$ does not exist at $t = 0$, showing that classical convergence assumptions based on higher derivatives are not satisfied.

This observation motivates the use of generalized majorant conditions rather than strict smoothness hypotheses. Moreover, method not only achieves high-order local convergence but also provides a semi-local convergence analysis by constructing suitable majorizing sequences.

The main contributions of this paper are as follows:

- A local convergence analysis is established that depends only on G, G', and suitably constructed auxiliary operators, thereby eliminating any need for higher-order derivatives.
- Semi-local convergence results are provided through the use of majorizing sequences, guaranteeing convergence even when the initial guess is relatively far from the solution.
- Computable radii of convergence and explicit error bounds are derived, permitting *a priori* estimates of the number of iterations required to achieve a prescribed accuracy.
- Conditions are specified that ensure the uniqueness of the solution within a neighborhood of the limit point.

The remainder of the paper is organized as follows. In Section 2, we introduce the assumptions and establish the local convergence theorems, including uniqueness and error estimates. Section 3 presents the semi-local convergence analysis via majorizing sequences. Section 4 discusses examples and practical aspects. Finally, conclusions are drawn in Section 5.

2. Convergence Analysis

2.1. **Local.** Some real functions which are defined on the interval $A = [0, +\infty)$ play a crucial role in the local convergence analysis of the method (2).

Suppose

(H_1) There exists a nondecreasing and continuous function $\phi_0 : A \to A$ such that the function $1 - \phi_0(t)$ has a smallest positive zero in A, which is denoted by s_0. Define the interval $A_0 = [0, s_0)$.

(H_2) There exists a nondecreasing and continuous function $\phi : A_0 \to A$ such that for $h_1 : A_0 \to A$ defined by

$$h_1(t) = \int_0^1 \phi((t - \eta)t) d\eta \tag{3}$$

the function $1-h_1(t)$ has a smallest positive zero in the interval A_0, which is denoted by r_1.

(H_3) For $p: A_0 \to A$ defined by

$$p(t) = \frac{1}{2}\left(\phi_0(t) + \phi_0(h_1(t)t)\right) \tag{4}$$

the function $1-p(t)$ has a smallest positive zero in the interval A_0, which is denoted by s_1. Define the interval $A_1 = [0, s_1)$.

(H_4) For $\overline{\phi}: A_1 \to A$, $h_2: A_1 \to A$, $p: A_1 \to A$, defined by

$$\overline{\phi}(t) = \begin{cases} \phi((1+h_1(t))t) \\ or \\ \phi_0(t) + \phi_0(h_1(t)t), \end{cases}$$

$$p(t) = \frac{1}{2}\left(\phi_0(t) + \phi_0(h_1(t)t)\right),$$

and

$$h_2(t) = \frac{\int\limits_0^1 \phi_0((1-\eta)t)d\eta}{1-\phi_0(t)} + \frac{\overline{\phi}(t)\left(1+\int\limits_0^1 \phi_0(\eta t)d\eta\right)}{2(1-\phi_0(t))(1-p(t))},$$

the function $1-h_2(t)$ has a smallest positive zero in the interval A_1, which is denoted by r_2.

(H_5) For $q: A_1 \to A$ defined by

$$q(t) = \frac{1}{2}\left(\phi_0(t) + 3\phi_0(h_1(t)t)\right)$$

the function $1-q(t)$ has a smallest positive zero, which is denoted by s_2. Define the interval $A_2 = [0, s_2)$.

(H_6) For $j = 3, \ldots, k$, $A_{j-1} = [0, s_{j-1})$, $h_j : A_{j-1} \to A$

the functions $1-\phi_0(h_{j-1}(t)t)$ and $1-h_j(t)$ have smallest positive zeros in the interval A_{j-1}, which are denoted by s_{j-1} and r_j, respectively, where

$$h_j(t) = \left[\frac{\int\limits_0^1 \phi_0((1-\eta)h_{j-1}(t)t)d\eta}{1-\phi_0(h_{j-1}(t)t)} + \frac{(1+\phi_0(h_1(t)t))\left(1+\int\limits_0^1 \phi_0(\eta h_{j-1}(t)t)d\eta\right)}{2(1-\phi_0(t))(1-q(t))}\right] h_{j-1}(t)$$

Define

$$r^* = \min\{r_j\}, m = 1, 2, \ldots, k \quad \text{and} \quad A^* = [0, r^*) \tag{5}$$

It follows by these definitions and conditions $(H_1)-(H_6)$ that for each $t \in A^*$

$$0 \le \phi_0(t) < 1, \tag{6}$$

$$0 \le p(t) < 1, \tag{7}$$

$$0 \le q(t) < 1, \tag{8}$$

$$0 \le \phi_0(h_{j-1}(t)t) < 1, \tag{9}$$

and (10)

$$0 \le h_j(t) < 1. \tag{11}$$

Notice also that the parameter r is shown to be a radius of convergence for the method (2) (see Theorem 1).

Next, we relate functions ϕ_0 and ϕ to the operators in the method (2).

(H_7) There exists a solution $x^* \in D$ and a linear operator $E \in \mathcal{L}(B_0, B)$ which is invertible such that for each $z \in D$

$$\|E^{-1}(G(z) - E)\| \le \phi_0(\|z - x^*\|).$$

Define the region $D_0 = D \cap U(x^*, s_0)$, where $U(x, \bar{s})$ stands for an open ball in B_0 centered at x and of some radius $s > 0$. The set $U[x, \bar{s}]$ denotes the closure of $U(x, \bar{s})$, which is a closed set.

(H_8) $\|E^{-1}(G'(z_2) - G'(z_1))\| \le \phi(\|z_2 - z_1\|)$ for each $z_1, z_2 \in D_0$.

(H_9) $U[x^*, r^*] \subset D$.

Remark 1. Some possible selections for the linear operator E can be $E = I$, the identity operator, or $E = G'(\bar{z})$ for some $\bar{z} \in D$ with $\bar{z} \ne x^*$ or $E = G'(x^*)$. The last choice of E implies x^* is a simple solution of the equation $G(x) = 0$. It is worth noting, though, that such an assumption is not made or implied by the conditions (H_1)–(H_9).

The local convergence analysis of the method (2) is provided in the next result. Let $U_0 = U(x^*, r^*) - \{x^*\}$.

Theorem 1. *Suppose that the conditions (H_1)–(H_9) hold. Then, the sequence $\{x_n\}$ generated for the starting point $x_0 \in U_0$ is convergent to the solution x^* of the equation $G(x) = 0$.*

Proof. The following assertions shall be established using induction on $n = 0, 1, 2, \ldots$

$$\|y_n^{(1)} - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r^*, \tag{12}$$

$$\|y_n^{(2)} - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{13}$$

$$\|y_n^{(j)} - x^*\| \le g_j(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{14}$$

$$\ldots$$

$$\|x_{n+1}-x^*\| = \|y_n^{(k)}-x^*\| \le g_k(\|x_n-x^*\|)\|x_n-x^*\| \le \|x_n-x^*\|. \tag{15}$$

Pick $u \in U_0$. It follows by conditions (H_1), (H_7), and (5), (6)

$$\|E^{-1}(G'(u)-E)\| \le \phi_0(\|u-x^*\|) \le \phi_0(r^*) < 1. \tag{16}$$

So, the linear operator $G'(u)$ is invertible by the lemma due to Banach [13] and

$$\|G'(u)^{-1}E\| \le \frac{1}{1-\phi_0(\|u-x^*\|)}. \tag{17}$$

In particular, if $u = x_0$, the iterate $y_0^{(1)}$ exists by the first substep of the method (2) for $n = 0$, and we can write

$$\begin{aligned} y_0^{(1)}-x^* &= x_0-x^*-G'(x_0)^{-1}G(x_0) \\ &= \left[G'(x_0)^{-1}E\right]\left(\int_0^1 E^{-1}(G'(x_0+\eta(x^*-x_0))-G'(x_0))d\eta(x_0-x^*)\right) \end{aligned} \tag{18}$$

Using the condition (H_8), (5), (11) (for $i = 1$), (17), and (18), we get from (18)

$$\|y_0^{(1)}-x^*\| \le \frac{\int_0^1 \phi((1-\eta)\|x_0-x^*\|)d\eta\|x_0-x^*\|}{1-\phi_0(\|x_0-x^*\|)} \le q_1(\|x_0-x^*\|)\|x_0-x^*\| \le \|x_0-x^*\| < r^*. \tag{19}$$

Thus, the assertion (12) holds if $n = 0$ and the iterate $y_0^{(1)} \in U_0$. Next, we show T_0 is also invertible. In view of the conditions (H_7), (5), (7), and (19), we can have

$$\begin{aligned} \|(2E)^{-1}(T_0-2E)\| &\le \frac{1}{2}\left(\phi_0(\|x_0-x^*\|)+\phi_0(\|y_0^{(1)}-x^*\|)\right) \\ &\le \frac{1}{2}\left(\phi_0(\|x_0-x^*\|)+\phi_0\left(g_1(\|x_0-x^*\|)\|x_0-x^*\|\right)\right) \\ &= p_0 < 1. \end{aligned}$$

Thus, the linear operator T_0 is invertible and

$$\|T_0^{-1}E\| \le \frac{1}{2(1-p_0)}. \tag{20}$$

Moreover, the iterate $y_0^{(2)}$ is well defined by the second substep of method (2), from which we can also write

$$\begin{aligned} y_0^{(2)}-x^* &= x_0-x^*-G'(x_0)^{-1}G(x_0)+(G'(x_0)^{-1}-2T_0^{-1})G(x_0) \\ &= x_0-x^*-G'(x_0)^{-1}G(x_0)-(2T_0^{-1}-G'(x_0)^{-1})G(x_0) \\ &= x_0-x^*-G'(x_0)^{-1}G(x_0)-T_0^{-1}(G'(x_0)-G'(y_0^{(1)})G'(x_0))^{-1}G(x_0) \\ &= x_0-x^*-G'(x_0)^{-1}G(x_0)+[T_0^{-1}E][E^{-1}(G'(y_0^{(1)})-G'(x_0))]G'(x_0)^{-1}G(x_0) \end{aligned} \tag{21}$$

By the conditions (H_1), (H_7), (H_8), (17) (for $u = x_0$), (19), (11) (for $i = 2$), (20) and (21), we obtain

$$\|y_0^{(2)} - x^*\| \leq \left[\frac{\int_0^1 \phi((1-\eta)\|x_0 - x^*\|)d\eta}{1 - \phi_0(\|x_0 - x^*\|)} + \frac{\overline{\phi}_0(1 + \int_0^1 \phi_0(\eta\|x_0 - x^*\|)d\eta)}{2(1 - \phi_0(\|x_0 - x^*\|))(1 - p_0)} \right] \|x_0 - x^*\| \tag{22}$$

$$\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|.$$

Hence, the assertion (13) holds if $n = 0$ and the iterate $y_0^{(2)} \in U_0$. Similarly, from (8) and (H_7) we can have

$$\|(2E^{-1})(L - 2E)\| = \frac{1}{2}\|E^{-1}\left(3(G'(y_0^{(1))} - E) + (G'(x_0) - E)\right)\|$$
$$\leq \frac{1}{2}\left(3\phi_0(\|y_0^{(1)} - x^*\|) + \phi_0(\|x_0 - x^*\|)\right) \leq q_0 < 1,$$

so

$$\|L^{-1}E\| \leq \frac{1}{2(1 - q_0)}. \tag{23}$$

Thus, the iterates $y_0^{(3)}, \ldots, y_0^{(k)} = x_1$ exist, since L is invertible and we can write for $j = 3, 4, \ldots, k$

$$\begin{aligned} y_0^{(j)} - x^* &= y_0^{(j-1)} - x^* - G'(y_0^{(j-1)})^{-1}G(y_0^{(j-1)}) + (I - M_0)G(y_0^{(j-1)}) \\ &= y_0^{(j-1)} - x^* - G'(y_0^{(j-1)})^{-1}G(y_0^{(j-1)}) - L^{-1}G'(y_0^{(j-1)})G'(x_0)^{-1}G(y_0^{(2)}) \end{aligned} \tag{24}$$

which can be implied by (5), (11), (22), and (23)

$$\|y_0^{(j)} - x^*\| \leq \left[\frac{\int_0^1 \phi_0((1-\eta)\|y_0^{(j-1)} - x^*\|)d\eta}{1 - \phi_0(\|y_0^{(j-1)} - x^*\|)} \right.$$
$$\left. + \frac{(1 + \overline{\phi}_0(\|y_0^{(j-1)} - x^*\|))(1 + \int_0^1 \phi_0(\eta\|y_0^{(j-1)} - x^*\|)d\eta)}{2(1 - \phi_0(\|x_0 - x^*\|))(1 - q_0)} \right] \|y_0^{(j-1)} - x^*\| \tag{25}$$
$$\leq g_j(\|y_0^{(j-1)} - x^*\|)\|y_0^{(j-1)} - x^*\| \leq \|x_0 - x^*\|$$

where we have also used the estimates

$$\|E^{-1}(G'(y_0^{(j)}) - G'(x_0))\| \leq \phi(\|y_0^{(1)} - x_0\|) \leq \phi(\|y_0^{(1)} - x^*\| + \|x_0 - x^*\|) \leq \overline{\phi}_0$$

or

$$\|E^{-1}(G'(y_0^{(1)}) - G'(x_0))\| \leq \|E^{-1}(G'(y_0^{(1)}) - G'(x^*))\| + \|E^{-1}(G'(x_0) - G'(x^*))\|$$
$$\leq \phi_0(\|y_0^{(1)} - x^*\|) + \phi_0(\|x_0 - x^*\|) \leq \overline{\phi}_0,$$

$$\|E^{-1}G(y_0^{(1)})\| = \|E^{-1}(G(y_0^{(1)}) - E + E)\| \le 1 + \|E^{-1}(G'(y_0^{(1)}) - E)\| \le 1 + \phi_0(\|y_0^{(1)} - x^*\|),$$

and

$$\begin{aligned}\|E^{-1}G(y_0^{(1)})\| &= \left\|\int_0^1 E^{-1}(G'(x^* + \eta(y_0^{(1)} - x^*)))d\eta(y_0^{(1)} - x^*)\right\| \\ &= \left\|\int_0^1 E^{-1}(G'(x^* + \eta(y_0^{(1)} - x^*)) - E + E)d\eta(y_0^{(1)} - x^*)\right\| \\ &\le \left(1 + \int_0^1 \phi_0(\eta\|y_0^{(1)} - x^*\|)d\eta\right)\|y_0^{(1)} - x^*\|.\end{aligned}$$

Therefore, the assertions (14) and (15) hold if $n = 0$ and the iterate $y_0^{(j,)}x_1 \in U_0$.

The induction for assertions (12)–(15) is completed if $x_0, y_0^{(1,)}y_0^{(2,)}y_0^{(j)}$ are replaced by $x_j, y_j^{(1,)}y_j^{(2,)}y_j^{(j)}$ respectively.

Furthermore, by estimate (15) and $d_k = g_k(\|x_0 - x^*\|) \in [0, 1)$, we can get

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|y_n^{(k)} - x^*\| \\ &\le d_k\|y_n^{(k-1)} - x^*\| \\ &\le d_k d_{k-1}\|y_n^{(k-2)} - x^*\| \\ &\le \cdots \le d_k d_{k-1} \cdots d_3\|y_n^{(2)} - x^*\| \\ &\le d_k d_{k-1} \cdots d_2\|x_n - x^*\|.\end{aligned} \tag{26}$$

It follows by the definition of d_k that there exists $d \in [0, 1)$ such that

$$d_2, d_3, \dots, d_k \le d \tag{27}$$

So, by (26) and (27), we get

$$\|x_{n+1} - x^*\| \le d^{k-1}\|x_n - x^*\| \le d^{(k-1)(n+1)}\|x_0 - x^*\| < r^*. \tag{28}$$

Finally, if we let $n \to +\infty$ in (28), we conclude that $\lim_{n\to+\infty} x_n = x^*$, and all the iterates $\{x_n\} \subseteq U_0$. □

The uniqueness of the solution x^* is established in a neighborhood of it next.

Proposition 1. *Suppose that the condition* (H_7) *holds in the ball* $U(x^*, R_1)$*, for some* $R_1 > 0$ *and there exists* $R_2 \ge R_1$ *such that*

$$\int_0^1 \phi_0(\eta R_2)d\eta < 1. \tag{29}$$

Define the region $D_1 = D \cap U[x^*, R_2]$.

Then, x^* *is the only solution of the equation* $G(x) = 0$ *in the region* D_1.

Proof. Suppose that there exists a solution $z^* \in D_1$ of the equation $G(x) = 0$ such that $z^* \neq x^*$. Then, define the linear operator

$$E_1 = \int_0^1 G'(x^* + \eta(z^* - x^*))d\eta.$$

By this definition, the condition (H_7) and (29) we get in turn

$$\|E_1^{-1}(E_1 - E)\| \leq \int_0^1 \phi_0(\eta\|z^* - x^*\|)d\eta \leq \int_0^1 \phi_0(\eta R_2)d\eta < 1.$$

Thus, the linear operator E_1 is invertible. It follows by the identity

$$z^* - x^* = E_1^{-1}(G(z^*) - G(x^*)) = E_1^{-1}(0) = 0,$$

and we conclude $z^* = x^*$. □

Remark 2. Under all the conditions (H_4)–(H_9), one can set $R_1 = r^*$ in Proposition 1.

2.2. **Semi-local.** The calculations and formulae are as in Section 2.1, but x^*, ϕ_0, ϕ are exchanged by x_0, ψ_0, and ψ, respectively.

Suppose

(C_1) There exists a nondecreasing and continuous function $\psi_0 : A \to A$ such that the function $1 - \psi_0(t)$ has a smallest positive solution in the interval A, which is denoted by t_0. Define the interval $S = [0, t_0)$.

(C_2) There exists a nondecreasing and continuous function $\psi : S \to A$.

Define the sequences $\{\alpha_n^i\}$ for $\alpha_0^0 = 0$, some $\alpha_0^1 \geq 0$, $i = 0, \ldots, k$, and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} \overline{\psi}_n &= \begin{cases} \psi(\alpha_n^1 - \alpha_n^0), \\ \text{or} \\ \psi_0(\alpha_n^0) + \psi_0(\alpha_n^1), \end{cases} \\ \overline{p}_n &= \frac{1}{2}\left(\psi_0(\alpha_n^0) + \psi_0(\alpha_n^1)\right), \\ \alpha_n^2 &= \alpha_n^1 + \frac{\overline{\psi}_n(\alpha_n^1 - \alpha_n^0)}{2(1 - \overline{p}_n)}, \end{aligned} \tag{30}$$

$$\lambda_n^{j-1} = \int_0^1 \psi((1-\eta)(\alpha_n^{j-1} - \alpha_n^0))d\eta \cdot (\alpha_n^{j-1} - \alpha_n^0) + (1 + \psi_0(\alpha_n^0))(\alpha_n^{j-1} - \alpha_n^1),$$

$$\overline{q}_n = \frac{1}{2}\left(3\psi_0(\alpha_n^1) + \psi_0(\alpha_n^0)\right),$$

$$\alpha_n^j = \alpha_n^{j-1} + \frac{(\psi_0(\alpha_n^0) + \psi_0(\alpha_n^1) + 2)\lambda_n^{j-1}}{2(1 - \psi_0(\alpha_n^0))(1 - \bar{q}_n)},$$

$$\mu_{n+1} = \int_0^1 \psi((1-\eta)(\alpha_{n+1}^0 - \alpha_n^0))d\eta(\alpha_{n+1}^0 - \alpha_n^0) + (1 + \psi_0(\alpha_n^0))(\alpha_n^0 - \alpha_n^0),$$

and

$$\alpha_{n+1}^1 = \alpha_{n+1}^0 + \frac{\mu_{n+1}}{1 - \psi_0(\alpha_{n+1}^0)},$$

where again $\alpha_{n+1}^0 = \alpha_n^k$.

The sequence $\{\alpha_n^i\}_n$ is shown to be majorizing for $\{y_n^{(i)}\}_n$ in Theorem 2.

But let us first provide a convergence condition for it.

(C_3) There exists $\bar{t} \in [0, t_0)$ such that for each $i = 0, 1, 2, \ldots, k$ and each $n = 0, 1, 2, \ldots$

$$\psi_0(\alpha_n^0) < 1, \quad \bar{p}_n < 1, \quad \bar{q}_n < 1, \quad \text{and} \quad \alpha_n^i \leq \bar{t}.$$

It follows by this condition and (30) that the sequence $\{\alpha_n^i\}$ is nondecreasing and bounded from above by $\bar{t}$ and as such it converges to some $\alpha^* \in [0, \bar{t}]$. The limit point α^* is the unique least upper bound of the sequence $\{\alpha_n^i\}$.

As in the local analysis, the operators on the method (2) connect to the functions ψ_0 and ψ.

(C_4) There exist $x_0 \in D$ and a linear operator E such that for each $u \in D$

$$\|E^{-1}(G'(u) - E)\| \leq \psi_0(\|u - x_0\|).$$

It follows by the conditions (C_1), (C_4), and (30) that if $u = x_0$, we get

$$\|E^{-1}(G'(x_0) - E)\| \leq \psi_0(0) < 1.$$

So, the linear operator $G'(x_0)$ is invertible, in which case we can take

$$\alpha_0^1 \geq \|G'(x_0)^{-1}G(x_0)\|.$$

Define the region

$$D_2 = U[x_0, \alpha^*] \cap D.$$

(C_5)

$$\|E^{-1}(G'(u_2) - G'(u_1))\| \leq \psi(\|u_2 - u_1\|), \quad \text{for each} \quad u_2, u_1 \in D_2.$$

(C_6)

$$U[x_0, \alpha^*] \subset D.$$

Remark 3. As in the local analysis, possible selections for E can be $E = I$ or $E = G'(\bar{z})$ for some auxiliary point $\bar{x} \in D$ such that $\bar{x} \neq x_0$, or $E = G'(x_0)$, or some other selection.

The semi-local analysis of the method (2) follows in the next result.

Theorem 2. *Suppose the conditions $(C_1)-(C_6)$ hold. Then, the sequence $\{x_n\}$ generated by the method (2) is well-defined in $U(x_0,\alpha^*)$, remains in $U(x_0,\alpha^*)$, and is convergent to a solution $x^*\in U[x_0,\alpha^*]$ of the equation $G(x)=0$ such that for each $n=0,1,2,\ldots$*

$$\|x^*-x_n\|\le \alpha^*\alpha_n.$$

Proof. As in the local analysis, induction is used to first establish the assertions

$$\|y_n^{(1)}-x_n\|\le \alpha_n^1-\alpha_n^0, \tag{31}$$

$$\|y_n^{(2)}-y_n^{(1)}\|\le \alpha_n^2-\alpha_n^1, \tag{32}$$

$$\|y_n^{(j)}-y_n^{(j-1)}\|\le \alpha_n^j-\alpha_n^{j-1}. \tag{33}$$

By switching the conditions $(H_1)-(H_9)$ by $(C_1)-(C_5)$ but using the same formulas, we get in turn

$$\begin{aligned} y_n^{(2)}-y_n^{(1)} &= T\left(G'(y_n^{(1)})-G'(x_n)\right)G'(x_n)^{-1}G(x_n)\\ &= -\left[TE^{-1}\right]\left[E^{-1}(G'(y_n^{(1)})-G'(x_n))\right](y_n^{(1)}-x_n),\\ \|y_n^{(2)}-y_n^{(1)}\| &\le \frac{\overline{\psi}_n(\alpha_n^1-\alpha_n^0)}{2(1-\overline{p}_n)}\le \alpha_n^2-\alpha_n^1,\\ \|y_n^{(2)}-x_0\| &\le \|y_n^{(2)}-y_n^{(1)}\|+\|y_n^{(1)}-x_0\|\\ &\le \alpha_n^2-\alpha_n^1+\alpha_n^1-\alpha_n^0=\alpha_n^2<\alpha^*. \end{aligned}$$

So, the estimate (32) holds and the iterate $y_n^{(2)}\in U[x_0,\alpha^*]$.

Then, by the identity

$$\begin{aligned} G(y_n^{(j-1)}) &= G(y_n^{(j-1)})-G(x_n)-G'(x_n)(y_n^{(1)}-x_n),\\ &= G(y_n^{(j-1)})-G(x_n)-G'(x_n)(y_n^{(j-1)}-x_n)+G'(x_n)(y_n^{(j-1)}-y_n^{(1)}), \end{aligned}$$

which can imply

$$\|E^{-1}G(y_n^{(j-1)})\|\le \int_0^1 \psi((1-\eta)(\alpha_n^{j-1}-\alpha_n^0))d\eta(\alpha_n^{j-1}-\alpha_n^0)+(1+\psi_0(\alpha_n^0))(\alpha_n^{j-1}-\alpha_n^1)=\lambda_n^{j-1} \tag{34}$$

$$\|y_n^{(j)}-y_n^{(j-1)}\|\le \frac{(\psi_0(\alpha_n^0)+\psi_0(\alpha_n^1)+2)\lambda_n^{j-1}}{2(1-\psi_0(\alpha_n^0))(1-\overline{q}_n)}=\alpha_n^j-\alpha_n^{j-1}.$$

Thus,

$$\|y_n^{(j)}-x_0\|\le \|y_n^{(j)}-y_n^{(j-1)}\|+\|y_n^{(j-1)}-x_0\|\le \alpha_n^j-\alpha_n^{j-1}+\alpha_n^{j-1}-\alpha_0^0=\alpha_n^j<\alpha^*.$$

Thus, the assertions (33) hold and all the iterates $\{y_n^{(j)}\}\subset U(x_0,\alpha^*)$.

It is left to show that assertion (31) holds if $n+1$ replaces n.

But we can write in turn

$$G(x_{n+1}) = G(x_{n+1}) - G(x_n) - G'(x_n)(y_1 - x_n)$$
$$= G(x_{n+1}) - G(x_n) - G'(x_n)(x_{n+1} - x_n) + G'(x_n)(x_{n+1} - y_n^{(1)}),$$

which can give, as in (34),

$$\|E^{-1}G(x_{n+1})\| \leq \int_0^1 \psi((1-\eta)(\alpha_{n+1}^0-\alpha_n^0))d\eta(\alpha_{n+1}^0-\alpha_n^0)+(1+\psi_0(\alpha_n^0))(\alpha_{n+1}^0-\alpha_n^1) = \mu_{n+1}. \quad (35)$$

Consequently, we obtain

$$\|y_{n+1}^{(1)} - x_{n+1}\| \leq \|G'(x_{n+1})^{-1}E\|\|E^{-1}G(x_{n+1})\|,$$
$$\leq \frac{\mu_{n+1}}{1-\psi_0(\alpha_{n+1}^0)} = \alpha_{n+1}^1 - \alpha_{n+1}^0$$

and

$$\|y_{n+1}^{(1)} - x_0\| \leq \|y_{n+1}^{(1)} - x_{n+1}\| + \|x_{n+1} - x_0\|$$
$$\leq (\alpha_{n+1}^1 - \alpha_{n+1}^0) + (\alpha_{n+1}^0 - \alpha_0^0) = \alpha_{n+1}^1 < \alpha^*.$$

Thus, the induction for assertions (31)–(33) is completed, and all the iterates $\{y_n^{(i)}\} \in U(x_0, \alpha^*)$.

It also follows that the sequence $\{x_n^j\}$ is complete in Banach space B_0, since $\{\alpha_n^i\}$ is also complete as convergent by the condition (C_4). Therefore, there exists $x^* \in U[x_0, \alpha^*]$ such that

$$\lim_{n\to+\infty} y_n^{(k)} = x^* \quad \text{or} \quad \lim_{n\to+\infty} x_n = x^*.$$

Moreover, by letting $n \to +\infty$ in (35), we obtain $G(x^*) = 0$, where the continuity of the operator G has also been used. Furthermore, by noticing that $\alpha_n^k = \alpha_{n+1}$ and $\alpha_n^k = \alpha_{n+1}^0$, estimate (33) can be rewritten for $j = k$ as

$$\|x_{n+1} - x_n\| \leq \alpha_{n+1} - \alpha_n,$$

so

$$\|x_{n+h} - x_n\| \leq \alpha_{n+h} - \alpha_n, \quad h = 0, 1, 2, \ldots \quad (36)$$

Finally, by letting $h \to +\infty$ in (36), we show the assertion (2).

□

Next, we study the uniqueness of a solution in a certain region.

Proposition 2. *Suppose there exists a solution $y^* \in U(x_0, R_3)$ of the equation $G(x) = 0$ for some $R_3 > 0$; the condition (C_4) holds in the ball $U(x_0, R_3)$, and there exists $R_4 \geq R_3$ such that*

$$\int_0^1 \psi_0((1-\eta)R_3 + \eta R_4)d\eta < 1. \quad (37)$$

Define the region $D_3 = D \cap U[x_0, R_4]$.

Then, y^* is the only solution of the equation $G(x) = 0$ in the region D_3.

Proof. Suppose there exists $y^{**} \in D_3$ solving the equation $G(x) = 0$ such that $y^{**} \neq y^*$.

Then define the linear operator

$$E_2 = \int_0^1 G'(y^* + \eta(y^{**} - y^*))d\eta.$$

By applying the condition (C_4) and (37), we obtain in turn

$$\|E^{-1}(E_2 - E)\| \leq \int_0^1 \psi_0((1-\eta)\|y^{**} - x_0\| + \eta\|y^* - x_0\|)d\eta \leq \int_0^1 \psi_0((1-\eta)R_3 + \eta R_4)d\eta < 1.$$

It follows that the linear operator E_2 is invertible. Hence, again we conclude $y^{**} = y^*$. □

Remark 4.

- The limit point a^* given in the condition (C_1) can be replaced by t_0 in (C_6).
- If all conditions $(C_1) - (C_6)$ hold, then we can set $R_3 = \alpha^*$ and $y^* = x^*$ in Proposition 2.

3. Numerical Results

To comprehensively evaluate the performance and robustness of the proposed high-order iterative methods, we present five numerical examples of different complexity and dimensionality. These test problems have been selected from the literature and include systems with diverse nonlinear characteristics, such as trigonometric, exponential, and polynomial structures. The examples are designed to assess the methods' accuracy, convergence speed, and stability.

In all numerical experiments, the stopping criterion was based on achieving a residual norm below certain ϵ, ensuring a high level of numerical precision. A maximum of 50 iterations was imposed to prevent excessive computational effort. This limit is justified by empirical evidence indicating that well-designed, high-order methods typically achieve convergence within this range. To ensure reliable performance metrics, CPU execution times were averaged over 50 independent runs, thereby mitigating the influence of background system noise and transient operational conditions

All simulations were conducted within a Google Colaboratory runtime environment. This environment was configured with an Intel Xeon CPU operating at 2.20 GHz, 13 GB of system RAM, and an NVIDIA Tesla K80 GPU equipped with 12 GB of VRAM. Numerical computations were performed using the Python library mpmath, with the arithmetic precision set to 100 decimal digits. This standardized setup was maintained across all test cases to ensure fair and reproducible comparisons.

We compare method (2) with several established iterative methods, specifically the sixth-order method (29) of Wang et al. [18], the method (14) by Hueso et al. [12], the scheme (6) of Cordero et al. [10] and the method (14) proposed by Abbasbandy et al. [1]. These benchmark techniques are well known in the literature for their high-order convergence properties and are frequently used for testing nonlinear solvers. Our method is evaluated against these in terms of number of iterations, residual norm $\|G(x)\|$, step difference $\|x_{n+1} - x_n\|$, and CPU time. The results are demonstrated in Tables 1–5.

Example 1. Consider the nonlinear system defined as:

$$G_i(x) = \arctan(x_i) + 1 - 2\sum_{\substack{j=1 \\ j\neq i}}^{20} x_j^2 = 0, \quad i = 1, 2, \dots, 20.$$

The methods converge to the zero $x^* = (0.1757683,\ 0.1757683,\ \dots,\ 0.1757683)^T$, starting from the initial approximation $x_0 = (0.15,\ 0.15,\ \dots,\ 0.15)^T$.

Table 1. Results for Example 1

Method	**Iterations**	$\|G(x)\|$	$\|x_{n+1} - x_n\|$	**CPU Time (s)**
Method (2)	2	6.7121×10^{-31}	8.1782×10^{-30}	0.342917
Wang et al.	3	1.8677×10^{-52}	1.4708×10^{-26}	0.656792
Hueso et al.	4	6.4643×10^{-44}	8.6687×10^{-46}	1.112088
Cordero et al.	2	1.8677×10^{-52}	1.0138×10^{-10}	0.333056
Abbasbandy et al.	3	1.2046×10^{-23}	2.0006×10^{-23}	0.307392

Example 2. Consider the nonlinear system:

$$G_i(x) = x_i - \cos\left(2\pi x_i - \sum_{j=1}^{50} x_j\right) = 0, \quad i = 1, 2, \dots, 50.$$

The solution is $x^* = (0.5018261,\ 0.5018261,\ \dots,\ 0.5018261)^T$, with the initial guess $x_0 = (0.51,\ 0.51,\ \dots,\ 0.51)^T$.

Table 2. Results for Example 2

Method	**Iterations**	$\|G(x)\|$	$\|x_{n+1} - x_n\|$	**CPU Time (s)**
Method (2)	2	2.221×10^{-25}	4.9027×10^{-12}	4.127071
Wang et al.	7	6.9532×10^{-22}	3.6049×10^{-20}	17.881168
Hueso et al.	8	5.4021×10^{-24}	2.8007×10^{-22}	29.956985
Cordero et al.	7	5.5536×10^{-23}	2.8792×10^{-21}	15.002068
Abbasbandy et al.	9	6.4707×10^{-16}	7.4808×10^{-16}	12.348433

Example 3. Let us consider the following system of 99 nonlinear equations:

$$G_i(x) = \begin{cases} x_i x_{i+1} - 1 = 0, & 1 \leq i \leq 98, \\ x_{99} x_1 - 1 = 0, & i = 99. \end{cases}$$

The exact solution is $x^* = (1, 1, \ldots, 1)^T$, with the starting vector $x_0 = (2, 2, \ldots, 2)^T$.

Table 3. Results for Example 3

Method	**Iterations**	$\|G(x)\|$	$\|x_{n+1} - x_n\|$	**CPU Time (s)**
Method (2)	2	0.0	9.3704×10^{-5}	21.838247
Wang et al.	4	0.0	4.0358×10^{-25}	56.250490
Hueso et al.	5	0.0	8.9985×10^{-39}	99.436670
Cordero et al.	3	0.0	7.0416×10^{-15}	33.705407
Abbasbandy et al.	4	0.0	1.3554×10^{-17}	24.382990

Example 4. Consider the following system of nonlinear equations

$$G(x) = \begin{cases} x_1 + x_2 - 1 & = 0, \\ 2x_1 + x_2 + 2x_3 - 2 & = 0, \\ x_1 + x_2 + x_3 - x_4 & = 0, \\ \frac{x_2^2 x_3}{x_1^2 x_4} - (0.647)^2 & = 0. \end{cases}$$

The solution vector is $x^* = (0.422499, 0.577501, 0.288751, 1.288751)^T$, obtained from the initial estimate $x_0 = (0.8, 0.2, 0.9, 1.8)^T$.

Table 4. Results for Example 4

Method	**Iterations**	$\|G(x)\|$	$\|x_{n+1} - x_n\|$	**CPU Time (s)**
Method (2)	4	1.4315×10^{-51}	4.334×10^{-16}	0.04373
Wang et al.	*N/A*	*N/A*	*N/A*	*N/A*
Hueso et al.	*N/A*	*N/A*	*N/A*	*N/A*
Cordero et al.	4	2.3346×10^{-53}	1.243×10^{-21}	0.01554
Abbasbandy et al.	*N/A*	*N/A*	*N/A*	*N/A*

N/A indicates that the method did not converge to the required solution within the prescribed iteration or tolerance limits.

Example 5. Consider the nonlinear system of equations of size 200

$$G_i(x) = e^{-x_i} - \sum_{\substack{j=1 \\ j\neq i}}^{200} x_j = 0, \quad i = 1, 2, \dots, 200.$$

The initial approximation is set as $x_0 = \left(\frac{3}{2}, \frac{3}{2}, \dots, \frac{3}{2}\right)^T$, with parameters $a = -2.0$ and $b = 2.0$, leading to the solution: $x^* = (0.0050, 0.0050, \dots, 0.0050)^T$.

Table 5. Results for Example 5

Method	Iterations	$\|G(x)\|$	$\|x_{n+1} - x_n\|$	CPU Time (s)
Method (2)	2	3.4388×10^{-51}	4.7064×10^{-37}	50.414524
Wang et al.	3	6.8725×10^{-52}	7.9328×10^{-26}	86.904696
Hueso et al.	3	1.1438×10^{-51}	1.4069×10^{-18}	129.807516
Cordero et al.	3	2.5137×10^{-51}	3.0846×10^{-48}	71.584528
Abbasbandy et al.	3	6.5891×10^{-52}	1.7579×10^{-20}	98.961347

4. Conclusions

This paper presented a general high-order iterative method for solving nonlinear systems without requiring higher-order derivatives. We established both local and semi-local convergence results using majorant conditions and majorizing sequences, providing rigorous guarantees even from distant initial guesses. Numerical experiments on benchmark problems confirmed the method's accuracy, fast convergence, and low residual errors compared to existing high-order methods. The results validate the theoretical findings and highlight the method's applicability to a wide range of nonlinear problems, suggesting potential for further extensions and refinements.

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