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Estimates of Variable Kernel Parameterized Littlewood–Paley Operators on Variable Herz Spaces

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Abstract. In this article, we prove some boundedness results for variable kernel parameterized Littlewood−Paley operators on the homogeneous Herz spaces $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Several known results are extended.

1. Introduction

Suppose that $\Psi(x_1, z_1) \in L^\infty(\mathbb{R}^n) \times L^b(S^{n-1})$ (where $b \geq 1$) satisfies:

(1) $\Psi(x_1, \alpha z_1) = \Psi(x_1, z_1)$ and $\int\limits_{S^{n-1}} \Psi(x_1, z_1')d\sigma(z_1') = 0$, for all $z_1, x_1 \in \mathbb{R}^n, \alpha > 0$;

(2) $\|\Psi\|_{L^\infty(\mathbb{R}^n)\times L^b(S^{n-1})} := \sup\limits_{x_1\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Psi(x_1, z_1')|^b dz_1'\right)^{\frac{1}{b}} < \infty$,

where S^{n-1} (for $n \geq 2$) is the unit sphere in $\mathbb{R}^n$ equipped with Lebesgue measure dz_1'. The parameterized Littlewood−Paley operators, denoted by $\mu^\sigma_{\Psi,s}$ and $\mu^{*,\sigma}_{\Psi,\lambda}$, are closely associated with the Lusin area integral and Littlewood−Paley g^*_λ function. These operators are defined as follows

$$\mu^\sigma_{\Psi,s}(f)(x) = \left(\int\int_{\Sigma(x)} \left|\frac{1}{t^\sigma}\int_{|y_1-z_1|\leq t} \frac{\Psi(y_1, y_1 - z_1)}{|y_1 - z_1|^{n-\sigma}} f(z_1)dz_1\right|^2 \frac{dy_1dt}{t^{n+1}}\right)^{\frac{1}{2}}$$

and

$$\mu^{*,\sigma}_{\Psi,\lambda}(f)(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x_1 - y_1|}\right)^{\lambda n} \left|\frac{1}{t^\sigma}\int_{|y_1-z_1|\leq t} \frac{\Psi(y_1, y_1 - z_1)}{|y_1 - z_1|^{n-\sigma}} f(z_1)dz_1\right|^2 \frac{dy_1dt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Sigma(x) = \{(y_1, t) \in \mathbb{R}^{n+1}_+ : |x_1 - y_1| < t \text{ and } \lambda > 1\}$.

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The parameterized Littlewood−Paley operators $\mu^{\sigma}_{\Psi,s}$ and $\mu^{*,\sigma}_{\Psi,\lambda}$ were initially investigated by Sakamoto and Yabuta in [1]. They proved that if $\Psi \in Lib_{\beta}(S^{n-1})$ and $1 < p < \infty$, then $\mu^{*,\sigma}_{\Psi,\lambda}$ and $\mu^{\sigma}_{\Psi,s}$ operators are bounded on $L^p(\mathbb{R}^n)$ space. Xue and Ding [2] established sharp $L^p(w)$ bounded for these operators ($\mu^{*,\sigma}_{\Psi,\lambda}$, $\mu^{\sigma}_{\Psi,s}$) in terms of the A_q characteristic of w, under the condition $\Psi \in L^b(S^{n-1})$. Deringoz, Guliyev and Ragusa [3] obtained the boundedness of intrinsic square functions and their commutators in the framework of Morrey-Orlicz spaces. The boundedness of parametric Littlewood−Paley operators on Musielak−Orlicz Hardy spaces was further studied in [4].

As is well known, over the past thirty years, variable kernel integral operators have become an increasingly active area of research. For example, Tao et al. [5] obtained the $L^p(\mathbb{R}^n)$ boundedness of variable kernel fractional integral operators $T_{\Psi,\alpha}$, Chen and Ding [6] proved the $L^p(\mathbb{R}^n)$ boundedness of variable kernel Littlewood−Paley operators, Shao [7] investigated the weighted estimates for variable kernel fractional integrals and their commutators on generalized Morrey spaces, in [8] the author proved the boundedness properties of Marcinkiewicz integral operator μ_{Ψ} with variable kernel on the Hardy space $H^p(\mathbb{R}^n)$. Recently, Abdalmonem et. al. [9] obtained the boundedness of littlewood−paley operators with variable kernel on the weighted variable herz-morrey spaces.

Moreover, variable exponents Herz spaces have been extensively studied by many authors using different methods([14–20, 24, 25]). Izuki [23] defined the variable exponent homogeneous Herz space $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and investigated the boundedness of some integral operators on these spaces. Wang [20] considered the boundedness results for certain rough kernel littlewood−paley operators in homogeneous and homogeneous Herz spaces $\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. In [21] the authors studied the boundedness of the vector-valued inequality for the intrinsic square function in variable exponents homogeneous Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$. Izuki and Noi [12] considered the generalized Herz spaces $\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ and obtained some boundedness results for integral operators and their commutators on those spaces. In [13] the author established the boundedness properties of the rough kernel fractional integral operators in $\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ spaces.

Motivated by the work of [9, 13, 19],this paper discusses the boundedness of variable kernel parameterized Littlewood-Paley operators on homogeneous Herz spaces $\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ with three variable exponents. The results are also new for the case when $\alpha(\cdot)$ is constant.

2. Mathematical background

Consider a Lebesgue measurable set $E \subset \mathbb{R}^n$ with positive measure $|E| > 0$. Denote by χ_E the characteristic function of E. In this paper, C represents a positive constant that may vary between occurrences. We write $g \lesssim f$ means $g \leq Cf$, for some constant $C > 0$.

Definition 2.1 ([22]). (variable Lebesgue space) Suppose that $p(\cdot) : \Gamma \to [1, \infty)$ is a measurable function. The $L^{p(\cdot)}(\Gamma)$ space is defined by

$$L^{p(\cdot)}(\Gamma) = \left\{ g \text{ is measurable} : \int_{\Gamma} \left(\frac{|g(x)|}{\beta} \right)^{p(x)} dx < \infty \text{ for some constant } \beta > 0 \right\}.$$

The local $L^{p(\cdot)}_{\text{loc}}(\Gamma)$ space is defined as

$$L^{p(\cdot)}_{\text{loc}}(\Gamma) = \{ g \text{ is measurable} : g \in L^{p(\cdot)}(K) \text{ for any compact set } K \subset \Gamma \}.$$

With the given norm, the Lebesgue space $L^{p(\cdot)}_{\text{loc}}(\Gamma)$ is a Banach space

$$\|f\|_{L^{p(\cdot)}(\Gamma)} = \inf \left\{ \eta > 0 : \int_{E} \left(\frac{|g(x)|}{\beta} \right)^{p(x)} dx \leq 1 \right\}.$$

Let $p_- = \text{ess inf}\{p(x) : x \in \Gamma\}$, $p_+ = \text{ ess sup}\{p(x) : x \in \Gamma\}$ denote the essential infimum and supremum of $p(\Gamma)$, respectively. $\mathcal{P}(\Gamma)$ represents the collection of all measurable functions $p(\cdot)$ with $p_- > 1$. $p_+ < +\infty$. $\mathcal{P}^0(\Gamma)$ consists of all measurable functions $p(\cdot)$ such that $p_- > 0$ and $p_+ < +\infty$. Furthermore, $\mathcal{B}(\mathbb{R}^n)$ is defined as the subset of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for which the Hardy−Littlewood maximal operator M_* is bounded in variable $L^{p(\cdot)}$ space.

We know that, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then the operator M_*,

$$M_* g(x) = \sup_{B \subseteq \mathbb{R}^n, B \ni x} \frac{1}{|B|} \int_B |g(y)| dy,$$

is bounded in variable $L^{p(\cdot)}$ space [24], where M_* denotes the Hardy−Littlewood maximal operator.

Let us now recall the definition of Herz space $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$. Let $B_k = \{y \in \mathbb{R}^n : |y| \leq 2^k\}$, $k \in \mathbb{Z}$, $C_k = B_k \backslash B_{k-1}$, $\chi_{C_k} = \chi_k$.

Definition 2.2 ([12]). Let $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$, $-\infty < \alpha_- \leq \alpha_+ < \infty$ and $q(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous variable exponents Herz $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ space is defined by

$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty \},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} := \left\| \{2^{k\alpha(\cdot)} |f\chi_k|\}_{k=-\infty}^{\infty} \right\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \beta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\beta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

The nonhomogeneous variable exponents Herz $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ space is defined by

$$K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty \},$$

where

$$\|f\|_{K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} := \left\| \{2^{k\alpha(\cdot)} |f\chi_k|\}_{k=0}^{\infty} \right\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \beta > 0 : \sum_{k=0}^{\infty} \left\| \left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\beta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

Remark. (1) If
$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n),$$
then, $q(\cdot)$ is a constant.

(2) If
$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n),$$
then, both $\alpha(\cdot)$, $q(\cdot)$ are constants.

(3) If
$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha,q}_{p}(\mathbb{R}^n),$$
then, $\alpha(\cdot)$, $p(\cdot)$, $q(\cdot)$ are all constants.

(4) Moreover, if $p(\cdot) = q(\cdot)$ and $\alpha(\cdot) = 0$, then
$$\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n).$$

Next, we present some key lemmas needed to prove our main theorems.

Lemma 2.3 ([22]). *(Generalized Hölder's inequality) Let $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_1'(\cdot)}(\mathbb{R}^n)$, and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, the following inequality is satisfied:*
$$\int_{\mathbb{R}^n} |g(x)f(x)|dx \leq C\|g\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},$$
here $C = 1 - \frac{1}{p_+} + \frac{1}{p_-}$.

Lemma 2.4 ([23]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For a given $C > 0$, the following inequality is satisfied:*
$$C \geq \frac{1}{|B|}\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)},$$
here $B \subset \mathbb{R}^n$.

Lemma 2.5 ([23]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For $n = 1, 2$, there are constants $\delta_{n1}, \delta_{n2} > 0$ for which the following inequalities hold:*
$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \frac{|B|}{|S|}, \ \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_{n1}}, \ \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_{n2}},$$
here $B \subset \mathbb{R}^n$, $S \subset B$.

Lemma 2.6 ([20]). *Let $p_1(\cdot), q_1(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, $g \in L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)$, and $0 < q_- \leq p_1(\cdot) \leq q_+$. Then, we have*
$$\min(\|g\|^{q_+}_{L^{p_1(\cdot)q_1(\cdot)}}, \ \|g\|^{q_-}_{L^{p_1(\cdot)q_1(\cdot)}}) \leq \||g|^{q_1(\cdot)}\|_{L^{p_1(\cdot)}} \leq \max(\|g\|^{q_+}_{L^{p_1(\cdot)q_1(\cdot)}}, \ \|g\|^{q_-}_{L^{p_1(\cdot)q_1(\cdot)}}).$$

Lemma 2.7 ([10]). *Suppose that $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $r_0 > 0$. If $\alpha(\cdot)$ be a function that is log-Hölder continuous both both at the origin and at infinity, then for any $x \in B(0, r_0) \setminus B(0, r_0/2)$, $x' \in B(0, r_1) \setminus B(0, r_1/2)$, we have*

$$r_0^{\alpha(x)} \lesssim r_1^{\alpha(x')} \times \begin{cases} [\frac{r_0}{r_1}]^{\alpha_+}, & 0 < r_1 \leq r_0/2, \\ 1, & r_0/2 < r_1 \leq 2r_0, \\ [\frac{r_0}{r_1}]^{\alpha_-}, & r_1 > 2r_0. \end{cases}$$

3. Boundedness of the parameterized Littlewood-Paley operators

In this section, we discuss the boundedness of variable kernel parameterized Littlewood-Paley operators on homogeneous Herz spaces $\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$. The results are also new for the case when $\alpha(\cdot)$ is constant.

Let $1 < q < \infty$, $q' = \frac{q}{q-1}$ and w be a weight. For every cube $Q \subseteq \mathbb{R}^n$, we say $w \in A_q$ if there exists $C > 0$, the following inequality is satisfied:

$$\left(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-q'}\mathrm{d}x\right)^{q-1} \leq C < \infty.$$

Xue et al. [2] proved the following L^p-boundedness of $\mu^{\sigma}_{\Psi,s}$ and $\mu^{*,\sigma}_{\Psi,\lambda}$.

Lemma 3.1 ([2]). *Let $1 < p < \infty$ and $\Psi \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies (1) and (2). Then, we have*

$$\|\mu^{\sigma}_{\Psi,s} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$$

and

$$\|\mu^{*,\sigma}_{\Psi,\lambda} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Lemma 3.2 ([21]). *Given a family of functions $\mathcal{F}$, if for some p_1, $1 < p_1 < \infty$, $p_1 \leq p_-$ and $(\frac{p(\cdot)}{p_1})' \in \mathcal{B}(E)$ and every $w_1 \in A_{p_1}$,*

$$\int\limits_{\mathbb{R}^n} f_1(x)^{p_1} w_1(x)\mathrm{d}x \lesssim \int\limits_{\mathbb{R}^n} g_1(x)^{p_1} w_1(x)\mathrm{d}x, \qquad (f,g) \in \mathcal{F}.$$

If $p(\cdot) \in \mathcal{P}(E)$ and $f_1 \in L^{p(\cdot)}(E)$, then for all $(f_1, g_1) \in \mathcal{F}$,

$$\|f_1\|_{L^{p(\cdot)}(E)} \lesssim \|g_1\|_{L^{p(\cdot)}(E)}.$$

Since $A_{q/s'} \subset A_\infty$, using Lemma 3.1 and Lemma 3.2, its simple to obtain the $L^{p(\cdot)}$-boundedness of $\mu^{\sigma}_{\Psi,s}$ and $\mu^{*,\sigma}_{\Psi,\lambda}$.

Theorem 3.3. *Assume that $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\lambda > 2$, $2\sigma - n > 0$, and $\Psi \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies (1) and (2). Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be a function that is log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_{11} < \alpha_- \leq \alpha_+ < n\delta_{12},$$

where δ_{n1}, δ_{n2} $(n = 1, 2)$ are the same as in Lemma 2.4. Then, $\mu^{\sigma}_{\Psi,s}$ operator is bounded from $\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ for all $f \in \dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$.

Theorem 3.4. *Assume that $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\lambda > 2$, $2\sigma - n > 0$, and $\Psi \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies (1) and (2). Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be a function that is log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_{11} < \alpha_- \leq \alpha_+ < n\delta_{12},$$

where δ_{n1}, δ_{n2} $(n = 1, 2)$ are the same as in Lemma 2.4. Then, $\mu^{*,\sigma}_{\Psi,\lambda}$ operator is bounded from $\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q_2(\cdot)}(\mathbb{R}^n)$ for all $f \in \dot{K}^{\alpha(\cdot)}_{p_1(\cdot),q_1(\cdot)}(\mathbb{R}^n)$.

Before proving Theorems, we first establish a necessary inequality.

Remark. Let $1 \leq p_m < \infty$, $a_m \geq 0$, $m \in \mathbb{N}$. We have

$$\sum_{m=0}^{\infty} a_m^{p_m} \leq \left(\sum_{m=0}^{\infty} a_m \right)^{p_\bullet},$$

here

$$p_\bullet = \begin{cases} \min\limits_{m\in\mathbb{N}} p_m & \text{if } \sum\limits_{m=0}^{\infty} a_m \leq 1, \\ \max\limits_{m\in\mathbb{N}} p_m & \text{if } \sum\limits_{m=0}^{\infty} a_m > 1. \end{cases}$$

Remark. From ([11], p.89]), we recall the estimate $\mu^{\sigma}_{\Psi,s} f(x) \leq 2^{n\lambda} \mu^{*,\sigma}_{\sigma,\lambda} f(x)$. Therefore, we present only the proof of Theorem 3.4.

Proof. We present the proof of $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (homogeneous case). The same argument holds true for $K^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (nonhomogeneous case).

Let $f \in \dot{K}^{\alpha(\cdot),q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)$. Decomposet f as:

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

From homogeneous $\dot{K}^{\alpha(\cdot),q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ (Definition 2.2), we have

$$\|\mu^{*,\sigma}_{\Psi,\lambda}(f)\|_{\dot{K}^{\alpha(\cdot),q_2(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha(\cdot)} |\mu^{*,\sigma}_{\Psi,\lambda}(f)\chi_k|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq 1 \right\}.$$

We have

$$\left\| \left(\frac{2^{k\alpha(\cdot)} |\mu^{*,\sigma}_{\Psi,\lambda}(f)\chi_k|}{\beta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}} \leq \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=-\infty}^{\infty} \mu^{*,\sigma}_{\Psi,\lambda}(f_j)\chi_k|}{\beta_{01}+\beta_{02}+\beta_{03}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_{01}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}+C\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_{02}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$+C\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_{03}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}},$$

where

$$\beta_{01}=\left\|\left\{2^{k\alpha(\cdot)}|\sum_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\beta_{02}=\left\|\left\{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})},$$

$$\beta_{03}=\left\|\left\{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_1(\cdot)})}.$$

If $\beta_0=\beta_{01}+\beta_{02}+\beta_{03}$ thus

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\lesssim 1.$$

Then

$$\|\mu_{\Psi,\lambda}^{*,\sigma}(f)\chi_k\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)}\lesssim\beta_0\lesssim[\beta_{01}+\beta_{02}+\beta_{03}].$$

Therefore, if we can conclude that

$$\beta_{01}\leq C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)},\qquad \beta_{02}\leq C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)},\qquad \beta_{03}\leq C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)},$$

we are finished. Let us set $\beta_0=\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)}$.

First, we estimate β_{02}. From Lemma 2.6 and Lemma 2.7, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right\|_{L^{p_1(\cdot)}}^{(q_2^0)_k}$$

$$\lesssim\sum_{k=-\infty}^{\infty}\left\|\frac{|\sum\limits_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(2^{\alpha j}f_j)\chi_k|}{\beta_0}\right\|_{L^{p_1(\cdot)}}^{(q_2^0)_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{k+1}\left\|\frac{|\mu_{\Psi,\lambda}^{*,\sigma}(2^{\alpha j}f_j)\chi_k|}{\beta_0}\right\|_{L^{p_1(\cdot)}}\right)^{(q_2^0)_k},$$

where

$$(q_2^0)_k=\begin{cases}(q_2)_+ & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\geq 1,\\ (q_2)_- & otherwise.\end{cases}$$

By the boundedness of $\mu_{\Psi,\lambda}^{*,\sigma}$ on $L^{p(\cdot)}$, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{k+1}\left\|\frac{|(2^{j\alpha(\cdot)}f_j)|}{\beta_0}\right\|_{L^{p_1(\cdot)}}\right)^{(q_2^0)_k},$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|f_k|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{\frac{L^{p_1(\cdot)}}{L^{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1)_+}}$$

$$\lesssim \left[\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|f_k|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{\frac{L^{p_1(\cdot)}}{L^{q_1(\cdot)}}}\right]^{q_\bullet}\lesssim 1,$$

Here $q_\bullet=\min\limits_{k\in\mathbb{N}}\frac{(q_2^0)_k}{(q_1)_+}\geq 1.$

The previous calculations imply that

$$\beta_{02}\lesssim\beta_0\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha(\cdot),q_1(\cdot)}(\mathbb{R}^n)}.$$

We must examine $\mu_{\Psi,\lambda}^{*,\sigma}f_j$. By applying the Minkowski inequality, we have

$$\begin{aligned}
&|\mu_{\Psi,\lambda}^{*,\sigma}(f_j)(x)|\\
&=\left(\int_0^\infty\int_{\mathbb{R}^n}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n}\left|\frac{1}{t^\sigma}\int_{|y_1-z_1|\leq t}\frac{\Psi(y_1,y_1-z_1)}{|y_1-z_1|^{n-\sigma}}f_j(z_1)\mathrm{d}z_1\right|^2\frac{\mathrm{d}y_1\mathrm{d}t}{t^{n+1}}\right)^{\frac12}\\
&\leq\int_{\mathbb{R}^n}f_j(z_1)\left(\int_0^\infty\int_{|y_1-z_1|\leq t}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{\mathrm{d}y_1\mathrm{d}t}{t^{2\sigma+n+1}}\right)^{\frac12}\mathrm{d}z_1\\
&\leq\int_{\mathbb{R}^n}f_j(z_1)\left(\int_0^{|x_1-z_1|}\int_{|y_1-z_1|\leq t}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{\mathrm{d}y_1\mathrm{d}t}{t^{2\sigma+n+1}}\right)^{\frac12}\mathrm{d}z_1
\end{aligned}$$

$$+\int_{\mathbb{R}^n} f_j(z_1)\left(\int_{|x_1-z_1|}^{\infty}\int_{|y_1-z_1|\le t}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{dy_1dt}{t^{2\sigma+n+1}}\right)^{\frac{1}{2}}dz_1.$$

Let $2\rho-n>0$ and $\Psi\in L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})$. Then, the following inequality is satisfied

$$\begin{aligned}\int_{|y_1-z_1|\le t}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}dy_1 &\le \int_{S^{n-1}}\int_0^t\frac{|\Psi(sy_1'+z_1,y_1')|^2}{s^{2n-2\sigma}}s^{n-1}dsd\sigma(y_1')\\ &\lesssim \|\Psi\|^2_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}t^{2\sigma-n}.\end{aligned}$$

Because $|x_1-z_1|\le|y_1-z_1|+|x_1-y_1|\le|x_1-y_1|+t$, for $\lambda>2$ and $0<\epsilon<(\lambda-2)n$, we obtain

$$\begin{aligned}&\int_0^{|x_1-z_1|}\int_{|y_1-z_1|\le t}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{dy_1dt}{t^{2\sigma+n+1}}\\ &\lesssim \int_0^{|x_1-z_1|}\int_{|y_1-z_1|\le t}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n-2n-\epsilon}\frac{1}{|x_1-z_1|^{2n+\epsilon}}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{dy_1dt}{t^{2\sigma-n-\epsilon+1}}\\ &\lesssim \frac{1}{|x_1-z_1|^{2n+\epsilon}}\int_0^{|x_1-z_1|}\int_{|y_1-z_1|\le t}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{dy_1dt}{t^{2\sigma-n-\epsilon+1}}\\ &\lesssim \frac{\|\Psi\|^2_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}}{|x_1-z_1|^{2n+\epsilon}}\int_0^{|x_1-z_1|}t^{\epsilon-1}dt\\ &\lesssim |x_1-z_1|^{-2n}.\end{aligned}$$

Let $\lambda_0 n-2n<0$, $\lambda_0 n-n>0$ and $1<\lambda_0<2$. Then, we get

$$\begin{aligned}&\int_{|x_1-z_1|}^{\infty}\int_{|y_1-z_1|\le t}\left((t)(t+|x_1-y_1|)^{-1}\right)^{\lambda n}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{dy_1dt}{t^{2\sigma+n+1}}\\ &\lesssim \int_{|x_1-z_1|}^{\infty}\int_{|y_1-z_1|\le t}|x_1-z_1|^{-\lambda_0 n}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-2\sigma}}\frac{dy_1dt}{t^{2\sigma-\lambda_0 n+n+1}}\\ &\lesssim \int_{|x_1-z_1|}^{\infty}|x_1-z_1|^{-\lambda_0 n}\int_{|y_1-z_1|\le t}\frac{|\Psi(y_1,y_1-z_1)|^2}{|y_1-z_1|^{2n-\lambda_0 n}}\frac{dy_1dt}{t^{n+1}}\\ &\lesssim \int_{|x_1-z_1|}^{\infty}|x_1-z_1|^{-\lambda_0 n}\int_{S^{n-1}}\int_0^t\frac{|\Psi(y_1',(y_1-z_1)')|^2}{s^{2n-\lambda_0 n}}s^{n-1}dsd\sigma(y_1')\frac{dt}{t^{n+1}}\\ &\lesssim \|\Psi\|^2_{L^{\infty}(\mathbb{R}^n)\times L^2(S^{n-1})}|x_1-z_1|^{-\lambda_0 n}\int_{|x_1-z_1|}^{\infty}t^{\lambda_0 n-2n-1}dt\\ &\lesssim |x_1-z_1|^{-2n}.\end{aligned}$$

By combining these estimates, we find

$$\mu^{*,\sigma}_{\Psi,\lambda}(f)(x)\lesssim\int_{\mathbb{R}^n}\frac{|f_j(z_1)|}{|x_1-z_1|^n}dz_1. \tag{$*$}$$

Next, we consider β_{01}. Since $j\le k-2$, by applying Hölder's inequality (Lemma 2.3), we obtain

$$\begin{aligned}\mu^{*,\sigma}_{\Psi,\lambda}(f)(x) &\lesssim \int_{\mathbb{R}^n}\frac{|f_j(z_1)|}{|x_1-z_1|^n}dz\\ &\lesssim 2^{-kn}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Then, by Lemmas 2.5 - 2.7, we deduce that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right\|_{L^{p_1(\cdot)}}^{(q_2^{01})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}\left\|\frac{f_j}{\beta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}2^{-kn}\right]^{(q_2^{01})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}\left\|\frac{f_j}{\beta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\right]^{(q_2^{01})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_{11}}\left\|\frac{f_j}{\beta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^{01})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(k-j)(-n\delta_{11}+\alpha_+)}\left\|\left(\frac{|2^{j\alpha(\cdot)}f\chi_j|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right]^{(q_2^{01})_k},$$

where

$$(q_2^{01})_k=\begin{cases}(q_2)_- & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\geq 1,\\ (q_2)_+ & otherwise.\end{cases}$$

If $(q_1)_+\leqslant 1$, then by Lemma 2.6, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(k-j)(q_1)_+(\alpha_+-n\delta_{11})}\left\|\left(\frac{|2^{j\alpha(\cdot)}f\chi_j|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{\frac{(q_2^{01})_k}{(q_1)_+}}$$

$$\lesssim \left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha(\cdot)}f\chi_j|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}2^{(k-j)(q_1)_+(\alpha_+-n\delta_{11})}\right]^{q_\bullet}\lesssim 1,$$

here $q_\bullet=\min\limits_{k\in\mathbb{N}}\frac{(q_2^{01})_k}{(q_1)_+}$.
If $(q_1)_+>1$, using Hölder's inequality, we deduce that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=-\infty}^{k-2}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})\frac{(q_1)_+}{2}}\left\|\left(\frac{|2^{j\alpha(\cdot)}f\chi_j|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{\frac{(q_2^{01})_k}{(q_1)_+}}$$

$$\times\left[\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})\frac{((q_1)_+)'}{2}}\right]^{\frac{(q_2^{01})_+}{((q_1)_+)'}}$$

$$\lesssim \left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{|2^{j\alpha(\cdot)}f\chi_j|}{\beta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha_+-n\delta_{11})\frac{(q_1)_+}{2}}\right]^{q_\bullet}\lesssim 1.$$

Therefore, we conclude the estimate

$$\beta_{01}\lesssim\beta_0\lesssim\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate β_{03}. Since $j\geq k+2$, by $(*)$ and applying Hölder's inequality (Lemma 2.3), we have

$$\begin{aligned}\mu_{\Psi,\lambda}^{*,\sigma}(f)(x)&\lesssim\int_{\mathbb{R}^n}\frac{|f_j(z_1)|}{|x_1-z_1|^n}dz_1\\&\lesssim 2^{-jn}\|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Then, by Lemma2.5, Lemma2.6 and Lemma2.7, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{k+2}^{\infty}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}$$

$$\lesssim\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right\|_{L^{p_1(\cdot)}}^{(q_2^{02})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{-jn}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\left\|\frac{f_j}{\beta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^{02})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{-jn}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}|B_j|\left\|\frac{f_j}{\beta_0}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^{02})_k}$$

$$\lesssim \sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_{12}+\alpha_-)}\left\|\left(\frac{2^{j\alpha(\cdot)}f\chi_j}{\beta_0}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}}^{\frac{1}{(q_1)_+}}\right]^{(q_2^{02})_k},$$

where

$$(q_2^{02})_k=\begin{cases}(q_2)_- & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}\mu_{\Psi,\lambda}^{*,\sigma}(f_j)\chi_k|}{\beta_0}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_2(\cdot)}}}\geq 1,\\ (q_2)_+, & otherwise.\end{cases}$$

Notice that $\alpha_- > -n\delta_{12}$, and applying the same estimation technique used for β_{01}, we get that

$$\beta_{03}\lesssim\beta_0\lesssim\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}.$$

Thus, the theorem 3.4 is proven. □

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References

[1] M. Sakamoto, K. Yabuta, Boundedness of Marcinkiewicz functions, Studia Math. 135 (1999), 103–142. http://eudml.org/doc/216646

[2] Q. Xue, Y. Ding, Wieghted L^p boundedness for parametrized Littlewood–Paley operator, Taiwan. J. Math. 11 (2007), 1143–1165. https://doi.org/10.11650/twjm/1500404809

[3] F. Deringoz, V.S. Guliyev, M.A. Ragusa, Intrinsic square functions on vanishing generalized Orlicz-Morrey spaces, Set-Valued Var. Anal. 25 (2017), 807–828. https://doi.org/10.1007/s11228-017-0422-y

[4] B. Li, Weighted norm inequalities for parametric Littlewood–Paley operators, Math. Inequal. Appl. 22 (2019), 1205–1220. https://doi.org/10.7153/mia-2019-22-34

[5] A. Abdalmonem, O. Abdalrhman, S. Tao, Commutators of fractional integral with variable kernel on variable exponent Herz and Lebesgue spaces, Int. J. Math. Appl. 4 (2016), 29–40. https://ijmaa.in/index.php/ijmaa/article/view/970

[6] J. Chen, Y. Ding, D. Fan, Littlewood–Paley operator with variable kernel, Sci. China Ser. A 49 (2006), 639–650. https://doi.org/10.1007/s11425-006-0639-y

[7] X. Shao, S. Tao, Weighted estimates of variable kernel fractional integral and its commutators on vanishing generalized Morrey spaces with variable exponent, Chin. Ann. Math. Ser. B 42 (2021), 451–470. https://doi.org/10.1007/s11401-021-0268-3

[8] H. Wang, Estimates of some integral operators with bounded variable kernels on the Hardy and weak Hardy spaces over $\mathbb{R}^n$, Acta Math. Sin. Engl. Ser. 32 (2016), 411–438. https://doi.org/10.1007/s10114-016-4617-1

[9] A. Abdalmonem, O. Abdalrhman, S. Tao, Boundedness of Littlewood–Paley operators with variable kernel on the weighted Herz–Morrey spaces with variable exponent, Surv. Math. Appl. 15 (2020), 295–313. https://www.utgjiu.ro/math/sma/v15/v15.html

[10] A. Almeida, D. Drihem, Maximal, potential and singular type operators on Herz spaces with variable exponents, J. Math. Anal. Appl. 394 (2012), 781–795. https://doi.org/10.1016/j.jmaa.2012.04.043

[11] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton (1970). https://doi.org/10.1112/blms/5.1.121

[12] M. Izuki, T. Noi, Boundedness of some integral operators and commutators on generalized Herz spaces with variable exponents, OCAMI Preprint Ser. 11 (2011), 1–23. https://ocu-omu.repo.nii.ac.jp/records/201676

[13] Y. Shu, L. Wang, D. Xia, Sublinear operators with rough kernel on Herz spaces with variable exponent, Anal. Theory Appl. 38 (2022), 79–91.

[14] M.A. Ragusa, Homogeneous Herz spaces and regularity results, Nonlinear Anal. 71 (2009), e1909–e1914. https://doi.org/10.1016/j.na.2009.02.075

[15] A. Scapellato, Homogeneous Herz spaces with variable exponents and regularity results, Electron. J. Qual. Theory Differ. Equ. 2018 (2018), 1–11. https://doi.org/10.14232/ejqtde.2018.1.82

[16] A. Scapellato, Regularity of solutions to elliptic equations on Herz spaces with variable exponents, Bound. Value Probl. 2019 (2019), 1–13. https://doi.org/10.1186/s13661-018-1116-6

[17] M.A. Ragusa, Parabolic Herz spaces and their applications, Appl. Math. Lett. 25 (2012), 1270–1273. https://doi.org/10.1016/j.aml.2011.11.022

[18] A. Abdalmonem, O. Abdalrhman, S. Tao, Boundedness of fractional integral with variable kernel and their commutators on variable exponent Herz spaces, Appl. Math. 7 (2016), 1160–1172. https://doi.org/10.4236/am.2016.710104

[19] A. Abdalmonem, A. Scapellato, Intrinsic square functions and commutators on Morrey–Herz spaces with variable exponents, Math. Methods Appl. Sci. 44 (2021), 12408–12425. https://doi.org/10.1002/mma.7487

[20] L. Wang, S. Tao, Parameterized Littlewood–Paley operators and their commutators on Herz spaces with variable exponents, Turk. J. Math. 40 (2016), 122–145. https://doi.org/10.3906/mat-1412-52

[21] O. Abdalrhman, M. Zainul Abidin, Boundedness of the vector-valued intrinsic square functions on variable exponents Herz spaces, Mathematics 10 (2022), 1168. https://doi.org/10.3390/math10071168

[22] D. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces. Foundations and Harmonic Analysis, Appl. Numer. Harmon. Anal., Springer, New York (2013). https://doi.org/10.1007/978-3-0348-0548-3

[23] M. Izuki, Commutators of fractional integrals on Lebesgue and Herz spaces with variable exponent, Rend. Circ. Mat. Palermo 59 (2010), 461–472. https://doi.org/10.1007/s12215-010-0034-y

[24] D. Cruz-Uribe, A. Fiorenza, J. Martell, C. Pérez, The boundedness of classical operators on variable L_p spaces, Ann. Fenn. Math. 31 (2006), 239–264. http://eudml.org/doc/126704

[25] A. Abdalmonem, O. Abdalrhman, H. Mohammed, Commutators of fractional integral with variable kernel on variable exponent Herz–Morrey spaces, Open J. Math. Anal. (2019), 19–29.