Eur. J. Math. Anal. 6 (2026) 6
doi: 10.28924/ada/ma.6.6

C-Class Functions and Fixed Points of Weakly Contractive Mappings in Rectangular b-Metric Spaces

Fakhr-dine Nhari[1], Mohamed Rossafi[1,*], Abdelkarim Kari[2], Arsalan Hojjat Ansari[3]

[1] *Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco*
nharidoc@gmail.com; rossafimohamed@gmail.com

[2] *Laboratory of Analysis, Modeling and Simulation Faculty of Sciences Ben M'Sik, Hassan II University, B.P. 7955 Casablanca, Morocco*
abdkrimkariprofes@gmail.com

[3] *Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, Pretoria, Medunsa-0204, South Africa*
mathanalsisamir4@gmail.com

[*] *Correspondence: rossafimohamed@gmail.com*

Abstract. In this paper, we introduce C class-F-generalized weakly contractive mapping and prove the existence and uniqueness of fixed points of such maps in the setting of rectangular b-metric spaces. We provide an example in support of our result.

1. Introduction

It is well known that the Banach contraction principle [3] is a fundamental result in the fixed point theory, several many interesting extensions and generalizations have obtained [10]. The well-known metric spaces have been generalized metric spaces introduced by Branciari [4]. Recently, George, Radenovic, Reshma, and Shukla [7] announced the notion of b-rectangular metric space and formulated some fixed point theorems in the b-rectangular metric space. Many authors initiated and studied many existing fixed point theorems in such spaces [8, 9].

The existence of fixed points of weakly contractive mappings is a generalization of the Banach contraction principle, which was first given by Alber and Delabriere in Hilbert spaces. [1]. Choudhury, Konar, Rhoades and Metiya [6] proved some fixed point results for weakly contractive mappings in complete metric spaces.

Very recently, Cho [5] introduced a special weakly contractive mappings called generalized weakly contractive mappings and proved some fixed point results for such mappings in complete metric spaces.

Received: 19 August 2025.
Key words and phrases. C-class functions, Fixed point, Weakly contractive mapping, Rectangular b-metric space.

In this work, we introduce C class F-generalized weakly contractive mappings and provide some fixed-point results for such mappings in complete b-rectangular metric spaces. We also present some special examples of generalized weakly contractive mappings on b-rectangular metric spaces. Also, we derive some useful corollaries.

2. Preliminaries

In the following we present the required literature that is needed to prove our result.

Definition 2.1. [7] Let X be a nonempty set and d: $X \times X \to [0, +\infty[$ be a function such that for all $x, y \in X$ and all distinct points $u, v \in X$. If there exists a real number $s \geq 1$ such that

1. $d(x, y) = 0$ if only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ $(b - rectangular\ inequality)$.

Then d is said to be a rectangular metric on X and (X, d) is called a b-rectangular metric space.

Example 2.2. [8] Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2, 3, 4, 5, 6, 7\}\}$ and $B = [1, 2]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x)\ for\ all\ x, y \in X; \\ d(x,y) = 0 \Leftrightarrow y = x \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = d\left(\frac{1}{6}, \frac{1}{7}\right) = 0,05 \\ d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{3}, \frac{1}{7}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0,08 \\ d\left(\frac{1}{2}, \frac{1}{6}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{7}\right) = 0,4 \\ d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{6}\right) = d\left(\frac{1}{4}, \frac{1}{7}\right) = 0,24 \\ d\left(\frac{1}{2}, \frac{1}{7}\right) = d\left(\frac{1}{3}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0,15 \\ \quad d(x, y) = (|x - y|)^2\ \ otherwise. \end{cases}$$

Then (X, d) is a b-rectangular metric space with coefficient $s = 3$.

Lemma 2.3. *[13] Let (X, d) be a b-rectangular metric space.*

(a) *Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to +\infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have*

$$\frac{1}{s} d(x, y) \leq \liminf_{n \to +\infty} d(x_n, y_n) \leq \limsup_{n \to +\infty} d(x_n, y_n) \leq s d(x, y).$$

(b) *If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then*

$$\frac{1}{s} d(x, y) \leq \liminf_{n \to +\infty} d(x_n, y) \leq \limsup_{n \to +\infty} d(x_n, y) \leq s d(x, y),$$

for all $x \in X$.

Lemma 2.4. *[8] Let (X, d) be a b-rectangular metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n \to +\infty} d(x_n, x_{n+1}) = \lim_{n \to +\infty} d(x_n, x_{n+2}) = 0. \tag{2.1}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\varepsilon \leq \liminf_{k \to +\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq \limsup_{k \to +\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \liminf_{k \to +\infty} d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq \limsup_{k \to +\infty} d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon,$$

$$\varepsilon \leq \liminf_{k \to +\infty} d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq \limsup_{k \to +\infty} d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leq s\varepsilon,$$

$$\frac{\varepsilon}{s} \leq \liminf_{k \to +\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq \limsup_{k \to +\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leq s^2\varepsilon.$$

Definition 2.5. A function $f : X \to \mathbb{R}^+$, where X is a b -rectangular metric space, is called lower semicontinuous if for all $x \in X$ and $x_n \in X$ with $lim_{n \to +\infty} x_n = x$, we have

$$f(x) \leq \liminf_{n \to +\infty} f(x_n).$$

Definition 2.6. A function $g : X \to \mathbb{R}^+$, where X is a b -rectangular metric space, is called is a right upper semicontinuous function if for all $x \in X$ and $x_n \in X$ with $lim_{n \to +\infty} x_n = x$, we have

$$g(x) \geq \limsup_{n \to +\infty} f(x_n).$$

Definition 2.7. [11] A function $\psi : [0, +\infty[\to [0, +\infty[$ is said to be an altering distance function if it satisfies the following conditions:

(a) ψ is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of altering distance functions by Φ.

Definition 2.8. [5] Let X be a complete metric space with metric d, and $T : X \to X$. Also let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi)),$$

where

$$m(x,y,d,T,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(x,Tx)+\varphi(x)+\varphi(Tx),d(y,Ty)+\varphi(y)+\varphi(Ty),$$
$$\frac{1}{2}\{d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Tx)+\varphi(Tx)+\varphi(y)\}\}$$

and

$$l(x,y,d,T,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

for all $x,y\in X$, where $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is continuous with $\psi(t)=0$ if and only if $t=0$ and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function with $\phi(t)=0$ if and only if $t=0$.

Theorem 2.9. *[5] Let X be complete. If T is a generalized weakly contractive mapping, then there exists a unique $z\in X$ such that $z=Tz$ and $\varphi(z)=0$.*

In 2014 the concept of C-class functions was introduced by H. Ansari in [2] .

Definition 2.10. [2] A mapping $F:[0,\infty)^2\to\mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(1) $F(s,t)\le s$;

(2) $F(s,t)=s$ implies that either $s=0$ or $t=0$; for all $s,t\in[0,\infty)$.

Note for some F we have that $F(0,0)=0$.

We denote C-class functions as $\mathcal{C}$.

Example 2.11. [2] The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t\in[0,\infty)$:

(1) $F(s,t)=s-t$, $F(s,t)=s\Rightarrow t=0$;

(2) $F(s,t)=ms$, $0<m<1$, $F(s,t)=s\Rightarrow s=0$;

(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;

(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;

(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,1)=s\Rightarrow s=0$;

(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1, r\in(0,\infty)$, $F(s,t)=s\Rightarrow t=0$;

(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;

(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$;

(9) $F(s,t)=s\beta(s)$, $\beta:[0,\infty)\to(0,1)$,and is continuous, $F(s,t)=s\Rightarrow s=0$;

(10) $F(s,t)=s-\frac{t}{k+t}, F(s,t)=s\Rightarrow t=0$;

(11) $F(s,t)=s-\varphi(s), F(s,t)=s\Rightarrow s=0$,here $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$;

(12) $F(s,t)=sh(s,t), F(s,t)=s\Rightarrow s=0$,here $h:[0,\infty)\times[0,\infty)\to[0,\infty)$is a continuous function such that $h(t,s)<1$ for all $t,s>0$;

(13) $F(s,t)=s-(\frac{2+t}{1+t})t$, $F(s,t)=s\Rightarrow t=0$.

(14) $F(s,t)=\sqrt[n]{\ln(1+s^n)}$, $F(s,t)=s\Rightarrow s=0$.

(15) $F(s,t)=\phi(s), F(s,t)=s\Rightarrow s=0$,here $\phi:[0,\infty)\to[0,\infty)$ is a upper semicontinuous function such that $\phi(0)=0$, and $\phi(t)<t$ for $t>0$,

(16)$F(s,t)=\frac{s}{(1+s)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$;

Definition 2.12. [2] Let Φ_u denote the class of the functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:

(a) φ is continuous ;
(b) $\varphi(t)>0, t>0$ and $\varphi(0)\geq 0$.

Definition 2.13. [12] Let X be a complete b-rectangular metric space with metric d and parameter s and $T:X\to X$. Also let $\varphi:X\to\mathbb{R}^+$ be a lower semicontinuous function. Then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\psi(M(x,y,d,T,\varphi))-\phi(M(x,y,d,T,\varphi)),$$

where

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(x,Tx)+\varphi(x)+\varphi(Tx),d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

for all $x,y\in X$, and ψ is an altering distance function and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function with $\phi(t)=0$ if and only if $t=0$.

Theorem 2.14. *[12] Let X be a complete b-rectangular metric space with parameter $s\geq 1$. If T is a generalized weakly contractive mapping, then T has a unique fixed point $z\in X$ such that $z=Tz$ and $\varphi(z)=0$.*

Definition 2.15. [12]Let X be a complete b-rectangular metric space with metric d and parameter s, and $T:X\to X$ be a mapping. Also let $\varphi:X\to\mathbb{R}^+$ be a lower semicontinuous function. Then T is called a generalized (ψ,φ,ϕ) contractive mapping if it satisfies the following condition:

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\phi(M(x,y,d,T,\varphi)), \tag{2.2}$$

where

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y),d(x,Tx)+\varphi(x)+\varphi(Tx),d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

for all $x,y\in X$, and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is an altering distance function and $\phi:\mathbb{R}^+\to\ \mathbb{R}^+$ is a right upper semi-continuous function with the condition: $\psi(t)>\phi(t)$ for all $t>0$ and $\phi(t)=0$ if and only if $t=0$.

Theorem 2.16. *[12] Let X be a complete b-rectangular metric space with parameter $s\geq 1$ and $T:X\to X$ be a mapping. If T is a generalized (ψ,φ,ϕ) contractive mapping then T has a unique fixed point $z\in X$ such that $z=Tz$ and $\varphi(z)=0$.*

3. Main results

Aspired by idea of the generalized weakly contractive mapping on metric space introduced by Cho [5], we introduce the notion of C-Class function for weakly contractive mapping on rectangular b-metric space and establish some fixed point on such mapping.

Definition 3.1. Let X be a complete b-rectangular metric space with metric d and parameter s and $T : X \to X$. Also let $\varphi : X \to R^+$ be a lower semicontinuous function. Then T is called a C class $-F-$generalized weakly contractive mapping if it satisfies the following condition:

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) \le F(\psi(M(x,y,d,T,\varphi)),\phi(M(x,y,d,T,\varphi))), \tag{3.1}$$

where

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

for all $x, y \in X$, and $\psi \in \Phi, \phi \in \Phi_u,\ \ F \in \mathcal{C}$.

Theorem 3.2. *Let X be a complete b-rectangular metric space with parameter $s \ge 1$. If T is a C class $-F-$generalized weakly contractive mapping, then T has a unique fixed point $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

Proof. Let $x_0 \in X$ be an arbitrary point in X. Then we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in\ \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = 0$, then x_{n_0} is a fixed point of T.

Next, we assume that $x_n \ne x_{n+1}$.

We claim that

$$\lim_{n\to+\infty} d(x_{n,}x_{n+1}) = 0$$

and

$$\lim_{n\to+\infty} d(x_{n,}x_{n+2}) = 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.1) for all $n \in \mathbb{N}$, we have

$$\psi\left(s^2 d(Tx_{n-1},Tx_n)+\varphi(Tx_{n-1})+\varphi(Tx_n)\right) \le F(\psi(M(x_{n-1},x_n,d,T,\varphi)),\phi(M(x_{n-1},x_n,d,T,\varphi))), \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1},x_n,d,T,\varphi) &= \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_{n-1},x_n)+\varphi(x_{n-1}) \\ &\quad +\varphi(x_n), d(x_n,Tx_n)+\varphi(x_n)+\varphi(Tx_n)\} \\ &= \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(Tx_{n+1})\}. \end{aligned}$$

If $M(x_{n-1},x_n,d,T,\varphi)=d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})$, then we have

$$\begin{aligned}\psi\left(d\left(Tx_{n-1},Tx_n\right)+\varphi\left(Tx_{n-1}\right)+\varphi\left(x_{n+1}\right)\right)&=\psi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\\ &\leq\psi\left(s^2d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\\ &\leq F(\psi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\\ &,\phi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)),\end{aligned}$$

which implies

$$\phi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)=0,$$

and so

$$d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)=0.$$

Hence

$$d\left(x_n,x_{n+1}\right)=0\ and\ \varphi\left(x_n\right)=\varphi\left(x_{n+1}\right)=0,$$

which is a contradiction. Thus we have

$$d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\leq d\left(x_{n-1},x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right),\ \ for\ all\ n\in\mathbb{N}\tag{3.3}$$

and

$$M\left(x_{n-1},x_n,d,T,\varphi\right)=d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),\ \ for\ all\ n\in\mathbb{N}\tag{3.4}$$

for all $for\ all\ n\in\mathbb{N}$. It follows from (3.2) that

$$\begin{aligned}\psi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)\leq F(\psi\left(d\left(x_{n-1},x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)\right)\\ ,\phi\left(d\left(x_{n-1},x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)\right)).\end{aligned}\tag{3.5}$$

It follows from (3.3) that the sequence $\{d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\}_{n\in\mathbb{N}}$ is nonincreasing. Hence $d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\to r$ as $n\to+\infty$ for some $r\geq 0$. Assume $r>0$ and letting $n\to+\infty$ in (3.5) and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned}\psi\left(r\right)\leq\psi\left(s^2r\right)&\leq F(\psi\left(r\right),\liminf_{n\to\infty}\phi\left(d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right))\\ &\leq F(\psi\left(r\right),\phi\left(r\right)).\end{aligned}$$

It follows that $\psi\left(r\right)=0$, or $\quad,\phi\left(r\right)=0$, hence we have $r=0$ and consequently,

$$\lim_{n\to+\infty}\left[d\left(x_n,x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)=0\right].$$

So

$$\lim_{n\to+\infty}d\left(x_n,x_{n+1}\right)=0,\tag{3.6}$$

$$\lim_{n\to+\infty}\varphi\left(x_n\right)=\lim_{n\to+\infty}\varphi\left(x_{n+1}\right)=0.\tag{3.7}$$

Now, we shall prove that T has a periodic point. Suppose that it is not the case. Then $x_n\neq x_m$ for all $n,m\in\mathbb{N},\ n\neq m$.

In (3.1), letting $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\psi\left(s^2 d\left(Tx_{n-1}, Tx_{n+1}\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(Tx_{n+1}\right)\right)$$
$$\leq F(\psi\left(M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right)\right), \phi\left(M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right)\right)),$$

where

$$\begin{aligned} & M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right) \\ = \; & \max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), \\ & d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n), d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})\} \\ = \; & \max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}. \end{aligned}$$

So we get

$$\begin{aligned} & \psi\left(d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right) \leq \psi\left(s^2 d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right) \\ & \leq F(\psi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right) \\ & , \phi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right)). \end{aligned}$$

Take $a_n = d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)$ and $b_n = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$. Then

$$\psi\left(a_n\right) \leq F(\psi\left(\max\left(a_{n-1}, b_{n-1}\right)\right), \phi\left(\max\left(a_{n-1}, b_{n-1}\right)\right)) \tag{3.8}$$
$$\leq \psi\left(\max\left(a_{n-1}, b_{n-1}\right)\right). \tag{3.9}$$

Since ψ is increasing, we get

$$a_n \leq \max\left\{a_{n-1}, b_{n-1}\right\}.$$

By (3.3), we have

$$b_n \leq b_{n-1} \leq \max\left\{a_{n-1}, b_{n-1}\right\},$$

which implies that

$$\max\left\{a_n, b_n\right\} \leq \max\left\{a_{n-1}, b_{n-1}\right\}, \ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\max\left\{a_{n-1}, b_{n-1}\right\}_{n\in \mathbb{N}}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\lambda \geq 0$ such that

$$\lim_{n\to+\infty} \max\left\{a_n, b_n\right\} = \lambda.$$

Assume that $\lambda > 0$. By (3.6), it is obvious that

$$\lambda = \lim_{n\to+\infty} \sup a_n = \lim_{n\to+\infty} \sup \max\left\{a_n, b_n\right\} = \lim_{n\to+\infty} \max\left\{a_n, b_n\right\}. \tag{3.10}$$

Taking $\limsup_n \to +\infty$ in (3.8), using (3.10) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}\psi(\lambda) &= \psi\left(\limsup_{n\to+\infty} a_n\right)\\ &= \limsup_{n\to+\infty} \psi\left(a_n\right)\\ &\le \limsup_{n\to+\infty} \psi\left(\max\left\{a_n, b_n\right\}\right) - \liminf_{n\to+\infty} \phi\left(\max\left\{a_n, b_n\right\}\right)\\ &\le F(\psi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right), \phi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right))\\ &= F(\psi(\lambda), \phi(\lambda)),\end{aligned}$$

which implies that $\psi\left(r\right) = 0$ o, $\phi\left(r\right) = 0$, a contradiction. Thus,

$$\limsup_{n\to+\infty} a_n = 0$$

and hence

$$\lim_{n\to+\infty} d\left(x_{n,} x_{n+2}\right) = 0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n,m\to+\infty} d\left(x_{n,} x_m\right) = 0$ for all $n, m \in \mathbb{N}$. Suppose to the contrary. By Lemma 2.4, there is an $\varepsilon > 0$ such that for an integer k there exist two sequences $\left\{n_{(k)}\right\}$ and $\left\{m_{(k)}\right\}$ such that

i) $\varepsilon \le \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \le \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \le s\varepsilon$,

ii) $\varepsilon \le \lim_{k\to+\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)}+1}\right) \le \lim_{k\to+\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)}+1}\right) \le s\varepsilon$,

iii) $\varepsilon \le \lim_{k\to+\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) \le \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) \le s\varepsilon$,

vi) $\frac{\varepsilon}{s} \le \lim_{k\to+\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)}+1}\right) \le \lim_{k\to+\infty} \sup d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) \le s^2\varepsilon$.

From (3.1) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\begin{aligned}M(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi) &= \max\{d(x_{m_{(k)}}, x_{n_{(k)}}) + \varphi(x_{m_{(k)}}) + \varphi(x_{m_{(k)}}),\\ &d(x_{m_{(k)}}, x_{m_{(k)}+1}) + \varphi(x_{m_{(k)}}) + \varphi(x_{m_{(k)}+1}), d(x_{n_{(k)}}, x_{n_{(k)}+1}) + \varphi(x_{n_{(k)}}) + \varphi(x_{n_{(k)}+1})\}.\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.6), (3.7) and (iii) of Lemma 2.4, we have

$$\lim_{k\to+\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi\right) \le s\varepsilon. \tag{3.11}$$

Now letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.1), we have

$$\begin{aligned}&\psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) + \varphi\left(x_{m_{(k)}+1}\right) + \varphi\left(x_{n_{(k)}+1}\right)\right]\\ &\quad\le F(\psi\left[d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) + \varphi\left(x_{m_{(k)}+1}\right) + \varphi\left(x_{n_{(k)}+1}\right)\right]\\ &\qquad, \phi\left[d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) + \varphi\left(x_{m_{(k)}}\right) + \varphi\left(x_{n_{(k)}}\right)\right]).\end{aligned}$$

Letting $k \to +\infty$, using (3.6), (3.7), (3.11), and applying the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\lim_{k\to+\infty} \psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right)\right] \le F(\psi(s\varepsilon), \phi(s\varepsilon)).$$

Using (3.11) and (iv) of Lemma 2.4, we obtain

$$\psi(s\varepsilon) = \psi\left(s^2\frac{\varepsilon}{s}\right) \leq \lim_{k\to+\infty} \sup \psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right)\right] \leq F(\psi(s\varepsilon), \phi(s\varepsilon)).$$

This is a contradiction. Thus

$$\lim_{n,m\to+\infty} d\left(x_m, x_n\right) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to+\infty} d\left(x_n, z\right) = 0.$$

Since φ is lower semicontinuous, we get

$$\varphi\left(z\right) \leq \liminf_{n\to+\infty} \varphi\left(x_n\right) \leq \lim_{n\to+\infty} \varphi\left(x_n\right) = 0,$$

which implies

$$\varphi\left(z\right) = 0. \tag{3.12}$$

Now, putting $x = x_n$ and $y = z$ in (3.1), we have

$$\begin{aligned} M\left(x_n, z, d, T, \varphi\right) &= \max\{d\left(x_n, z\right) + \varphi\left(x_n\right) + \varphi\left(z\right), \\ &\quad d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right), d\left(z, Tz\right) + \varphi\left(z\right) + \varphi\left(Tz\right)\}. \end{aligned}$$

Taking the limit as $n \to +\infty$ and using (3.6), (3.7) and (3.12), we have

$$\lim_{n\to+\infty} M\left(x_n, z, d, T, \varphi\right) = d\left(z, Tz\right) + \varphi\left(Tz\right).$$

Since $x_n \to z$ as $n \to +\infty$, from Lemma 2.3 , we conclude that

$$\frac{1}{s} d\left(z, Tz\right) \leq \lim_{n\to+\infty} \sup d\left(Tx_n, Tz\right) \leq s d\left(z, Tz\right).$$

Hence

$$s d\left(z, Tz\right) = s^2 \frac{1}{s} d\left(z, Tz\right) \leq \lim_{n\to+\infty} \sup s^2 d\left(Tx_n, Tz\right),$$

which implies

$$\lim_{n\to+\infty} \sup\left[s d\left(z, Tz\right) + \varphi\left(x_{n+1}\right) + \varphi\left(Tz\right)\right] \leq \lim_{n\to+\infty} \sup\left[s^2 d\left(Tx_n, Tz\right) + \left(x_{n+1}\right) + \varphi\left(Tz\right)\right].$$

Then using (3.1), we have

$$\begin{aligned} \psi\left[s^2 d\left(Tx_n, Tz\right) + \varphi\left(Tx_n\right) + \varphi\left(Tz\right)\right] &= \psi\left[s^2 d\left(x_{n+1}, Tz\right) + \varphi\left(x_{n+1}\right) + \varphi\left(Tz\right)\right] \\ &\leq F(\psi\left[M\left(x_n, z, d, T, \varphi\right)\right], \phi\left[M\left(x_n, z, d, T, \varphi\right)\right]). \end{aligned}$$

Letting $n \to +\infty$ and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned} &\psi\left[\lim_{n\to+\infty} \sup\left(s d\left(z, Tz\right) + \varphi\left(x_{n+1}\right) + \varphi\left(Tz\right)\right)\right] \\ &\leq \psi\left[\lim_{n\to+\infty} \sup\left(s^2 d\left(Tx_n, Tz\right) + \left(x_{n+1}\right) + \varphi\left(Tz\right)\right)\right] \\ &\leq F(\psi\left[\lim_{n\to+\infty} \sup M\left(x_n, z, d, T, \varphi\right)\right], \lim_{n\to+\infty} \phi\left[M\left(x_n, z, d, T, \varphi\right)\right]), \end{aligned}$$

which implies

$$\psi\left[sd\left(z,Tz\right)+\varphi\left(Tz\right)\right]\leq F(\psi\left[d\left(z,Tz\right)+\varphi\left(Tz\right)\right],\phi\left[d\left(z,Tz\right)+\varphi\left(Tz\right)\right]).$$

This holds if and only if $\psi\left(d\left(z,Tz\right)+\varphi\left(Tz\right)\right)=0$ or $\phi\left(d\left(z,Tz\right)+\varphi\left(Tz\right)\right)=0$ and from the property of ψ,ϕ, we have

$$d\left(z,Tz\right)+\varphi\left(Tz\right)=0.$$

Hence $d\left(z,Tz\right)=0$ and so $z=Tz$ and $\varphi\left(Tz\right)=0$. It is a contradiction to the assumption: that T does not have a periodic point. Thus T has a periodic point, say, z of period n. Suppose that the set of fixed points of T is empty. Then we have

$$q>0 \text{ and } d(z,Tz)>0.$$

Since T has a periodic point, $z=T^nz$. Letting $x=T^{n-1}z$ and $y=T^nz$, we obtain

$$\begin{aligned}M\left(T^nz,T^{n-1}z,d,T,\varphi\right)=\max\{&d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right),\\&d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right),d\left(T^nz,TT^nz\right)+\varphi\left(T^nz\right)+\varphi\left(TT^nz\right)\}.\end{aligned}$$

By a similar method to (3.4), we conclude that

$$M\left(T^nz,T^{n-1}z,d,T,\varphi\right)=d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right).$$

From (3.1), we have

$$\begin{aligned}\psi\left[s^2d\left(z,Tz\right)+\varphi\left(T^nz\right)+\varphi\left(T^{n+1}z\right)\right]&=\psi\left[s^2d\left(T^nz,T^{n+1}z\right)+\varphi\left(T^nz\right)+\varphi\left(T^{n+1}z\right)\right]\\&\leq F(\psi\left[d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\&,\phi\left[d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right])\\&\leq\psi\left[s^2d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\&\vdots\\&\leq F(\psi\left[d\left(z,Tz\right)+\varphi\left(z\right)+\varphi\left(Tz\right)\right]\\&,\phi\left[d\left(z,Tz\right)+\varphi\left(z\right)+\varphi\left(Tz\right)\right])\end{aligned}$$

Taking the limit as $n\to+\infty$ and applying the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\psi\left[s^2d\left(z,Tz\right)\right]\leq F(\psi\left[d\left(z,Tz\right)\right],\phi\left[d\left(z,Tz\right)\right]).$$

Hence $d(z,Tz)=0$, which is a contradiction. Thus the set of fixed points of T is non-empty, that is, T has at least one fixed point.

Suppose that $z,u\in X$ are two fixed points of T such that $u\neq z$. Then $Tz=z$ and $Tu=u$.

Letting $x=z$ and $y=u$ in (3.1), we have

$$\psi\left(s^2d\left(Tz,Tu\right)+\varphi\left(Tz\right)+\varphi\left(Tu\right)\right)=\psi\left(s^2d\left(z,u\right)\right)\leq F(\psi\left(M\left(z,u,d,T,\varphi\right)\right),\phi\left(M\left(z,u,d,T,\varphi\right)\right)),$$

where

$$\begin{aligned} M(z,u,d,T,\varphi) &= \max\{d(z,u)+\varphi(z)+\varphi(u), d(z,Tz)+\varphi(z)+\varphi(Tz), d(u,Tu)+\varphi(u)+\varphi(Tu)\} \\ &= d(z,u). \end{aligned}$$

So

$$\psi\left(s^2 d\left(z,u\right)\right) \leq F(\psi\left(d\left(z,u\right)\right), \phi\left(d\left(z,u\right)\right)).$$

This holds if $\phi\left(d\left(z,u\right)\right)=0$ and so we have $(d\left(z,u\right)=0$. Hence $z=u$ and T has a unique fixed point. □

Corollary 3.3. *Let (X,d) be a complete b-rectangular metric space and $T:X\to X$ be a mapping. Suppose that there exists $k\in\,]0,1[$ such that for all $x,y\in X$,*

$$\begin{aligned} &s^2 d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right) \leq \\ &\qquad k\max\{d\left(x,y\right)+\varphi\left(x\right)+\varphi\left(y\right), d\left(x,Tx\right)+\varphi\left(x\right)+\varphi\left(Tx\right), d\left(y,Ty\right)+\varphi\left(y\right)+\varphi\left(Ty\right)\}, \end{aligned}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function. Then T has a unique fixed point.

Proof. It suffices to take Take $\psi(t)=t$ and $F(s,t)=ks$ in Theorem 3.2. □

Corollary 3.4. *Let (X,d) be a complete b-rectangular metric space and $T:X\to X$ be a mapping. Suppose that there exists $\alpha\in\left]0,\frac{1}{2}\right[$ such that for all $x,y\in X$,*

$$s^2 d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right) \leq \alpha\left[\left(d\left(Tx,x\right)+\varphi\left(x\right)+\varphi\left(Tx\right)+d+\varphi\left(y\right)+\varphi\left(Ty\right)+\left(Ty,y\right)\right)\right], \tag{3.13}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function. Then T has a unique fixed point.

Proof. Let $k=2\alpha$. Then $k\in\,]0,1[$. Also, if (3.13) holds, then

$$\begin{aligned} s^2 d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right) &\leq \alpha\left[d\left(Tx,x\right)+\varphi\left(x\right)+\varphi\left(Tx\right)+d+\varphi\left(y\right)+\varphi\left(Ty\right)+\left(Ty,y\right)\right] \\ &= k\frac{\left[d\left(Tx,x\right)+\varphi\left(x\right)+\varphi\left(Tx\right)+d+\varphi\left(y\right)+\varphi\left(Ty\right)+\left(Ty,y\right)\right]}{2} \\ &\leq k\max\{d\left(x,Tx\right)+\varphi\left(x\right)+\varphi\left(Tx\right), d\left(y,Ty\right)+\varphi\left(y\right)+\varphi\left(Ty\right)\} \\ &\leq k\max\{d\left(x,y\right)+\varphi\left(x\right)+\varphi\left(y\right), d\left(x,Tx\right)+\varphi\left(x\right)+\varphi\left(Tx\right), d\left(y,Ty\right)+\varphi\left(y\right)+\varphi\left(Ty\right)\}. \end{aligned}$$

Thus it suffices to apply Corollary 3.3. □

Corollary 3.5. *Let (X,d) be a complete b-rectangular metric space and $T:X\to X$ be a mapping. Suppose that there exists $\lambda\in\left]0,\frac{1}{3}\right[$ such that for all $x,y\in X$,*

$$\begin{aligned} &s^2 d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right) \leq \\ &\quad \lambda\left[d\left(x,y\right)+\varphi\left(x\right)+\varphi\left(y\right)+d\left(Tx,x\right)+\varphi\left(x\right)+\varphi\left(Tx\right)+\varphi\left(y\right)+\varphi\left(Ty\right)+d\left(Ty,y\right)\right], \end{aligned} \tag{3.14}$$

where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a lower semicontinuous function. Then T has a unique fixed point.

Proof. Let $k = 3\lambda$. Then $k \in]0,1[$. Also, if (3.14) holds, then

$$\begin{aligned} & s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty) \\ & \leq \lambda \left[d(x,y) + \varphi(x) + \varphi(y) + d(Tx,x) + \varphi(x) + \varphi(Tx) + \varphi(y) + \varphi(Ty) + d(Ty,y) \right] \\ & = k \frac{\left[d(x,y) + \varphi(x) + \varphi(y) + d(Tx,x) + \varphi(x) + \varphi(Tx) + \varphi(y) + \varphi(Ty) + d(Ty,y) \right]}{3} \\ & \leq k \max\{ d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty) \}. \end{aligned}$$

Thus it suffices to apply Corollary 3.3. ☐

Corollary 3.6. *Let $d(X,d)$ be a complete b-rectangular metric space with parameter $s > 1$ and T be a self mapping on X. If there exists $k \in]0,1[$ such that for all $x,y \in X$,*

$$s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Tx) \leq$$

$$k \left[\beta_1 \left(d(x,y) + \varphi(x) + \varphi(y) \right) + \beta_2 \left(d(Tx,x) + \varphi(x) + \varphi(Tx) \right) + \beta_3 \left(d(Ty,y) + \varphi(y) + \varphi(Ty) \right) \right],$$

where $\beta_i \geq 0$ for $i \in \{1,2,3\}$, $\sum_{i=0}^{i=3} \beta_i \leq 1$, φ is a lower semicontinuous function. Then T has a unique fixed point.

Proof. Take $\psi(t) = t$ and $F(s,t) = ks$. Then it suffices to apply Corollary 3.3. ☐

Corollary 3.7. *Let (X,d) be a complete b-rectangular metric space and $T : X \to X$ be a mapping. Suppose that for all $x,y \in X$,*

$$\psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) \leq \psi(M(x,y,d,T,\varphi)) \log_{a+\phi(M(x,y,d,T,\varphi))} a, \tag{3.15}$$

where

$$M(x,y,d,T,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function also $\psi, \varphi \in \Phi$, $a > 1$. Then T has a unique fixed point.

Proof. It suffices to take $F(s,t) = s \log_{t+a} a$, $a > 1$, in Theorem 3.2. ☐

Corollary 3.8. *Let (X,d) be a complete b-rectangular metric space and $T : X \to X$ be a mapping. Suppose that for all $x,y \in X$,*

$$\psi\left(s^2 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)\right) \leq \frac{\psi(M(x,y,d,T,\varphi))}{1 + \phi(M(x,y,d,T,\varphi))}, \tag{3.16}$$

where

$$M(x,y,d,T,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), d(y,Ty)+\varphi(y)+\varphi(Ty)\}$$

and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function also $\psi \in \Phi, \phi \in \Phi_u$, . Then T has a unique fixed point.

Proof. It suffices to take $F(s,t) = \frac{s}{1+t}$, in Theorem 3.2. ☐

Example 3.9. Let $X = A \cup B$, where $A = \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$ and $B = \left[\frac{1}{2}, 1\right]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x,y) = d(y,x) \ for \ all \ x, y \in X; \\ d(x,y) = 0 \Leftrightarrow y = x \end{cases}$$

and

$$\begin{cases} d\left(0, \dfrac{1}{9}\right) = d\left(\dfrac{1}{5}, \dfrac{1}{16}\right) = 0,1 \\ d\left(0, \dfrac{1}{5}\right) = d\left(\dfrac{1}{5}, \dfrac{1}{9}\right) = 0,5 \\ d\left(0, \dfrac{1}{16}\right) = d\left(\dfrac{1}{9}, \dfrac{1}{16}\right) = 0,05 \\ d(x,y) = (|x-y|)^2 \ \ otherwise. \end{cases}$$

Then (X, d) is a b-rectangular metric space with coefficient $s = 3$. However we have the following:

1) (X, d) is not a metric space, since $d\left(\frac{1}{5}, \frac{1}{9}\right) = 0.5 > 0.15 = d\left(\frac{1}{5}, \frac{1}{16}\right) + d\left(\frac{1}{16}, \frac{1}{9}\right)$.
2) (X, d) is not a b-metric space for s=3, since $d\left(\frac{1}{5}, \frac{1}{9}\right) = 0.5 > 0.45 = 3\left[d\left(\frac{1}{5}, \frac{1}{16}\right) + d\left(\frac{1}{16}, \frac{1}{9}\right)\right]$.
3) (X, d) is not a rectangular metric space, since $d\left(\frac{1}{5}, \frac{1}{9}\right) = 0.5 > 0.25 = d\left(\frac{1}{5}, \frac{1}{16}\right) + d\left(\frac{1}{16}, 0\right) + d\left(0, \frac{1}{9}\right)$.

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{1}{16} \ if \ x \in \left[\dfrac{1}{2}, 1\right] \\ 0 \ if \ x \in A. \end{cases}$$

Then $T(x) \in X$ for all $x \in X$. Let

$$\varphi(t) = \begin{cases} t \ if \ t \in [0,1] \\ 2t \ if \ t > 1 \end{cases}$$

$$\varphi(t) = \begin{cases} \dfrac{t}{16} \ if \ t \in [0,1] \\ \dfrac{t}{8} \ if \ t > 1 \end{cases}$$

and

$$\psi(t) = \frac{3t}{2}$$

and

$$F(s,t) = s - t.$$

Then ψ is an altering distance function and φ is a lower semicontinuous function and ϕ is a lower semicontinuous function such that $\psi(t) = 0 \Leftrightarrow t = 0$, $\phi(t) = 0 \Leftrightarrow t = 0$ and $\varphi(t) = 0 \Leftrightarrow t = 0$ and F C-class function .

Consider the following possibilities:

Case I: $x, y \in \left\{0, \frac{1}{5}, \frac{1}{9}, \frac{1}{16}\right\}$.

Assume that $x \geq y$. Then

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)=\psi\left(9.d(0,0)+\varphi(0)+\varphi()\right)=\psi(0)=0.$$

Also

$$d(x,y)+\varphi(x)+\varphi(y)=d(x,y)+x+y,$$
$$d(x,Tx)+\varphi(x)+\varphi(Tx)=d(x,0)+x,$$
$$d(y,Ty)+\varphi(y)+\varphi(Ty)=d(y,0)+y$$

and

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+x+y,d(x,0)+x,d(y,0)+y\}.$$

Since $x \geq y$, we have

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+x+y,d(x,0)+x\}.$$

If

$$M(x,y,d,T,\varphi)=d(x,y)+x+y \geq \frac{1}{20},$$

then

$$\begin{aligned}\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right) &= \psi\left(d(x,y)+x+y\right)-\phi\left(d(x,y)+x+y\right)\\ &= \frac{23}{16}\left(d(x,y)+x+y\right)\geq 0\end{aligned}$$

and so

$$0=\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If

$$M(x,y,d,T,\varphi)=d(x,0)+x \geq \frac{1}{20},$$

then

$$\begin{aligned}\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) &\leq \frac{3}{2}\cdot\frac{1}{20}-\frac{1}{20}\cdot\frac{1}{16}=\frac{23}{320}\\ &\leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).\end{aligned}$$

Assume that $x < y$. Then

$$M(x,y,d,T,\varphi)=\max\{d(x,y)+x+y,d(y,0)+y\}.$$

If

$$M(x,y,d,T,\varphi)=d(x,y)+x+y \geq \frac{1}{20}+\frac{1}{16}=\frac{9}{80},$$

then

$$\begin{aligned}\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right) &\leq \frac{3}{2}\cdot\frac{9}{80}-\frac{1}{16}\cdot\frac{9}{80}=\frac{207}{1280}\\ &\leq \psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).\end{aligned}$$

If

$$M(x,y,d,T,\varphi)=d(y,0)+y,$$

then

$$d(y,0)+y\geq\frac{9}{80},$$

since $x<y$ and $0<y$. Thus

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case II: $x\in\left\{0,\frac{1}{5},\frac{1}{9},\frac{1}{16}\right\}$ and $y\in\left[\frac{1}{2},1\right]$. This implies $x<y$. Then

$$\psi\left(s^2\left(d\left(Tx,Ty\right)+\varphi(Tx)+\varphi(Ty)\right)\right)=\frac{3}{2}\left[9d\left(\frac{1}{16},0\right)+\frac{1}{16}\right]=\frac{123}{160}.$$

Also

$$d(x,y)+\varphi(x)+\varphi(y)=(x-y)^2+x+y,$$
$$d(x,Tx)+\varphi(x)+\varphi(Tx)=d(x,0)+x$$
$$d(y,Ty)+\varphi(y)+\varphi(Ty)=d\left(y,\frac{1}{16}\right)+y+\frac{1}{16},$$

and

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d(x,0)+x,d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}$$

Since $x<y$, we have

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}.$$

If

$$M(x,y,d,T,\varphi)=d(x-y)^2+x+y\geq\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{5}\right)^2=\frac{59}{100},$$

then

$$\psi\left(M(x,y,d,T,\varphi)\right)-\phi\left(M(x,y,d,T,\varphi)\right)=\frac{23}{16}\left(d(x,y)^2+x+y\right)\geq\frac{23}{16}\cdot\frac{59}{100}=\frac{1357}{1600}\geq\frac{123}{160}.$$

Then

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If

$$M(x,y,d,T,\varphi)=d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\geq\frac{1}{2}+\frac{1}{16}+\left(\frac{1}{2}-\frac{1}{16}\right)^2=\frac{193}{256},$$

then

$$\psi\left(M(x,y,d,T,\varphi)\right)-\phi\left(M(x,y,d,T,\varphi)\right)=\frac{23}{16}d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\geq\frac{23}{16}\cdot\frac{193}{256}=\frac{4439}{4096}\geq\frac{123}{160}.$$

Then

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case III: $y\in\left\{0,\frac{1}{5},\frac{1}{9},\frac{1}{16}\right\}$ and $x\in\left[\frac{1}{2},1\right]$.

By a similar method to Case II, we deduce that

$$\psi\left(s^2d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Case IV: $x,y\in\left[\frac{1}{2},1\right]$.

If $x \geq y$, then

$$\psi\left(s^2 d(Tx,Ty)+\varphi(Tx)+\varphi(Ty)\right)=\psi\left(9d\left(\frac{1}{16},\frac{1}{16}\right)+\varphi\left(\frac{1}{16}\right)+\varphi\left(\frac{1}{16}\right)\right)=\frac{3}{16}.$$

Also

$$d(x,y)+\varphi(x)+\varphi(y)=(x-y)^2+x+y,$$

$$d(x,Tx)+\varphi(x)+\varphi(Tx)=d\left(x,\frac{1}{16}\right)+x+\frac{1}{16},$$

$$d(y,Ty)+\varphi(y)+\varphi(Ty)=d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}$$

and

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d\left(x,\frac{1}{16}\right)+x+\frac{1}{16},d\left(y,\frac{1}{16}\right)+y+\frac{1}{16}\}.$$

Since $x \geq y$, we have

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d\left(x,\frac{1}{16}\right)+x+\frac{1}{16}\}.$$

If

$$M(x,y,d,T,\varphi)=(x-y)^2+x+y\geq 1,$$

then

$$\frac{3}{16}\leq\frac{23}{16}\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If

$$M(x,y,d,T,\varphi)=d\left(x,\frac{1}{16}\right)+x+\frac{1}{16}\geq d\left(\frac{1}{2},\frac{1}{16}\right)+\frac{1}{2}+\frac{1}{16}=\frac{193}{256},$$

then

$$\frac{3}{16}\leq\frac{23}{16}\cdot\frac{193}{256}=\frac{4439}{4096}\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

If $x,y\in A$ and $x<y$, then

$$M(x,y,d,T,\varphi)=\max\{(x-y)^2+x+y,d\left(y,\frac{1}{16}\right)+y\}.$$

By a similar method to the condition $x \geq y$, we have

$$\frac{3}{16}\leq\frac{4439}{4096}\leq\psi\left(M(x,y,d,T,\varphi)-\phi\left(M(x,y,d,T,\varphi)\right)\right).$$

Hence

$$\psi\left(s^2 d\left(Tx,Ty\right)+\varphi\left(Tx\right)+\varphi\left(Ty\right)\right)\leq F(\psi\left(M\left(x,y,d,T,\varphi\right)\right),\phi\left(M\left(x,y,d,T,\varphi\right)\right)),\qquad(3.17)$$

Thus all the conditions of Theorem 3.2 are satisfied and 0 is the unique fixed point of T.

Definition 3.10. [2]A tripled (ψ,ϕ,F) where $\psi\in\Psi$, $\phi\in\Phi_u$ and $F\in\mathcal{C}$ is say to be monotone if for any $x,y\in[0,\infty)$

$$x\leqslant y\Longrightarrow F(\psi(x),\phi(x))\leqslant F(\psi(y),\phi(y)).$$

Example 3.11. let $F(s,t) = s - t, \phi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ x^2, & \text{if x} > 1 \end{cases},$$

then (ψ, ϕ, F) is monotone.

Example 3.12. let $F(s,t) = s - t, \phi(x) = x^2$

$$\psi(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1, \\ \sqrt{x} & \text{if x} > 1 \end{cases},$$

then (ψ, ϕ, F) is not monotone. Indeed, let $x \in [0,1]$ and $y > 1$ so, $x \leqslant y$ Therefore,

$$F(\psi(x), \phi(x)) = F(x^2, x^2) = 0$$

and

$$F(\psi(y), \phi(y)) = F(\sqrt{x}, x^2) = \sqrt{x} - x^2 < 0.$$

Definition 3.13. Let (X, d) be a complete b-rectangular metric space and
$T : X \to X$. Also let $\varphi : X \to R^+$ be a lower semicontinuous function. Then T is called a convex C class $-F-$generalized weakly contractive mapping if it satisfies the following condition:

$$\psi\left(s^2 d\left(Tx, Ty\right) + \varphi\left(Tx\right) + \varphi\left(Ty\right)\right) \le F(\psi\left(M\left(x, y, d, T, \varphi\right)\right), \phi\left(M\left(x, y, d, T, \varphi\right)\right)), \quad (3.18)$$

where

$$M(x, y, d, T, \varphi) = \frac{a[d(x,y) + \varphi(x) + \varphi(y)] + b[d(x,Tx) + \varphi(x) + \varphi(Tx)] + c[d(y,Ty) + \varphi(y) + \varphi(Ty)]}{a + b + c}$$

for all $x, y \in X$,$a, b, c \in [0, \infty)$,$a + b + c > 0$ and ,(ψ, ϕ, F) is not monotone, $psi \in \Phi, \phi \in \Phi_u$, $F \in \mathcal{C}$.

Theorem 3.14. *Let X be a complete b-rectangular metric space with parameter $s \ge 1$. If T is a is called a convex C class $-F-$generalized weakly contractive mapping, then T has a unique fixed point $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

Proof. Let $x_0 \in X$ be an arbitrary point in X. Then we define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = 0$, then x_{n_0} is a fixed point of T.

Next, we assume that $x_n \neq x_{n+1}$.

We claim that

$$\lim_{n \to +\infty} d\left(x_n, x_{n+1}\right) = 0$$

and

$$\lim_{n \to +\infty} d\left(x_n, x_{n+2}\right) = 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.1) for all $n \in \mathbb{N}$, we have

$$\psi\left(s^2 d\left(Tx_{n-1}, Tx_n\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(Tx_n\right)\right) \le F(\psi\left(M\left(x_{n-1}, x_n, d, T, \varphi\right)\right), \phi\left(M\left(x_{n-1}, x_n, d, T, \varphi\right)\right)), \tag{3.19}$$

where

$$M\left(x_{n-1}, x_n, d, T, \varphi\right) = \frac{1}{a+b+c}\{[ad(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] + b[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] + c[d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n)]\}$$

$$= \frac{1}{a+b+c}\{(a+b)[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] + c[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1})]\}.$$

then we have

$$\begin{aligned}
&\psi\left(d\left(Tx_{n-1}, Tx_n\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(x_{n+1}\right)\right) \\
&= \psi\left(d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \\
&\le \psi\left(s^2 d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \\
&\le F(\psi\begin{pmatrix} \frac{1}{a+b+c}\{(a+b)[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] \\ +c[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1})]\} \end{pmatrix} \\
&, \phi\begin{pmatrix} \frac{1}{a+b+c}\{(a+b)[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] \\ +c[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1})]\} \end{pmatrix}) \\
&\le \psi\begin{pmatrix} \frac{1}{a+b+c}\{(a+b)[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] \\ +c[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1})]\} \end{pmatrix},
\end{aligned}$$

which implies

$$\phi\left(d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right) \le \begin{pmatrix} \frac{1}{a+b+c}\{(a+b)[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] \\ +c[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(Tx_{n+1})]\} \end{pmatrix}$$

and so

$$d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) \le d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)$$

Hence the sequence $\{d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\}_{n\in\mathbb{N}}$ is nonincreasing.
Hence $d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) \to r$ as $n \to +\infty$ for some $r \ge 0$. Assume $r > 0$ and letting $n \to +\infty$ in above inequality and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned}
\psi\left(r\right) \le \psi\left(s^2 r\right) &\le F(\psi\left(r\right), \liminf_{n\to\infty} \phi\left(d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right)\right)) \\
&\le F(\psi\left(r\right), \phi\left(r\right)).
\end{aligned}$$

It follows that $\psi\left(r\right) = 0$, or , $\phi\left(r\right) = 0$, hence we have $r = 0$ and consequently, $\lim_{n\to+\infty} d\left(x_n, x_{n+1}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+1}\right) = 0$. So

$$\lim_{n\to+\infty} d\left(x_n, x_{n+1}\right) = 0, \tag{3.20}$$

$$\lim_{n\to+\infty} \varphi\left(x_n\right) = \lim_{n\to+\infty} \varphi\left(x_{n+1}\right) = 0. \tag{3.21}$$

Now, we shall prove that T has a periodic point. Suppose that it is not the case. Then $x_n \neq x_m$ for all $n, m \in \mathbb{N},\ n \neq m$.

In (3.1), letting $x = x_{n-1}$ and $y = x_{n+1}$, we have

$$\begin{aligned}&\psi\left(s^2 d\left(Tx_{n-1}, Tx_{n+1}\right) + \varphi\left(Tx_{n-1}\right) + \varphi\left(Tx_{n+1}\right)\right)\\&\quad\leq F(\psi\left(M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right)\right), \phi\left(M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right)\right)),\end{aligned}$$

where

$$\begin{aligned}M\left(x_{n-1}, x_{n+1}, d, T, \varphi\right) &= \frac{1}{a+b+c}\{a[d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1})]+\\&\quad b[d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)] + c[d(x_{n+1}, x_{n+2}) + \varphi(x_{n+1}) + \varphi(x_{n+2})]\}\\&= \frac{1}{a+b+c}\{a[d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1})], d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}.\end{aligned} \tag{3.22}$$

So we get

$$\begin{aligned}&\psi\left(d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right) \leq \psi\left(s^2 d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)\right)\\&\quad\leq F(\psi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right)\\&\quad, \phi\left(\max\{d(x_{n-1}, x_{n+1}) + \varphi(x_{n-1}) + \varphi(x_{n+1}), d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)\}\right)).\end{aligned} \tag{3.23}$$

Take $a_n = d\left(x_n, x_{n+2}\right) + \varphi\left(x_n\right) + \varphi\left(x_{n+2}\right)$ and $b_n = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})$.
Then by (3.23), one can write

$$\begin{aligned}\psi\left(a_n\right) &\leq F(\psi\left(\max\left(a_{n-1}, b_{n-1}\right)\right), \phi\left(\max\left(a_{n-1}, b_{n-1}\right)\right))\\&\leq \psi\left(\max\left(a_{n-1}, b_{n-1}\right)\right).\end{aligned}$$

Since ψ is increasing, we get

$$a_n \leq \max\left\{a_{n-1}, b_{n-1}\right\}.$$

By (3.3), we have

$$b_n \leq b_{n-1} \leq \max\left\{a_{n-1}, b_{n-1}\right\},$$

which implies that

$$\max\left\{a_n, b_n\right\} \leq \max\left\{a_{n-1}, b_{n-1}\right\},\ \forall n \in \mathbb{N}.$$

Therefore, the sequence $\max\left\{a_{n-1}, b_{n-1}\right\}_{n\in\ \mathbb{N}}$ is a nonnegative decreasing sequence of real numbers. Thus there exists $\lambda \geq 0$ such that

$$\lim_{n\to+\infty} \max\left\{a_n, b_n\right\} = \lambda.$$

Assume that $\lambda > 0$. By (3.6), it is obvious that

$$\lambda = \lim_{n\to+\infty} \sup a_n = \lim_{n\to+\infty} \sup \max\left\{a_n, b_n\right\} = \lim_{n\to+\infty} \max\left\{a_n, b_n\right\}. \tag{3.24}$$

Taking $\limsup_n \to +\infty$ in (3.23), using (3.24) and using the properties of ψ and ϕ, we obtain

$$\begin{aligned}\psi(\lambda) &= \psi\left(\limsup_{n\to+\infty} a_n\right)\\ &= \limsup_{n\to+\infty} \psi\left(a_n\right)\\ &\leq \limsup_{n\to+\infty} \psi\left(\max\left\{a_n, b_n\right\}\right) - \liminf_{n\to+\infty} \phi\left(\max\left\{a_n, b_n\right\}\right)\\ &\leq F(\psi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right), \phi\left(\lim_{n\to+\infty} \max\left\{a_n, b_n\right\}\right))\\ &= F(\psi(\lambda), \phi(\lambda)),\end{aligned}$$

which implies that $\psi(r) = 0$, or $, \phi(r) = 0$, a contradiction. Thus, from (3.24),

$$\limsup_{n\to+\infty} a_n = 0$$

and hence

$$\lim_{n\to+\infty} d\left(x_n, x_{n+2}\right) = 0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n,m\to+\infty} d\left(x_n, x_m\right) = 0$ for all $n, m \in \mathbb{N}$. Suppose to the contrary.

From (3.1) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\begin{aligned}M(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi) = \max\{& d(x_{m_{(k)}}, x_{n_{(k)}}) + \varphi(x_{m_{(k)}}) + \varphi(x_{m_{(k)}}),\\ & d(x_{m_{(k)}}, x_{m_{(k)}+1}) + \varphi(x_{m_{(k)}}) + \varphi(x_{m_{(k)}+1}), d(x_{n_{(k)}}, x_{n_{(k)}+1}) + \varphi(x_{n_{(k)}}) + \varphi(x_{n_{(k)}+1})\}.\end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.6), (3.7) and (iii) of Lemma 2.4, we have

$$\lim_{k\to+\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}, d, T, \varphi\right) \leq s\varepsilon. \tag{3.25}$$

Now letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.1), we have

$$\begin{aligned}&\psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) + \varphi\left(m_{(k)} + 1\right) + \varphi\left(n_{(k)} + 1\right)\right]\\ &\qquad \leq F(\psi\left[d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right) + \varphi\left(m_{(k)} + 1\right) + \varphi\left(n_{(k)} + 1\right)\right]\\ &\qquad\qquad , \phi\left[d\left(x_{m_{(k)}}, x_{n_{(k)}+1}\right) + \varphi\left(m_{(k)}\right) + \varphi\left(n_{(k)}\right)\right]).\end{aligned}$$

Letting $k \to +\infty$, using (3.6), (3.7), (3.25), and applying the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\lim_{k\to+\infty} \psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right)\right] \leq F(\psi(s\varepsilon), \phi(s\varepsilon)).$$

Using (3.25) and (iv) of Lemma 2.4, we obtain

$$\psi(s\varepsilon) = \psi\left(s^2 \frac{\varepsilon}{s}\right) \leq \lim_{k\to+\infty} \sup \psi\left[s^2 d\left(x_{m_{(k)}+1}, x_{n_{(k)}+1}\right)\right] \leq F(\psi(s\varepsilon), \phi(s\varepsilon)).$$

This is a contradiction. Thus

$$\lim_{n,m\to+\infty} d\left(x_m, x_n\right) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d), there exists $z \in X$ such that

$$\lim_{n\to+\infty} d(x_n, z) = 0.$$

Since φ is lower semicontinuous, we get

$$\varphi(z) \leq \liminf_{n\to+\infty} \varphi(x_n) \leq \lim_{n\to+\infty} \varphi(x_n) = 0,$$

which implies

$$\varphi(z) \quad = \quad 0. \tag{3.26}$$

Now, putting $x = x_n$ and $y = z$ in (3.1), we have

$$\begin{aligned} M(x_n, z, d, T, \varphi) \quad = \quad & \max\{d(x_n, z) + \varphi(x_n) + \varphi(z), \\ & d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(z, Tz) + \varphi(z) + \varphi(Tz)\}. \end{aligned}$$

Taking the limit as $n \to +\infty$ and using (3.6), (3.7) and (3.26), we have

$$\lim_{n\to+\infty} M(x_n, z, d, T, \varphi) = d(z, Tz) + \varphi(Tz).$$

Since $x_n \to z$ as $n \to +\infty$, from Lemma 2.3 , we conclude that

$$\frac{1}{s} d(z, Tz) \leq \lim_{n\to+\infty} \sup d(Tx_n, Tz) \leq s d(z, Tz).$$

Hence

$$s d(z, Tz) = s^2 \frac{1}{s} d(z, Tz) \leq \lim_{n\to+\infty} \sup s^2 d(Tx_n, Tz),$$

which implies

$$\lim_{n\to+\infty} \sup\left[s d(z, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right] \leq \lim_{n\to+\infty} \sup\left[s^2 d(Tx_n, Tz) + (x_{n+1}) + \varphi(Tz)\right].$$

Then using (3.1), we have

$$\begin{aligned} \psi\left[s^2 d(Tx_n, Tz) + \varphi(Tx_n) + \varphi(Tz)\right] &= \psi\left[s^2 d(x_{n+1}, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right] \\ &\leq F(\psi\left[M(x_n, z, d, T, \varphi)\right], \phi\left[M(x_n, z, d, T, \varphi)\right]). \end{aligned}$$

Letting $n \to +\infty$ and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned} &\psi\left[\lim_{n\to+\infty} \sup\left(s d(z, Tz) + \varphi(x_{n+1}) + \varphi(Tz)\right)\right] \\ &\qquad \leq \psi\left[\lim_{n\to+\infty} \sup\left(s^2 d(Tx_n, Tz) + (x_{n+1}) + \varphi(Tz)\right)\right] \\ &\qquad\qquad \leq F(\psi\left[\lim_{n\to+\infty} \sup M(x_n, z, d, T, \varphi)\right], \lim_{n\to+\infty} \phi\left[M(x_n, z, d, T, \varphi)\right]), \end{aligned}$$

which implies

$$\psi\left[s d(z, Tz) + \varphi(Tz)\right] \leq F(\psi\left[d(z, Tz) + \varphi(Tz)\right], \phi\left[d(z, Tz) + \varphi(Tz)\right]).$$

This holds if and only if $\psi(d(z, Tz) + \varphi(Tz)) = 0$ or $\phi(d(z, Tz) + \varphi(Tz)) = 0$ and from the property of ψ, ϕ, we have

$$d(z, Tz) + \varphi(Tz) = 0.$$

Hence $d(z,Tz)=0$ and so $z=Tz$ and $\varphi(Tz)=0$. It is a contradiction to the assumption: that T does not have a periodic point. Thus T has a periodic point, say, z of period n. Suppose that the set of fixed points of T is empty. Then we have

$$q>0 \text{ and } d(z,Tz)>0.$$

Since T has a periodic point, $z=T^nz$. Letting $x=T^{n-1}z$ and $y=T^nz$, we obtain

$$\begin{aligned}M\left(T^nz,T^{n-1}z,d,T,\varphi\right)&=\max\{d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right),\\&\quad d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right),d\left(T^nz,TT^nz\right)+\varphi\left(T^nz\right)+\varphi\left(TT^nz\right)\}.\end{aligned}$$

By a similar method to (3.4), we conclude that

$$M\left(T^nz,T^{n-1}z,d,T,\varphi\right)=d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right).$$

From (3.1), we have

$$\begin{aligned}\psi\left[s^2d(z,Tz)+\varphi\left(T^nz\right)+\varphi\left(T^{n+1}z\right)\right]&=\psi\left[s^2d\left(T^nz,T^{n+1}z\right)+\varphi\left(T^nz\right)+\varphi\left(T^{n+1}z\right)\right]\\&\leq F(\psi\left[d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\&,\phi\left[d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right])\\&\leq\psi\left[s^2d\left(T^{n-1}z,T^nz\right)+\varphi\left(T^{n-1}z\right)+\varphi\left(T^nz\right)\right]\\&\vdots\\&\leq F(\psi\left[d(z,Tz)+\varphi(z)+\varphi(Tz)\right]\\&,\phi\left[d(z,Tz)+\varphi(z)+\varphi(Tz)\right])\end{aligned}$$

Taking the limit as $n\to+\infty$ and applying the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\psi\left[s^2d(z,Tz)\right]\leq F(\psi\left[d(z,Tz)\right],\phi\left[d(z,Tz)\right]).$$

Hence $d(z,Tz)=0$, which is a contradiction. Thus the set of fixed points of T is non-empty, that is, T has at least one fixed point.

Suppose that $z,u\in X$ are two fixed points of T such that $u\neq z$. Then $Tz=z$ and $Tu=u$.

Letting $x=z$ and $y=u$ in (3.1), we have

$$\psi\left(s^2d(Tz,Tu)+\varphi(Tz)+\varphi(Tu)\right)=\psi\left(s^2d(z,u)\right)\leq F(\psi\left(M(z,u,d,T,\varphi)\right),\phi\left(M(z,u,d,T,\varphi)\right)),$$

where

$$\begin{aligned}M(z,u,d,T,\varphi)&=\max\{d(z,u)+\varphi(z)+\varphi(u),d(z,Tz)+\varphi(z)+\varphi(Tz),d(u,Tu)+\varphi(u)+\varphi(Tu)\}\\&=d(z,u).\end{aligned}$$

So

$$\psi\left(s^2d(z,u)\right)\leq F(\psi\left(d(z,u)\right),\phi\left(d(z,u)\right)).$$

This holds if $\phi\left(d\left(z,u\right)\right)=0$ and so we have $\left(d\left(z,u\right)=0\right.$. Hence $z=u$ and T has a unique fixed point. □

4. Conclusion

In this paper, inspired by the concept of generalized weakly contractive mappings in metric spaces, we introduced the concept of C-class function for generalized weakly contractive mappings in rectangular b-metric spaces to study the existence of fixed point for the mappings in this spaces. Furthermore, we provided some useful examples.

References

[1] Ya.I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, in: I. Goldberg, Yu. Lyubich (Eds.), New Results in Operator Theory, Adv. Appl., 98 (1997), 7–22.

[2] A.H. Ansari, Note on ϕ-ψ-contractive type mappings and related fixed point, Proc. 2nd Reg. Conf. Math. Appl., Payame Noor Univ., 2014, 377–380.

[3] S. Banach, Sur les operations dans les ensembles abstraits et leur applications integrales, Fund. Math. 3 (1922), 133–181.

[4] A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen 57 (2000), 31–37.

[5] S. Cho, Fixed point theorems for generalized weakly contractive mappings in metric spaces with applications, Fixed Point Theory Appl. 2018 (2018), Paper No. 3.

[6] B.S. Choudhury, P. Konar, B.E. Rhoades, N. Metiya, Fixed point theorems for generalized weakly contractive mappings, Nonlinear Anal. 74 (2011), 2116–2126.

[7] R. George, S. Radenovic, K.P. Reshma, S. Shukla, Rectangular b-metric spaces and contraction principle, J. Nonlinear Sci. Appl. 8 (2015), 1005–1013.

[8] A. Kari, M. Rossafi, E. Marhrani, M. Aamri, θ-ϕ-contraction on (α,η)-complete rectangular b-metric spaces, Int. J. Math. Math. Sci. 2020 (2020), Article ID 5689458.

[9] A. Kari, M. Rossafi, E. Marhrani, M. Aamri, New fixed point theorems for θ-ϕ-contraction on complete rectangular b-metric spaces, Abstr. Appl. Anal. 2020 (2020), Article ID 8833214.

[10] A. Kari, M. Rossafi, E. Marhrani, M. Aamri, Fixed-point theorem for nonlinear F-contraction via w-distance, Adv. Math. Phys. 2020 (2020), Article ID 6617517.

[11] M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distances between the points, Bull. Aust. Math. Soc. 30 (1984), 1–9.

[12] M. Rossafi, A. Kari, Fixed points for weakly contractive mappings in rectangular b-metric spaces, Int. J. Nonlinear Anal. Appl. 14 (2023), 763–783.

[13] J.R. Roshan, V. Parvaneh, Z. Kadelburg, N. Hussain, New fixed point results in b-rectangular metric spaces, Nonlinear Anal. Model. Control 21 (2016), 614–634.