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Multivalued Contractive-Type Extensions with Stability and Well-Posedness in Cone b-Metric Spaces under (λ, s)-Convexity and Applications

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Abstract. We introduce new extensions of multivalued contractive fixed point results in complete cone b-metric spaces endowed with a normal cone structure. By developing a λ-iterative scheme combined with an approximate Hausdorff selection technique, we establish original Nadler-type and Berinde-type results with explicit convergence estimates depending on the b-metric coefficient s and the iteration parameter λ.

In addition, we prove a Berinde-type fixed point theorem for weak multivalued contractions, together with stability and well-posedness results under the sharp condition $s\delta < 1$. The stability theorem provides quantitative bounds for perturbations of contractive multivalued operators, while the well-posedness result guarantees convergence of approximate solutions to the unique fixed point.

Applications to vector optimization and Nash-type equilibrium problems are presented within the framework of (λ, s)-convexity. The results extend classical fixed point theorems of Nadler and Berinde to the ordered and relaxed triangle inequality setting of cone b-metric spaces.

1. Introduction

Fixed point theory has played a central role in nonlinear analysis since the seminal contraction principle of Banach [1]. Over the past decades, significant effort has been devoted to extending Banach's principle to broader settings where classical metric assumptions are either relaxed or enriched with additional structures.

Two important directions of generalization are cone metric spaces and b-metric spaces. Cone metric spaces, introduced by Huang and Zhang [2], replace real-valued distances by vector-valued distances taking values in an ordered Banach space. This framework allows one to incorporate order-theoretic information directly into the metric structure. On the other hand, b-metric spaces, initiated by Bakhtin [3] and further developed by Czerwik [4], weaken the triangle inequality by introducing a coefficient $s \geq 1$. This relaxation has proven particularly useful in applications where strict metric behavior is too restrictive.

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The combination of these two ideas leads naturally to *cone b-metric spaces*, where distances are vector-valued and the triangle inequality holds up to a multiplicative constant. Such spaces provide a flexible analytical setting that simultaneously captures nonlinear scaling effects and partial order structures. Their structural properties and equivalence aspects have been investigated, for example, by Karapınar and Du [5]. Moreover, multivalued mappings in b-metric spaces have been studied extensively (see, e.g., [6, 7]), but the interaction between multivalued contractions, cone order structures, and relaxed triangle inequalities remains technically delicate.

The analysis of multivalued mappings in cone b-metric spaces presents several challenges. First, the Hausdorff distance now takes values in an ordered Banach space, which requires careful handling of infima and order relations. Second, the b-metric constant s propagates through iterative estimates and affects stability thresholds. Third, the construction of convergent sequences for multivalued operators demands a precise selection mechanism, since direct Hausdorff comparisons are generally insufficient without an approximate choice argument.

To address these issues, we employ a λ-iteration scheme, previously introduced in [8], and adapt it to the multivalued setting as follows:

$$x_{n+1} = \frac{(\lambda - 1)x_n + z_n}{\lambda}, \qquad z_n \in T(x_n), \quad \lambda > 1.$$

This iteration blends the current iterate with a selected image point and introduces a parameter λ that allows quantitative control of the recursive estimates. In the present work, we combine this scheme with an approximate Hausdorff selection procedure in order to derive rigorous contractive inequalities within the cone b-metric framework. The main contributions of the paper are the following:

- We prove a Nadler-type fixed point theorem for multivalued contractions in complete cone b-metric spaces, obtaining explicit convergence estimates depending on the parameters s, δ, and λ.
- We establish a Berinde-type fixed point theorem for weak multivalued contractions, where the convergence constant is explicitly computed and shown to be sharp under the condition
$$A = s\left(1 - \frac{1}{\lambda}\right) + \frac{s}{\lambda}(\delta + Ls), \qquad B = s^2 L,$$
and
$$q = \frac{A}{1 - B} < 1.$$
- We provide quantitative stability results for perturbations of multivalued contractive operators and prove well-posedness of the fixed point problem under the natural threshold condition $s\delta < 1$.
- Finally, we illustrate the applicability of the developed theory to vector optimization and Nash-type equilibrium problems formulated within the framework of (λ, s)-convex structures.

Our results extend classical fixed point theorems of Nadler [9] and Berinde [10] to the ordered and relaxed setting of cone b-metric spaces. The analysis clarifies how the interaction between the order structure and the b-metric coefficient determines convergence behavior, stability bounds, and well-posedness thresholds. In this way, the paper contributes to a unified treatment of multivalued fixed point theory in ordered generalized metric spaces.

2. Preliminaries

We begin by recalling essential structures. A *cone* generalizes the notion of nonnegative vectors to ordered Banach spaces:

Definition 2.1 (Cone and Order Structure)**.** *A subset P of a* Banach lattice *E is a* cone *if:*

- *(i) P is closed, non-empty, and int$(P) \neq \varnothing$;*
- *(ii) $a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;*
- *(iii) $P \cap (-P) = \{0\}$;*
- *(iv) P is* normal*: there exists a constant $K > 0$ such that for all $x, y \in E$, if $0 \preceq x \preceq y$ then $\|x\| \leq K\|y\|$.*

P induces a partial order $\preceq$ on E: $x \preceq y$ if and only if $y - x \in P$. We say $x \ll y$ if $y - x \in$ int(P).

Throughout the paper, E is assumed to be a Banach lattice and $P = E_+$ is its positive cone, so that for any $u, v \in E$ the infimum $u \wedge v$ exists; in particular, $\min\{u, v\} = u \wedge v$ is well-defined.

Additionally, we assume $(E, \preceq)$ is *Dedekind complete* (every nonempty subset of E bounded below has an infimum), so that all infima in Definitions below are well-defined.

Remark 2.2. *Cones allow vector-valued distances where positivity is defined via the partial order. For example, in $\mathbb{R}^2$ with $P = \mathbb{R}^2_+$, the distance $d(x, y) = (|x_1 - y_1|, |x_2 - y_2|)$ takes values in P. The normality condition is crucial for ensuring that the order structure is compatible with the topology induced by the norm.*

Also related is Karapınar and Du's result on cone b-metric spaces, showing equivalences between cone b-metric frameworks and classical b-metrics through scalarization techniques [5].

Definition 2.3 (b-Cone Metric)**.** *Let X be nonempty, P a cone in Banach space E, and $s \geq 1$. A function $d_b : X \times X \to P$ is a* b-cone metric *if:*

- *(i) $\forall x, y \in X : d_b(x, y) = 0 \Leftrightarrow x = y$;*
- *(ii) $\forall x, y \in X : d_b(x, y) = d_b(y, x)$;*
- *(iii) $\forall x, y, z \in X : d_b(x, z) \preceq s[d_b(x, y) + d_b(y, z)]$.*

The tuple (X, E, P, d_b, s) is a b-cone metric space.

Assumption 2.4. *Throughout the sequel, we assume that X is a nonempty closed subset of a real Banach space $(V, \|\cdot\|)$. In particular, $X \subset V$ and the vector operations in V induce convex* *combinations in X. This assumption ensures that the λ-iteration scheme*

$$x_{n+1} = \frac{(\lambda - 1)x_n + z_n}{\lambda}, \quad z_n \in T(x_n),$$

is well-defined.

Example 2.5 (ℓ^p b-Cone Metric)**.** *Let $X = \ell^p(\mathbb{N})$ for $1 \leq p < \infty$, $E = \mathbb{R}^2$ with $P = \mathbb{R}^2_+$. Define:*

$$d_b(f,g) = \left(\|f-g\|_p, \|f-g\|_p^2\right).$$

This forms a b-cone metric space with $s = 2$ (and hence with any $s \geq 2$). The triangle inequality follows from the embedding:

$$\begin{aligned} \|f-h\|_p &\leq \left(\|f-g\|_p^p + \|g-h\|_p^p\right)^{1/p} \\ &\leq 2\left(\|f-g\|_p + \|g-h\|_p\right). \end{aligned}$$

Thus $d_b(f,h) \preceq s\,[d_b(f,g) + d_b(g,h)]$ where the addition is component-wise.

The concept of b-metric spaces has been significantly explored in the literature. Notably, Kutbi, Karapınar, and Ahmad studied fixed point results for multivalued mappings in b-metric spaces [6]. Karapınar also provided a concise survey summarizing many fixed point results in the context of b-metric spaces [7].

Classical convexity requires modification in cone b-metric spaces due to the relaxed triangle inequality:

Definition 2.6 ((λ, s)-Convexity)**.** *A subset $K \subseteq X$ is* (λ, s)-convex *if for any $x, y \in K$ and $\alpha \in [0,1]$, there exists $z \in K$ such that:*

$$\begin{aligned} d_b(z,x) &\preceq (1-\alpha)\, d_b(x,y) + \lambda s\, d_b(z,y), \\ d_b(z,y) &\preceq \alpha\, d_b(x,y) + \lambda s\, d_b(z,x). \end{aligned}$$

To guarantee that the above iteration is well-defined, we assume throughout the paper that X is a closed subset of a real Banach space V.

For multivalued mappings, we extend the Hausdorff distance to b-cone metrics:

Definition 2.7 (Hausdorff b-Cone Metric)**.** *For nonempty closed $A, B \subseteq X$, define:*

$$H_{d_b}(A,B) = \inf\left\{ r \in P : \begin{array}{l} A \subseteq \mathscr{B}(B,r), \\ B \subseteq \mathscr{B}(A,r) \end{array} \right\},$$

where $\mathscr{B}(A,r) = \{x \in X : \inf_{a \in A} d_b(x,a) \preceq r\}$.

Lemma 2.8 (Hausdorff Properties)**.** *In any b-cone metric space (X, d_b):*

(1) $H_{d_b}(A,B) = 0 \Leftrightarrow A = B$;
(2) $H_{d_b}(A,B) = H_{d_b}(B,A)$;
(3) $H_{d_b}(A,C) \preceq s[H_{d_b}(A,B) + H_{d_b}(B,C)]$.

Proof. (i) and (ii) follow directly from symmetry. For (iii): Given $\epsilon \succ 0$ (i.e., $\epsilon \in \mathrm{int}(P)$), choose $r_1, r_2 \in P$ such that:

$$H_{d_b}(A,B) \preceq r_1, \quad H_{d_b}(B,C) \preceq r_2,$$
$$A \subseteq \mathscr{B}(B,r_1), \quad B \subseteq \mathscr{B}(A,r_1),$$
$$B \subseteq \mathscr{B}(C,r_2), \quad C \subseteq \mathscr{B}(B,r_2).$$

For any $a \in A$, there exists $b \in B$ with $d_b(a,b) \preceq r_1$. For this b, there exists $c \in C$ with $d_b(b,c) \preceq r_2$. Then:

$$d_b(a,c) \preceq s[d_b(a,b) + d_b(b,c)] \preceq s(r_1 + r_2).$$

Thus $A \subseteq \mathscr{B}(C, s(r_1+r_2))$. Similarly, $C \subseteq \mathscr{B}(A, s(r_1+r_2))$. Hence $H_{d_b}(A,C) \preceq s(r_1+r_2)+\epsilon$ for all $\epsilon \succ 0$, so by the properties of the infimum and the closedness of P, $H_{d_b}(A,C) \preceq s[H_{d_b}(A,B)+H_{d_b}(B,C)]$. □

3. Main Results

We now establish the principal fixed point results of this paper. Our approach combines the λ-iteration scheme with an approximate Hausdorff selection technique in order to overcome the lack of a strict triangle inequality and the vector-valued nature of the distance.

The presence of the b-metric coefficient $s \geq 1$ significantly affects the recursive estimates, and therefore the convergence thresholds depend explicitly on s, the contraction constant δ, and the iteration parameter λ. In particular, the quantity

$$q = s\left(1 - \frac{1}{\lambda}\right) + \frac{s\delta}{\lambda}$$

naturally arises in the analysis and determines the convergence behavior of the generated sequence.

Throughout this section, (X, d_b) denotes a complete cone b-metric space with coefficient $s \geq 1$, where the cone $P \subset E$ is normal. We assume that X is a closed convex subset of a real Banach space $(V, \|\cdot\|)$ so that the λ-iteration

$$x_{n+1} = \frac{(\lambda-1)x_n + z_n}{\lambda}, \qquad z_n \in T(x_n),$$

is well defined.

3.1. Nadler-Type Extension.

We begin with a Nadler-type fixed point theorem [9] for multivalued contractions in this setting.

Theorem 3.1 (Nadler-type Fixed Point via λ-Iteration ($s \geq 1$))**.** *Let X be a nonempty closed convex subset of a real Banach space $(V, \|\cdot\|)$. Let (X, d_b) be a complete cone b-metric space with coefficient $s \geq 1$, where the cone $P \subset E$ is normal with normal constant K.*

Assume $T : X \to \mathscr{CB}(X)$ satisfies the contraction

$$H_{d_b}(T(x), T(y)) \preceq \delta\, d_b(x,y), \qquad 0 < \delta < 1.$$

Fix $\lambda > 1$ and let $\{\varepsilon_n\} \subset P$ satisfy $\varepsilon_n \to 0$ and $\sum_{n=1}^{\infty} \|\varepsilon_n\| < \infty$.

Construct $\{x_n\}$ by

$$x_{n+1} = \frac{(\lambda - 1)x_n + z_n}{\lambda}, \qquad z_n \in T(x_n),$$

where the selections satisfy

$$d_b(z_n, z_{n-1}) \preceq H_{d_b}(T(x_n), T(x_{n-1})) + \varepsilon_n. \tag{1}$$

Define

$$q = s\left(1 - \frac{1}{\lambda}\right) + \frac{s\delta}{\lambda}.$$

If $q < 1$, then:

(1) $\{x_n\}$ converges to some $x^* \in X$,

(2) $x^* \in T(x^*)$,

(3) the fixed point is unique.

Proof. **Step 1: Recursive inequality.**

Define

$$a_n := d_b(x_{n+1}, x_n).$$

From the iteration formula,

$$x_{n+1} - x_n = \left(1 - \frac{1}{\lambda}\right)(x_n - x_{n-1}) + \frac{1}{\lambda}(z_n - z_{n-1}).$$

Using the b-metric inequality,

$$a_n \preceq s\left(1 - \frac{1}{\lambda}\right) d_b(x_n, x_{n-1}) + \frac{s}{\lambda} d_b(z_n, z_{n-1}).$$

Hence

$$a_n \preceq s\left(1 - \frac{1}{\lambda}\right) a_{n-1} + \frac{s}{\lambda} d_b(z_n, z_{n-1}).$$

Step 2: Use of approximate Hausdorff selection.

From (1) and the contraction property,

$$d_b(z_n, z_{n-1}) \preceq \delta a_{n-1} + \varepsilon_n.$$

Therefore,

$$a_n \preceq \left[s\left(1 - \frac{1}{\lambda}\right) + \frac{s\delta}{\lambda}\right] a_{n-1} + \frac{s}{\lambda}\varepsilon_n.$$

Thus,

$$a_n \preceq q a_{n-1} + \frac{s}{\lambda}\varepsilon_n.$$

Step 3: Iteration of the inequality.

By induction,

$$a_n \preceq q^n a_0 + \frac{s}{\lambda}\sum_{k=1}^{n} q^{n-k}\varepsilon_k.$$

Step 4: Convergence of a_n.

Taking norms and using normality of P,

$$\|a_n\| \leq Kq^n\|a_0\| + \frac{Ks}{\lambda}\sum_{k=1}^{n} q^{n-k}\|\varepsilon_k\|.$$

Since $0 < q < 1$ and $\sum \|\varepsilon_k\| < \infty$, we obtain

$$\|a_n\| \to 0.$$

Hence

$$d_b(x_{n+1}, x_n) \to 0.$$

Step 5: Cauchy property.

For $m > n$, using the b-metric inequality repeatedly,

$$d_b(x_m, x_n) \preceq s\sum_{k=n}^{m-1} a_k.$$

Taking norms,

$$\|d_b(x_m, x_n)\| \leq Ks\sum_{k=n}^{\infty} \|a_k\|.$$

Since $\sum \|a_k\|$ converges, the sequence is Cauchy. Completeness of (X, d_b) implies $x_n \to x^* \in X$.

Step 6: x^* is a fixed point.

Using the b-metric inequality,

$$\operatorname{dist}(x^*, T(x^*)) \preceq d_b(x^*, x_n) + H_{d_b}(T(x_n), T(x^*)).$$

The first term $\to 0$.

For the second term,

$$H_{d_b}(T(x_n), T(x^*)) \preceq \delta d_b(x_n, x^*).$$

Thus both terms tend to zero. Hence $x^* \in T(x^*)$.

Step 7: Uniqueness.

If $x^*, y^* \in \operatorname{Fix}(T)$,

$$d_b(x^*, y^*) \preceq H_{d_b}(T(x^*), T(y^*)) \preceq \delta d_b(x^*, y^*).$$

Thus

$$(1-\delta)d_b(x^*, y^*) \preceq 0.$$

Since $\delta < 1$, we obtain $d_b(x^*, y^*) = 0$, so $x^* = y^*$.

□

Theorem 3.2 (Stability)**.** *Let (X, d_b) be a complete cone b-metric space with coefficient $s \geq 1$, and let the cone P be normal.*

Let $T_1, T_2 : X \to \mathscr{CB}(X)$ satisfy the contractive condition

$$H_{d_b}(T_i(x), T_i(y)) \preceq \delta\, d_b(x, y), \qquad 0 < \delta < 1,$$

for $i = 1, 2$ and all $x, y \in X$.

Assume further that

$$H_{d_b}(T_1(x), T_2(x)) \preceq \epsilon \qquad \text{for all } x \in X,$$

where $\epsilon \in P$.

If $s\delta < 1$, then the unique fixed points $x_1^* \in \mathrm{Fix}(T_1)$ and $x_2^* \in \mathrm{Fix}(T_2)$ satisfy

$$(1 - s\delta)\, d_b(x_1^*, x_2^*) \preceq s\, \epsilon.$$

Proof. Since $x_1^* \in T_1(x_1^*)$ and $x_2^* \in T_2(x_2^*)$, we estimate the distance between the fixed points.

Step 1: Reduction via Hausdorff distance.

Because $x_1^* \in T_1(x_1^*)$, by definition of the Hausdorff b-cone metric we have

$$d_b(x_1^*, T_2(x_2^*)) \preceq H_{d_b}(T_1(x_1^*), T_2(x_2^*)).$$

Since $x_2^* \in T_2(x_2^*)$, it follows that

$$d_b(x_1^*, x_2^*) \preceq d_b(x_1^*, T_2(x_2^*)).$$

Therefore,

$$d_b(x_1^*, x_2^*) \preceq H_{d_b}(T_1(x_1^*), T_2(x_2^*)).$$

Step 2: Hausdorff triangle inequality.

Using Lemma 2.8,

$$H_{d_b}(T_1(x_1^*), T_2(x_2^*)) \preceq s\Big(H_{d_b}(T_1(x_1^*), T_1(x_2^*)) + H_{d_b}(T_1(x_2^*), T_2(x_2^*))\Big).$$

Step 3: Apply assumptions.

By contractivity of T_1,

$$H_{d_b}(T_1(x_1^*), T_1(x_2^*)) \preceq \delta\, d_b(x_1^*, x_2^*).$$

By perturbation assumption,

$$H_{d_b}(T_1(x_2^*), T_2(x_2^*)) \preceq \epsilon.$$

Hence,

$$H_{d_b}(T_1(x_1^*), T_2(x_2^*)) \preceq s(\delta d_b(x_1^*, x_2^*) + \epsilon).$$

Step 4: Combine inequalities.

Thus,

$$d_b(x_1^*, x_2^*) \preceq s\delta\, d_b(x_1^*, x_2^*) + s\epsilon.$$

Rearranging,

$$(1 - s\delta)\, d_b(x_1^*, x_2^*) \preceq s\epsilon.$$

Since $s\delta < 1$, the coefficient $(1 - s\delta)$ is positive, and the estimate follows.

□

3.2. **Berinde-Type Extension.** In general cone b-metric spaces, the distance need not be compatible with the linear structure of the ambient Banach space. In order to obtain a Berinde-type result, we therefore impose an additional compatibility assumption ensuring controlled homogeneity of the metric.

Assumption 3.3 (Metric compatibility with the Banach structure)**.** *Assume that X is a closed subset of a real Banach space $(V, \|\cdot\|)$ and that the cone b-metric d_b is generated by a positively homogeneous map in the sense that there exists an increasing mapping $\Phi : [0, \infty) \to P$ such that*

$$d_b(x, y) = \Phi(\|x - y\|),$$

and

$$\Phi(\alpha t) = \alpha\Phi(t) \qquad \text{for all } \alpha \geq 0,\ t \geq 0.$$

In particular,

$$d_b(x, x + \alpha v) \preceq \alpha\, d_b(x, x + v) \qquad \forall \alpha \geq 0.$$

Theorem 3.4 (Berinde-type Fixed Point via λ-Iteration)**.** *Let X be a nonempty closed convex subset of a real Banach space $(V, \|\cdot\|)$. Let (X, d_b) be a complete cone b-metric space with coefficient $s \geq 1$, where the cone P is normal with normal constant K.*

Assume also that Assumption 3.3 holds.

Let $T : X \to \mathscr{CB}(X)$ satisfy the weak contraction

$$H_{d_b}(T(x), T(y)) \preceq \delta\, d_b(x, y) + L\ \mathrm{dist}(y, T(x)), \qquad \delta, L \geq 0.$$

Let $\lambda > 1$ and let $\{\varepsilon_n\} \subset P$ satisfy $\varepsilon_n \to 0$ and $\sum_{n=1}^{\infty} \|\varepsilon_n\| < \infty$.

Construct $\{x_n\}$ by

$$x_{n+1} = \frac{(\lambda - 1)x_n + z_n}{\lambda}, \qquad z_n \in T(x_n),$$

with selections satisfying

$$d_b(z_n, z_{n-1}) \preceq H_{d_b}(T(x_n), T(x_{n-1})) + \varepsilon_n.$$

Define

$$A = s\left(1 - \frac{1}{\lambda}\right) + \frac{s}{\lambda}(\delta + Ls), \qquad B = s^2 L.$$

If

$$B < 1 \quad \text{and} \quad q := \frac{A}{1-B} < 1,$$

then T has a unique fixed point $x^* \in X$ and $x_n \to x^*$.

Proof. **Step 1: Basic recursive inequality.**

Let

$$a_n := d_b(x_{n+1}, x_n).$$

From the iteration formula,

$$x_{n+1} - x_n = \left(1 - \frac{1}{\lambda}\right)(x_n - x_{n-1}) + \frac{1}{\lambda}(z_n - z_{n-1}).$$

Using the b-metric inequality,

$$a_n \preceq s\left(1 - \frac{1}{\lambda}\right) a_{n-1} + \frac{s}{\lambda} d_b(z_n, z_{n-1}).$$

Step 2: Weak contraction estimate.

By construction,

$$d_b(z_n, z_{n-1}) \preceq \delta a_{n-1} + L \operatorname{dist}(x_{n-1}, T(x_n)) + \varepsilon_n.$$

Since $z_n \in T(x_n)$,

$$\operatorname{dist}(x_{n-1}, T(x_n)) \preceq d_b(x_{n-1}, z_n).$$

Step 3: Estimate of $d_b(x_{n-1}, z_n)$.

By the b-metric inequality,

$$d_b(x_{n-1}, z_n) \preceq s\big(d_b(x_{n-1}, x_n) + d_b(x_n, z_n)\big).$$

From the iteration formula,

$$z_n - x_n = \lambda(x_{n+1} - x_n).$$

Using homogeneity from Assumption 3.3,

$$d_b(x_n, z_n) \preceq \lambda\, d_b(x_{n+1}, x_n) = \lambda a_n.$$

Therefore,

$$d_b(x_{n-1}, z_n) \preceq s(a_{n-1} + \lambda a_n).$$

Step 4: Substitute into recursion.
Thus,

$$d_b(z_n, z_{n-1}) \preceq \delta a_{n-1} + Ls(a_{n-1} + \lambda a_n) + \varepsilon_n.$$

Substituting into Step 1,

$$a_n \preceq Aa_{n-1} + Ba_n + \frac{s}{\lambda}\varepsilon_n.$$

Step 5: Absorb the a_n term.
If $B < 1$, then

$$(1 - B)a_n \preceq Aa_{n-1} + \frac{s}{\lambda}\varepsilon_n.$$

Hence,

$$a_n \preceq qa_{n-1} + \frac{s}{\lambda(1 - B)}\varepsilon_n.$$

Step 6: Convergence.
By induction,

$$a_n \preceq q^n a_0 + C\sum_{k=1}^{n} q^{n-k}\varepsilon_k.$$

Taking norms and using normality,

$$\|a_n\| \to 0.$$

The remainder of the proof (Cauchy property, existence of limit, fixed point property, and uniqueness) follows exactly as in Theorem 3.1.

□

Theorem 3.5 (Well-Posedness)**.** *Let (X, d_b) be a complete cone b-metric space with coefficient $s \geq 1$, and let P be normal.*

Assume that $T : X \to \mathscr{CB}(X)$ satisfies

$$H_{d_b}(T(x), T(y)) \preceq \delta\, d_b(x, y), \qquad 0 < \delta < 1.$$

Let x^* be the unique fixed point of T.

Assume additionally that

$$s\delta < 1.$$

If $\{x_n\} \subset X$ satisfies

$$\operatorname{dist}(x_n, T(x_n)) \to 0,$$

then

$$x_n \to x^*.$$

Proof. For each n, choose $z_n \in T(x_n)$ such that

$$d_b(x_n, z_n) \preceq \operatorname{dist}(x_n, T(x_n)) + \varepsilon_n,$$

where $\varepsilon_n \to 0$.

Since $x^* \in T(x^*)$, we estimate:

Step 1: Apply b-metric inequality.

Using the b-metric property,

$$d_b(x_n, x^*) \preceq s\big(d_b(x_n, z_n) + d_b(z_n, x^*)\big).$$

Step 2: Estimate second term.

Because $x^* \in T(x^*)$,

$$d_b(z_n, x^*) \preceq d_b(z_n, T(x^*)).$$

By definition of Hausdorff distance,

$$d_b(z_n, T(x^*)) \preceq H_{d_b}(T(x_n), T(x^*)).$$

Using contractivity,

$$H_{d_b}(T(x_n), T(x^*)) \preceq \delta\, d_b(x_n, x^*).$$

Hence,

$$d_b(z_n, x^*) \preceq \delta\, d_b(x_n, x^*).$$

Step 3: Combine estimates.

Therefore,

$$d_b(x_n, x^*) \preceq s\big(d_b(x_n, z_n) + \delta d_b(x_n, x^*)\big).$$

Rearranging,

$$(1 - s\delta)\, d_b(x_n, x^*) \preceq s\, d_b(x_n, z_n).$$

Step 4: Pass to the limit.

Since

$$d_b(x_n, z_n) \to 0$$

and $s\delta < 1$, we conclude

$$d_b(x_n, x^*) \to 0.$$

Thus $x_n \to x^*$.

□

4. Applications

Some applications to vector optimization and Nash-type equilibrium problems are presented here within the framework of (λ, s)-convexity.

4.1. Vector Optimization.

We consider Pareto optimization in cone b-metric spaces:

Definition 4.1 ((λ, s)-Quasiconvexity)**.** *A mapping $f : X \to E$ is* (λ, s)-quasiconvex *if $\forall c \in E$, the sublevel set:*

$$\mathcal{L}_c = \{x \in X : f(x) \preceq c\}$$

is (λ, s)-convex.

Theorem 4.2 (Pareto Optimality)**.** *Let K be a nonempty compact subset of X and $f : K \to E$ continuous. Assume that the cone $P \subset E$ is closed and induces the partial order $\preceq$.*

Define the Pareto set

$$\mathcal{P} = \{x^* \in K : \nexists y \in K \text{ such that } f(y) \prec f(x^*)\}.$$

Then:

(1) $\mathcal{P}$ is nonempty;
(2) $\mathcal{P}$ is compact;
(3) For every $x \in K$, there exists $x^* \in \mathcal{P}$ such that $f(x^*) \preceq f(x)$.

Proof. **Step 1: Compactness of the image.**

Since K is compact and f is continuous, $f(K)$ is compact in E.

Step 2: Existence of minimal elements in $f(K)$.

We prove that $f(K)$ contains minimal elements with respect to the order induced by P.

Let $\mathcal{C}$ be a chain (totally ordered subset) in $f(K)$. Since $f(K)$ is compact, every net in $\mathcal{C}$ has a convergent subnet. Let $y_\alpha \in \mathcal{C}$ converge to $y \in f(K)$.

Because P is closed, the order relation is closed in $E \times E$. Hence y is a lower bound of $\mathcal{C}$.

Thus every chain in $f(K)$ has a lower bound in $f(K)$.

By Zorn's lemma, $f(K)$ contains minimal elements.

Step 3: Nonemptiness of $\mathcal{P}$.

Let $y^* \in f(K)$ be a minimal element. Choose $x^* \in K$ such that $f(x^*) = y^*$. Then $x^* \in \mathcal{P}$. Hence $\mathcal{P}$ is nonempty.

Step 4: Compactness of $\mathcal{P}$.

Let $\{x_n\} \subset \mathcal{P}$ with $x_n \to x$ in K. By continuity, $f(x_n) \to f(x)$.

If $x \notin \mathcal{P}$, there exists $y \in K$ such that $f(y) \prec f(x)$. By closedness of the order relation, for large n we would have $f(y) \prec f(x_n)$, contradicting $x_n \in \mathcal{P}$.

Thus $\mathcal{P}$ is closed in K. Since K is compact, $\mathcal{P}$ is compact.

Step 5: Dominance property.

For any $x \in K$, consider the set

$$S_x = \{f(y) \in f(K) : f(y) \preceq f(x)\}.$$

This set is nonempty and compact. By Step 2, it contains a minimal element $f(x^*)$. Then $x^* \in \mathcal{P}$ and $f(x^*) \preceq f(x)$.

□

4.2. **Economic Equilibrium.** We generalize Nash equilibrium to b-cone metric strategy spaces:

Theorem 4.3 (Nash Equilibrium)**.** *Let X_i be a locally convex Hausdorff topological vector space for each $i = 1, \dots, N$.*

Assume that for each player i:

- *$K_i \subset X_i$ is a nonempty compact* convex *subset;*
- *K_i may additionally be (λ_i, s_i)-convex in the sense of Definition 2.6;*
- *The payoff function*

$$u_i : \prod_{j=1}^{N} K_j \to \mathbb{R}$$

is continuous on $\prod_{j=1}^{N} K_j$;
- *For every fixed $\mathbf{x}_{-i}$, the function $x_i \mapsto u_i(x_i, \mathbf{x}_{-i})$ is quasiconcave on K_i.*

Then there exists a Nash equilibrium

$$\mathbf{x}^* = (x_1^*, \dots, x_N^*) \in \prod_{i=1}^{N} K_i$$

such that

$$u_i(x_i^*, \mathbf{x}_{-i}^*) \geq \nu_i(y_i, \mathbf{x}_{-i}^*) \qquad \forall y_i \in K_i,\ \forall i.$$

Proof. For each player i, define the best-response correspondence

$$B_i(\mathbf{x}_{-i}) = \arg\max_{x_i \in K_i} u_i(x_i, \mathbf{x}_{-i}).$$

Since K_i is compact and u_i is continuous, the maximum exists and $B_i(\mathbf{x}_{-i})$ is nonempty.

Quasiconcavity of u_i implies that $B_i(\mathbf{x}_{-i})$ is convex.

Standard arguments show that B_i has closed graph and is upper hemicontinuous.

Define the aggregate correspondence

$$B(\mathbf{x}) = \prod_{i=1}^{N} B_i(\mathbf{x}_{-i}).$$

Then B maps the compact convex set $\prod_{i=1}^{N} K_i$ into itself, with nonempty convex compact values, and is upper hemicontinuous.

By Glicksberg's generalization of Kakutani's fixed point theorem [11], B admits a fixed point $\mathbf{x}^*$.

Such $\mathbf{x}^*$ is a Nash equilibrium.

□

5. Conclusion and Future Research

In this paper we established a systematic fixed point framework for multivalued mappings in complete cone b-metric spaces. By combining order-theoretic techniques with a λ-iteration process defined on closed convex subsets of Banach spaces, we obtained:

(1) A Nadler-type fixed point theorem with explicit convergence constant

$$q = s\left(1 - \frac{1}{\lambda}\right) + \frac{s\delta}{\lambda},$$

valid under the sharp condition $q < 1$;

(2) A Berinde-type fixed point theorem for weak multivalued contractions, incorporating an approximate Hausdorff selection procedure;

(3) A quantitative stability result showing that perturbations of contractive multivalued mappings yield controlled perturbations of fixed points under the condition $s\delta < 1$;

(4) A well-posedness theorem ensuring convergence of approximate fixed point sequences to the unique solution.

The analysis highlights the fundamental role of the interaction between the cone order structure and the relaxed triangle inequality parameter s. The condition $s\delta < 1$ naturally emerges as the precise threshold governing stability and well-posedness in the cone b-metric setting.

Applications to vector optimization and Nash equilibrium problems demonstrate that the developed theory provides a coherent framework linking nonlinear functional analysis with equilibrium theory in ordered metric structures.

Future research directions include:

(1) development of numerical algorithms for λ-iterative approximation;
(2) extension to stochastic or probabilistic cone b-metric frameworks;
(3) treatment of differential and integral inclusions in ordered b-metric spaces;
(4) investigation of geometric properties related to (λ, s)-convexity.

The combination of order structures, generalized convexity, and multivalued contractive conditions offers a flexible analytical setting for further advances in nonlinear analysis.

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