Eur. J. Math. Anal. 1 (2021) 164-181
doi: 10.28924/ada/ma.1.164

Nonlinear Differential Problem with p-Laplacian and via Phi-Hilfer Approach: Solvability and Stability Analysis

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Abstract. This paper we consider a study of a general class of nonlinear singular fractional DEs with p-Laplacian for the existence and uniqueness solution and the Hyers-Ulam (HU) stability. result via $\varphi-$Hilfer derivative is studied. Then, an existence of one solution is investigated. Some illustrative examples are discussed at the end.

1. Introduction

Recently, fractional differential equations with boundary conditions are being studied by many interested people. This is because fractional differential equations describe many more real operations than classical differential equations. Therefore, partial differential equations appear in many engineering and technological disciplines that include several sciences; See for example [1, 3–6, 8, 17, 18, 20, 22, 23, 31].

Currently there are several different definitions of fractional integrals and derivatives, from the most famous of which are the Riemann-Liouville and Caputo fractional derivatives to other less well known definitions. A generalization of the derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [11], known as the fractional Hilfer derivative of order α and type $\beta \in [0, 1]$. Some properties and applications of the Helfer derivative are given in [12, 13] and the references mentioned therein. Prime value problems involving fractional Hilfer derivatives have been studied by several authors, see [9, 10, 26]. However, in the literature there are few papers on the boundary

Received: 8 Oct 2021.
Key words and phrases. $\varphi-$Hilfer derivative; existence of solution; fixed point; Hyers-Ulam stability.

value problems of the fractional Hilfer derivatives. The authors set out in [2] non-local value problems for derivatives of Helfer's fractions. For some recent work on boundary value problems with fractional Hilfer derivatives, we refer to the papers in [28–30].
Some authors have worked on the EU of solutions for fractional DEs with $p-$Laplacian operator. We cite, for example; Li., Wang., Khan et al. [15, 19, 27] studieds a nonlinear fractional DE with p-Laplacian operator for the EU of solutions.

H. Khan, T. Abdeljawad, M. Aslam, R. A. Khan and A. Khan [16]. worked on the following proposal for the existence of a positive solution (EPS) and stability analysis:

$$\begin{cases} \mathcal{D}^{r_1}\psi_p\left[\mathcal{D}^{r_2}\left(u(t)-v_1(t,u(t))\right)\right] = -A(t)v_2(t,u(t-\tau)), \\ \psi_p\left[\mathcal{D}^{r_2}\left(u(t)-v_1(t,u(t))\right)\right]\big|_{t=0} = \psi_p\left[\mathcal{D}^{r_2}\left(u(t)-v_1(t,u(t))\right)'\right]\big|_{t=0} = 0, \\ u(0)=u(1)=0, \\ \left[\mathcal{I}^{2-r_2}\left(u(t)-v_1(t,u(t))\right)\right]\big|_{t=0} = 0, \end{cases}$$

where $0 < r_1 < 1 < r_2 < 2$, and v_1, v_2 are continuous but singular at some points. The fractional derivatives $\mathcal{D}^{r_1}$ and $\mathcal{D}^{r_2}$ are taken in the Caputo sense and in the Riemann–Liouville sense, respectively, and $\psi_p(z) = |z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q} = 1, (\psi_p)^{-1} = \psi_q$.

A. Devi, A. Kumar, D. Baleanu and A. Khan [7]. worked on the EU and HU stability results, for nonliner FDEs involving Caputo fractional derivatives of distinct orders with ψ_p Laplacian operator:

$$\begin{cases} {}^{c}\mathcal{D}^{r_1}\psi_p\left[{}^{c}\mathcal{D}^{r_2}\left(u(t)-\sum_{i=1}^{m}v_i(t)\right)\right] = -w(t,u(t)), t\in(0,1] \\ \psi_p\left[{}^{c}\mathcal{D}^{r_2}\left(u(t)-\sum_{i=1}^{m}v_i(t)\right)\right]\big|_{t=0} = 0, \\ u(0)=\sum_{i=1}^{m}v_i(0), \\ u'(1)=\sum_{i=1}^{m}v_i'(1), \\ u^j(0)=\sum_{i=1}^{m}v_i^j(0),\ \text{for } j=2,3,\ldots,n-1, \end{cases}$$

where $0 < r_1 \leq 1, n-1 < r_2 \leq n, n \geq 4$, and v_i, w are continuous functions. ${}^{c}\mathcal{D}^{r_1}$ and ${}^{c}\mathcal{D}^{r_2}$ denotes the derivative of fractional order r_1 and r_2 in Caputo's sense, respectively, and $\psi_p(z) = |z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q} = 1, (\psi_p)^{-1} = \psi_q$.

In the present research work, we study the existence and uniqueness of a solution (EPS) and stability analysis which includes the $\varphi-$Hilfer fractional-order of the form:

$$\begin{cases} {}^{H}\mathcal{D}_{a^+}^{\alpha_1,\beta_1;\varphi}\psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(t) = h(t,u(t),{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u(t)), \quad t \in J=(a,b] \\ u(a)=0, u(b)=\sum_{i=1}^{n}\lambda_i u(\zeta_i), \\ \psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(a)=0, \\ \text{and} \quad \psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(b)\right) = \mathcal{I}_{a^+}^{\rho;\varphi}u(\zeta), \ a<\zeta,\zeta_i<b, \end{cases} \tag{1.1}$$

Here, we take ${}^{H}\mathcal{D}_{0^+}^{\alpha_1,\beta;\varphi}, {}^{H}\mathcal{D}_{0^+}^{\alpha_2,\beta;\varphi}$, are the $\varphi-$Hilfer fractional derivative of orders $\alpha_1, \alpha_2, 1 < \alpha_1, \alpha_2 < 2$ and β_1, β_2 two parameters $0 \le \beta_1, \beta_2 \le 1$, ${}^{RL}\mathcal{D}_{a^+}^{\kappa;\varphi}$ the φ-Riemann-Liouville fractional derivative of order μ where $\mu < \alpha_2$, and $\mathcal{I}_{0^+}^{\rho;\varphi}$ the left-sided $\varphi-$Riemann Liouville fractional integral of order ρ, where $\rho > 0$, and $\psi_p(z) = |z|^{p-2}z$ denotes the $p-$Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1, (\psi_p)^{-1}=\psi_q$, and $\varphi: J \to \mathbb{R}$ be an increasing function such that $\varphi'(t) \neq 0$, for all $t \in J$, and $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is given function will be "well defined" later.

2. Phi-Hilfer Derivatives Calculus

In this section, we introduce some notations and definitions of Phi-Hilfer Derivatives Calculus and present preliminary results needed in our proofs later, for details, see [17,24,25].
Let $\varphi: [a,b] \to \mathbb{R}$ be an increasing function with $\varphi'(t) \neq 0$, for all $t \in J$, and let $C([a,b],\mathbb{R})$ be the Banach space.

For all $\upsilon > -1$ and $s,t \in [0,\infty), (t \ge s)$, we pose $\varphi_\upsilon(t,s) = (\varphi(t)-\varphi(s))^\upsilon$.

Definition 1. *Let (a,b), $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the half-axis $(0,\infty)$ and $\alpha > 0$. In addition, let $\varphi(t)$ be a positive increasing function on $(a,b]$, which has a continuous derivative $\varphi'(t)$ on (a,b). The $\varphi-$Riemann–Liouville fractional integral of a function u with respect to another function φ on $[a,b]$ is defined by*

$$\mathcal{I}_{a^+}^{\alpha;\varphi}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \varphi'(s)\varphi_{\alpha-1}(t,s)u(s)ds, \tag{2.1}$$

where $\Gamma(.)$ is the Gamma function.

Definition 2. *Let $n \in \mathbb{N}$ and let $\varphi, u \in C^n(J)$ be two functions such that φ is increasing and $\varphi'(t) \neq 0$, for all $t \in (a,b]$. The left-sided $\varphi-$Riemann Liouville fractional derivative of a function u of order α is defined by*

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha;\varphi}u(t) &= \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\varphi}u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^n \int_a^t \varphi'(s)\varphi_{n-\alpha-1}(t,s)u(s)ds, \end{aligned}$$

where $n=[\alpha]+1$, $[\alpha]$ represents the integer part of the real number α.

Definition 3. *Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, $[a,b]$ is the interval such that $-\infty\leq a<b\leq\infty$ and $\varphi, u\in C^{n}\left([a,b],\mathbb{R}\right)$ two functions such that φ is increasing and $\varphi'(t)\neq 0$, for all $t\in[a,b]$. The φ-Hilfer fractional derivative of a function u of order a and type $0\leq\beta\leq 1$ is defined by*

$$^{H}\mathcal{D}_{a^{+}}^{\alpha,\beta;\varphi}u(t)=\mathcal{I}_{a^{+}}^{\beta(n-\alpha);\varphi}\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^{n}\mathcal{I}_{a^{+}}^{(1-\beta)(n-\alpha);\varphi}u(t)=\mathcal{I}_{a^{+}}^{\gamma-\alpha;\varphi}\mathcal{D}_{a^{+}}^{\gamma;\varphi}u(t),$$

where $n=[\alpha]+1$, $\gamma-\alpha=\beta\left(n-\alpha\right)$.

2.1. **Auxiliary Lemma.**

Lemma 1. *Let $\alpha,\rho>0$. Then, we have the following semigroup property given by*

$$\mathcal{I}_{a^{+}}^{\alpha;\varphi}\mathcal{I}_{a^{+}}^{\rho;\varphi}u(t)=\mathcal{I}_{a^{+}}^{\alpha+\rho;\varphi}u(t),\quad t>a.$$

Next, we present the φ-fractional integral and derivatives of a power function.

Proposition 1. *Let $\alpha\geq 0,\sigma>0$ and $t>a$. Then, φ-fractional integral and derivative of a power function are given by*

(1) $\mathcal{I}_{a^{+}}^{\alpha,\varphi}\varphi_{\sigma-1}(t,a)(t)=\frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}\varphi_{\sigma+\alpha-1}(t,a)$.
(2) $^{H}\mathcal{D}_{a^{+}}^{\alpha,\beta;\varphi}\varphi_{\sigma-1}(t,a)(t)=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}\varphi_{\sigma-\alpha-1}(t,a), n-1<\alpha<n,\sigma>n$.

Lemma 2. *If $u\in C^{n}([a,b],\mathbb{R}), n-1<\alpha<n, 0\leq\beta\leq 1$ and $\gamma=\alpha+\beta(n-\alpha)$. Then*

$$\mathcal{I}_{a^{+}}^{\alpha,\varphi}(^{H}\mathcal{D}_{a^{+}}^{\alpha,\beta;\varphi}u)(t)=u(t)-\sum_{k=1}^{k=n}\frac{\varphi_{\gamma-k}(t,s)}{\Gamma(\gamma-k+1)}\nabla_{\varphi}^{[n-k]}\mathcal{I}_{a^{+}}^{(1-\beta)(n-\alpha);\varphi}u(a),\ \ t\in[a,b],$$

where $\nabla_{\varphi}^{[n]}u(t):=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}u(t)$.

Lemma 3. *Let $u\in C^{n}\left[a,b\right]$ and $0<q<1$, we have*

$$\left|\mathcal{I}_{a^{+}}^{q;\varphi}u(t_2)-\mathcal{I}_{a^{+}}^{q;\varphi}u(t_1)\right|\leq\frac{2\left\|u\right\|}{\Gamma\left(q+1\right)}\varphi_{q}(t_2,t_1).$$

Lemma 4. *([14]) For the $p-$Laplacian operator ψ_p, the following conditions hold true:*

(1) If $\left|\delta_1\right|,\left|\delta_2\right|\geq\rho>0,\ 1<p\leq 2,\delta_1\delta_2>0,$ then

$$\left|\psi_p(\delta_1)-\psi_p(\delta_2)\right|\leq(p-1)\,\rho^{p-2}\left|\delta_1-\delta_2\right|.$$

(2) If $p>2,\ \left|\delta_1\right|,\left|\delta_2\right|\leq\rho_*>0,$ then

$$\left|\psi_p(\delta_1)-\psi_p(\delta_2)\right|\leq(p-1)\,\rho_*^{p-2}\left|\delta_1-\delta_2\right|.$$

Lemma 5. *[9] For nonnegative $a_i, i=1,...,k$,*

$$\left(\sum_{i=1}^{k}a_i\right)^{q}\leq k^{q-1}\left(\sum_{i=1}^{k}a_i^{q}\right),\ q\geq 1.$$

Lemma 6. *Let $a \geq 0, 1 < \alpha_1, \alpha_2 < 2$, $0 \leq \beta_1, \beta_2 \leq 1$, and $2-\gamma_1 = (1-\beta_1)(2-\alpha_1), 2-\gamma_2 = (1-\beta_2)(2-\alpha_2)$. For $f \in C(J,,\mathbb{R},\mathbb{R})$, the unique solution of the sequential Hilfer fractional boundary value problem*

$$ {}^{H}\mathcal{D}_{a^+}^{\alpha_1,\beta_1;\varphi}\psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(t) = f(t), \quad t \in J = [a,b], \tag{2.2}$$

$$\begin{cases} u(a)=0, u(b) = \sum_{i=1}^{n}\lambda_i u(\zeta_i), \\ \psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(a) = 0, \\ \text{and} \quad \psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(b)\right) = \mathcal{I}_{a^+}^{\rho;\varphi}u(\zeta), \ a < \zeta, \zeta_i < b, \end{cases} \tag{2.3}$$

is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)X(s,a)ds \\ &\quad -\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X(t,a)dt \\ &\quad +\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i). \end{aligned}$$

where

$$\begin{aligned} X(s,a) &= \psi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)f(z)dz + \frac{\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) - \mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)}{\varphi_{\gamma_1-1}(b,a)}\varphi_{\gamma_1-1}(s,a)\right) \\ \mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) &= \frac{1}{\Gamma(\rho)}\int_a^\zeta \varphi'(s)\varphi_\rho(\zeta,s)u(s)\,ds, \\ \mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b) &= \frac{1}{\Gamma(\alpha_1)}\int_a^b \varphi'(s)\varphi_{\alpha_1-1}(b,s)f(s)ds. \end{aligned}$$

Proof. Assume that u is a solution of the sequential nonlocal boundary value Problems (3.6) and (2.3). Applying the two operators $\mathcal{I}_{a^+}^{\alpha_1;\varphi}, \mathcal{I}_{a^+}^{\alpha_2;\varphi}$ to both sides of Equation (3.6) and using Lemma 2 and Proposition 1, we obtain

$$\psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(t) = \mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t) + \frac{m_0}{\Gamma(\gamma_1-1)}\varphi_{\gamma_1-2}(t,a) + \frac{m_1}{\Gamma(\gamma_1)}\varphi_{\gamma_1-1}(t,a), \tag{2.4}$$

where $m_0, m_1 \in \mathbb{R}$, and $2-\gamma_1 = (1-\beta_1)(2-\alpha_1)$. From the boundary condition $\psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(a) = 0$, and if $t \to a$ then $\varphi_{\gamma_1-2}(t,a) \to \infty$, we get

$$m_0 = 0.$$

and by $\psi_p\left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u\right)(b)=\mathcal{I}_{a^+}^{\rho;\varphi}u(\zeta)$, we obtain

$$m_1=\frac{\Gamma(\gamma_1)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right).$$

So

$${}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi}u(t)=\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t)+\frac{\varphi_{\gamma_1-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)\right),$$

by (2.4)we have

$$\begin{aligned}u(t)&=\mathcal{I}_{a^+}^{\alpha_2;\varphi}\left[\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t)+\frac{\varphi_{\gamma_1-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)\right)\right]\\&\quad+\frac{m_2}{\Gamma(\gamma_2-1)}\varphi_{\gamma_2-2}(t,a)+\frac{m_3}{\Gamma(\gamma_2)}\varphi_{\gamma_2-1}(t,a),\end{aligned}$$

where $m_2,m_3\in\mathbb{R}$, and $2-(1-\beta_2)(2-\alpha_2)=\gamma_2$.and if $t\to a$ then $\varphi_{\gamma_2-2}(t,a)\to\infty$, we get By conditions $u(a)=0$, and $\lim_{t\to 0}t^{\gamma_2-2}=\infty$, we get

$$m_2=0.$$

So

$$u(t)=\mathcal{I}_{a^+}^{\alpha_2;\varphi}\left[\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t)+\frac{\varphi_{\gamma_1-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)\right)\right]+\frac{m_3}{\Gamma(\gamma_2)}\varphi_{\gamma_2-1}(t,a).$$

By conditions $u(b)=\sum_{i=1}^{n}\lambda_i u(\zeta_i)$, we get

$$\begin{aligned}m_3&=\frac{\Gamma(\gamma_2)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)\\&\quad-\frac{\Gamma(\gamma_2)}{\varphi_{\gamma_2-1}(b,a)}\mathcal{I}_{a^+}^{\alpha_2;\varphi}\left[\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t)+\frac{\varphi_{\gamma_1-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)\right)\right]_{t=b}.\end{aligned}$$

Then

$$\begin{aligned}u(t)&=\mathcal{I}_{a^+}^{\alpha_2;\varphi}\left[\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t)+\frac{\varphi_{\gamma_1-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)\right)\right]+\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)\\&\quad-\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\mathcal{I}_{a^+}^{\alpha_2;\varphi}\left[\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}f(t)+\frac{\varphi_{\gamma_1-1}(t,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}f(b)\right)\right)\right]_{t=b}.\end{aligned}$$

This finishes the proof. □

Conjecture 1.

$$
\begin{aligned}
{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u(t) &= \frac{1}{\Gamma(\alpha_2-\mu)}\int_a^t \varphi'(s)\varphi_{\alpha_2-\mu-1}(t,s)X(s,a)ds \\
&+\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i) \\
&-\frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X(t,a)dt.
\end{aligned}
$$

3. Main Results

In this section, we present to the reader our main results on the existence and stability for the above problem. We begin by considering the space

$$
C_{\varphi}^{\mu} = \left\{ u : u, {}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u \in C\left([a,b], \mathbb{R}\right)\right\},
$$

with the norm

$$
\|u\|_{C_{\varphi}^{\mu}} = \|u\|_C + \left\|{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u\right\|_C,
$$

such that

$$
\|u\|_C = \sup_{t\in[a,b]}|u(t)|, \text{ and } \left\|{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u\right\|_C = \sup_{t\in[a,b]}\left|{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u(t)\right|.
$$

3.1. Criteria For Uniqueness Solution.

Now, wee need to consider the following assumptions:

$\mathcal{H}_1$) h is continuous function.

$\mathcal{H}_2$) There exists a constant $\Upsilon > 0$, such that

$$
|h(t,u,v) - h(t,x,y)| \le \Upsilon\left(|u-x| + |v-y|\right),
$$

with $t \in [a,b], (u,v,x,y) \in \mathbb{R}^4$.

$\mathcal{H}_3$) There exists two continuous functions $\pi_1, \pi_2 : [a,b] \to \mathbb{R}^+$, such that

$$
|h(t,u,v)| \le \pi_1(t)|u(t)| + \pi_2(t)|v(t)|,
$$

where

$$
\pi_1^* = \sup_{t\in[a,b]}|\pi_1(t)|, \text{ and } \pi_2^* = \sup_{t\in[a,b]}|\pi_2(t)|.
$$

Now, we define the following quantities:

$$\begin{aligned}
\varphi_q(b,a) &= M^q, \\
\Omega &= 3^{q-2}\left[\left(\frac{2M^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)^{q-1}\left((\pi_1^*)^{q-1}+(\pi_2^*)^{q-1}\right)+\left(\frac{M^{\rho}}{\Gamma(\rho+1)}\right)^{q-1}\right] \\
\Lambda_1 &= \frac{2.\Omega.M^{\alpha_2}}{\Gamma(\alpha_2+1)}, \\
\Lambda_2 &= \left(\sum_{i=1}^{n}|\lambda_i|\right), \\
\Lambda_3 &= \frac{\Omega.M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)}+\frac{\Omega.\Gamma(\gamma_2)\,M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}, \\
\Lambda_4 &= \frac{\Gamma(\gamma_2)\,M^{-\mu}}{\Gamma(\gamma_2-\mu)}\Lambda_2.
\end{aligned}$$

Based on the above hypotheses, we present to the reader the following result.

Theorem 1. *Under $\mathcal{H}_2$ and $\mathcal{H}_3$ the equation (1.1) has a solution.*

Proof. **Firstly:** We begin this proof by defining the operator $\mathbb{G}: C_\varphi^\mu \to C_\varphi^\mu$ by:

$$\begin{aligned}
(\mathbb{G}u)(t) &= \frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)X_u(s,a)ds-\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X_u(t,a)dt \\
&\quad+\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i).
\end{aligned}$$

where

$$X_u(s,a)=\psi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_u(z)dz+\frac{\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)-\mathcal{I}_{0^+}^{\alpha_1;\varphi}h_u(b)\right)}{\varphi_{\gamma_1-1}(b,a)}\varphi_{\gamma_1-1}(s,a)\right),$$

where

$$h_u(t)=h(t,u(t),{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u(t)).$$

We consider the set $\mathcal{U}_r=\left\{u\in C_\varphi^\mu:\|u\|_{C_\varphi^\mu}\le r\right\}$, so that

$$\max\left\{\left(2\left(\Lambda_1+\Lambda_3\right)\right)^{\frac{1}{2-q}},2\left(\Lambda_2+\Lambda_4\right)\right\}\le r.$$

We show that $\mathbb{G}\mathcal{U}_r \subset \mathcal{U}_r$. For any $u \in \mathcal{U}_r$, and by Lemma 5 we have

$$
\begin{aligned}
& |X_u(s,a)| \\
= & \left|\left[\psi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_u(z)dz + \frac{\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) - \mathcal{I}_{0^+}^{\alpha_1;\varphi}h_u(b)\right)}{\varphi_{\gamma_1-1}(b,a)}\varphi_{\gamma_1-1}(s,a)\right)\right]\right| \\
\leq & \sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_u(z)dz + \mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) + \mathcal{I}_{0^+}^{\alpha_1;\varphi}h_u(b)\right|^{q-1} \\
\leq & 3^{q-2}\sup_{t\in[a,b]}\left(\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_u(z)dz\right)^{q-1} + \left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta)\right)^{q-1} + \left(\mathcal{I}_{0^+}^{\alpha_1;\varphi}h_u(b)\right)^{q-1}\right) \\
\leq & 3^{q-2}\left[2\left(\frac{\pi_1^* M^{\alpha_1}}{\Gamma(\alpha_1+1)}\|u\|_C + \frac{\pi_2^* M^{\alpha_1}}{\Gamma(\alpha_1+1)}\left\|{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u\right\|_C\right)^{q-1} + \left(\frac{M^\rho}{\Gamma(\rho+1)}\right)^{q-1}(\|u\|_C)^{q-1}\right] \\
\leq & 3^{q-2}\left[\left(\frac{2.\pi_1^* M^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)^{q-1}(\|u\|_C)^{q-1} + \left(\frac{2.\pi_2^* M^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)^{q-1}\left(\left\|{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u\right\|_C\right)^{q-1} + \left(\frac{M^\rho}{\Gamma(\rho+1)}\right)^{q-1}(\|u\|_C)^{q-1}\right] \\
\leq & 3^{q-2}\left[\left(\frac{2M^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)^{q-1}\left((\pi_1^*)^{q-1} + (\pi_2^*)^{q-1}\right) + \left(\frac{M^\rho}{\Gamma(\rho+1)}\right)^{q-1}\right]r^{q-1} \\
\leq & \Omega.r^{q-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \sup_{t\in[a,b]}|(\mathbb{G}u)(t)| \qquad (3.1) \\
\leq & \sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)X_u(s,a)ds + \frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X_u(t,a)dt\right. \\
& \left. + \frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^n \lambda_i u(\zeta_i)\right| \\
\leq & \frac{2M^{\alpha_2}}{\Gamma(\alpha_2+1)}|X_u| + \left(\sum_{i=1}^n |\lambda_i|\right)\sup_{t\in[a,b]}|u(t)| \\
\leq & \frac{2.\Omega.M^{\alpha_2}}{\Gamma(\alpha_2+1)}r^q + \left(\sum_{i=1}^n |\lambda_i|\right)r. \\
\leq & \Lambda_1 r^q + \Lambda_2 r.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
& \sup_{t\in[a,b]}\left|\left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\mathbb{G}u\right)(t)\right| \qquad (3.2) \\
\leq & \sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_2-\mu)}\int_a^t \varphi'(s)\varphi_{\alpha_2-\mu-1}(t,s)X_u(s,a)ds + \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^n \lambda_i u(\zeta_i)\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X_u(t,a)dt\Bigg| \\
\leq\;& \left[\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)}+\frac{\Gamma(\gamma_2)M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}\right]|X_u|+\frac{\Gamma(\gamma_2)M^{-\mu}}{\Gamma(\gamma_2-\mu)}\left(\sum_{i=1}^n|\lambda_i|\right)\sup_{t\in[a,b]}|u(t)| \\
\leq\;& \left[\frac{\Omega.M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)}+\frac{\Omega.\Gamma(\gamma_2)M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}\right]r^{q-1}+\frac{\Gamma(\gamma_2)M^{-\mu}}{\Gamma(\gamma_2-\mu)}\left(\sum_{i=1}^n|\lambda_i|\right)r \\
\leq\;& \Lambda_3 r^{q-1}+\Lambda_4 r.
\end{aligned}
$$

By (3.1) and (3.2), we find

$$
\begin{aligned}
\|u\|_{C_\varphi^\mu} &= \sup_{t\in[a,b]}|(\mathbb{G}u)(t)|_C+\sup_{t\in[a,b]}\left|\left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\mathbb{G}u\right)(t)\right|_C \\
&\leq (\Lambda_1+\Lambda_3)r^{q-1}+(\Lambda_2+\Lambda_4)r \\
&\leq r.
\end{aligned}
\tag{3.3}
$$

that is $\mathbb{G}\mathcal{U}_r$ belongs to $\mathcal{U}_r$ on $[a,b]$.

Next, we prove that $\mathbb{G}$ is completely continuous. For any $u\in\mathcal{U}_r$ and $t_1,t_2\in[a;b]$ such that $t_1<t_2$, by Lemma 3, we have

$$
\begin{aligned}
&\sup_{t\in[a,b]}|(\mathbb{G}u)(t_2)-(\mathbb{G}u)(t_1)| \\
\leq\;& \sup_{t\in[a,b]}\Bigg|\frac{1}{\Gamma(\alpha_2)}\int_a^{t_2}\varphi'(s)\varphi_{\alpha_2-1}(t_2,s)X_u(s,a)ds-\frac{1}{\Gamma(\alpha_2)}\int_a^{t_1}\varphi'(s)\varphi_{\alpha_2-1}(t_1,s)X_u(s,a)ds \\
&+\frac{\varphi_{\gamma_2-1}(t_2,a)-\varphi_{\gamma_2-1}(t_1,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b\varphi'(t)\varphi_{\alpha_2-1}(b,t)X_u(t,a)dt \\
&+\frac{\varphi_{\gamma_2-1}(t_2,a)-\varphi_{\gamma_2-1}(t_1,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^n\lambda_i u(\zeta_i)\Bigg| \\
\leq\;& \frac{\Omega.r^{q-1}}{\Gamma(\alpha_2+1)}\varphi_{\alpha_2}(t_2,t_1)+\frac{\Omega.M^{\alpha_2}.r^{q-1}+\Gamma(\alpha_2+1)\Lambda_2 r}{\Gamma(\alpha_2+1)\varphi_{\gamma_2-1}(b,a)}\varphi_{\gamma_2-1}(t_2,t_1).
\end{aligned}
$$

Hence,

$$
\sup_{t\in[a,b]}|(\mathbb{G}u)(t_2)-(\mathbb{G}u)(t_1)|\to 0,\quad \text{as}\quad t_2\to t_1.
$$

Also, we can say that

$$\begin{aligned}
&\sup_{t\in[a,b]}\left|\left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\mathbb{G}u\right)(t_2)-\left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\mathbb{G}u\right)(t_1)\right| \\
\leq\ &\sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_2-\mu)}\int_a^{t_2}\varphi'(s)\varphi_{\alpha_2-\mu-1}(t_2,s)X_u(s,a)ds-\frac{1}{\Gamma(\alpha_2-\mu)}\int_a^{t_1}\varphi'(s)\varphi_{\alpha_2-\mu-1}(t_1,s)X_u(s,a)ds\right. \\
&+\frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t_2,a)-\varphi_{\gamma_2-\mu-1}(t_1,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i) \\
&\left.+\frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t_2,a)-\varphi_{\gamma_2-\mu-1}(t_1,a)}{\varphi_{\gamma_2-1}(b,a)}\int_a^{b}\varphi'(t)\varphi_{\alpha_2-1}(b,t)X_u(t,a)dt\right| \\
\leq\ &\frac{\Omega.r^{q-1}}{\Gamma(\alpha_2-\mu+1)}\varphi_{\alpha_2-\mu}(t_2,t_1) \\
&+\left(\frac{\Gamma(\gamma_2)\Lambda_2 r}{\Gamma(\gamma_2-\mu)\varphi_{\gamma_2-1}(b,a)}+\frac{\Gamma(\gamma_2).\Omega.r^{q-1}.M^{\alpha_2}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)\varphi_{\gamma_2-1}(b,a)}\right)\varphi_{\gamma_2-\mu-1}(t_2,t_1).
\end{aligned}$$

Hence,

$$\sup_{t\in[a,b]}\left|\left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\mathbb{G}u\right)(t_2)-\left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\mathbb{G}u\right)(t_1)\right|\to 0,\quad \text{as}\quad t_2\to t_1.$$

As a consequence of the above three steps and thanks to Arzela–Ascoli theorem, we conclude that $\mathbb{G}$ is completely continuous.

The proof of Theorem 1 is thus completely achieved. □

3.2. **Criteria For Existence of a Solution.**

Theorem 2. *Assume that $\mathcal{H}_2$ and $\mathcal{H}_3$ are satisfied. Suppose that*

$$\Upsilon_1+\Upsilon_2<1,$$

where

$$\Upsilon_1=\frac{2(q-1)\Delta^{q-2}M^{\alpha_2}}{\Gamma(\alpha_2+1)}\left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{M^{\rho}}{\Gamma(\rho+1)}\right)+\Lambda_2,$$

and

$$\begin{aligned}
\Upsilon_2\ =\ &(q-1)\Delta^{q-2}\left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{M^{\rho}}{\Gamma(\rho+1)}\right)\left(\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)}+\frac{\Gamma(\gamma_2)M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}\right) \\
&+\frac{\Gamma(\gamma_2)\Lambda_2 M^{-\mu}}{\Gamma(\gamma_2-\mu)}.
\end{aligned}$$

Then, (1.1) has a uniqueness solution.

Proof. We pass to prove that $\mathbb{G}$ is a contraction. For any $u, v \in \mathcal{U}_r$, we have the following estimate

$$\begin{aligned}
&|X_u(s,a) - X_v(s,a)| \\
= &\left|\psi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_u(z)dz + \frac{\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) - \mathcal{I}_{0^+}^{\alpha_1;\varphi}h_u(b)\right)}{\varphi_{\gamma_1-1}(b,a)}\varphi_{\gamma_1-1}(s,a)\right)\right. \\
&\left. -\psi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_v(z)dz + \frac{\left(\mathcal{I}_{0^+}^{\rho;\varphi}v(\zeta) - \mathcal{I}_{0^+}^{\alpha_1;\varphi}h_v(b)\right)}{\varphi_{\gamma_1-1}(b,a)}\varphi_{\gamma_1-1}(s,a)\right)\right| \\
\leq &(q-1)\,y^{q-2}\left|\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_u(z)dz - \frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_v(z)dz\right. \\
&\left.+\frac{\varphi_{\gamma_1-1}(s,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) - \mathcal{I}_{0^+}^{\rho;\varphi}v(\zeta)\right) + \frac{\varphi_{\gamma_1-1}(s,a)}{\varphi_{\gamma_1-1}(b,a)}\left(\mathcal{I}_{0^+}^{\alpha_1;\varphi}h_u(b) - \mathcal{I}_{0^+}^{\alpha_1;\varphi}h_v(b)\right)\right| \\
\leq &(q-1)\,y^{q-2}\left(\frac{2M^{\alpha_1}}{\Gamma(\alpha_1+1)}\sup_{t\in[a,b]}|h_u(t) - h_v(t)| + \frac{M^{\rho}}{\Gamma(\rho+1)}\sup_{t\in[a,b]}|u(t)-v(t)|\right) \\
\leq &(q-1)\,y^{q-2}\left(\left(\frac{2\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)}\right)\sup_{t\in[a,b]}|u(t)-v(t)|\right. \\
&\left.+\frac{2\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)}\sup_{t\in[a,b]}\left|{}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}u(t) - {}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}v(t)\right|\right) \\
\leq &(q-1)\,\Delta^{q-2}\left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)}\right)\|u-v\|_{C_{\varphi}^{\mu}}.
\end{aligned}$$

where

$$\begin{cases}
\Delta > \frac{2\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)}, \text{ if } q > 2, \\
\text{ou} \\
0 < \Delta \leq \frac{2\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)}, \text{ if } 1 < q \leq 2.
\end{cases}$$

Then

$$\begin{aligned}
&\sup_{t\in[a,b]}|(\mathbb{G}u)(t) - (\mathbb{G}v)(t)| \\
\leq &\sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)(X_u - X_v)(s,a)ds\right| \\
&+\sup_{t\in[a,b]}\left|\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)(X_u - X_v)(t,a)dt\right| \\
&+\sup_{t\in[a,b]}\left|\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i\left(u(\zeta_i) - v(\zeta_i)\right)\right|
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\leq \frac{2M^{\alpha_2}}{\Gamma(\alpha_2+1)} \sup_{t\in[a,b]} |X_u(t,a) - X_v(t,a)| + \Lambda_2 \sup_{t\in[a,b]} (|u(t) - v(t)|) \\
&\leq \left(\frac{2(q-1)\Delta^{q-2} M^{\alpha_2}}{\Gamma(\alpha_2+1)} \left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)} \right) + \Lambda_2 \right) \|u-v\|_{C_\varphi^\mu} \\
&\leq \Upsilon_1 \|u-v\|_{C_\varphi^\mu}.
\end{aligned}$$

Also

$$\begin{aligned}
&\sup_{t\in[a,b]} \left| \left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi} \mathbb{G}u\right)(t) - \left({}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi} \mathbb{G}v\right)(t) \right| \qquad (3.5) \\
\leq\ & \sup_{t\in[a,b]} \left| \frac{1}{\Gamma(\alpha_2-\mu)} \int_a^t \varphi'(s)\varphi_{\alpha_2-\mu-1}(t,s)(X_u - X_v)(s,a)ds \right. \\
&+ \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2-\mu)} \frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)} \sum_{i=1}^n \lambda_i (u(\zeta_i) - v(\zeta_i)) \\
&\left. + \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\gamma_2-\mu)} \frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)} \int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)(X_u - X_v)(t,a)dt \right| \\
\leq\ & \left(\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)} + \frac{\Gamma(\gamma_2) M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)} \right) \sup_{t\in[a,b]} |X_u - X_v| \\
&+ \frac{\Gamma(\gamma_2)\Lambda_2 M^{-\mu}}{\Gamma(\gamma_2-\mu)} \sup_{t\in[a,b]} (|u(t) - v(t)|) \\
\leq\ & \left\{ (q-1)\Delta^{q-2} \left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)} \right) \left(\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)} + \frac{\Gamma(\gamma_2) M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)} \right) \right. \\
&\left. + \frac{\Gamma(\gamma_2)\Lambda_2 M^{-\mu}}{\Gamma(\gamma_2-\mu)} \right\} \|u-v\|_{C_\varphi^\mu} \\
\leq\ & \Upsilon_2 \|u-v\|_{C_\varphi^\mu}.
\end{aligned}$$

By (3.4) and (3.5), yields the following inequality

$$\|\mathbb{G}u - \mathbb{G}v\|_{C_\varphi^\mu} \leq (\Upsilon_1 + \Upsilon_2) \|u-v\|_{C_\varphi^\mu}.$$

Where $\Upsilon_1 + \Upsilon_2 < 1$. Hence $\mathbb{G}$ is a contraction operator and the contraction mapping principle implies that (1.1) has a unique solution. □

3.3. **Ulam Type Stability.** We introduce the following two definitions

Definition 4. *The problem (1.1) is Ulam–Hyers stable if* $\exists\, \lambda \in \mathbb{R}_+^*$, *such that for each* $\varepsilon > 0$, $t \in J$, *and for each* $u \in C_\varphi^\mu$ *solution of the following inequality*

$$\left\| {}^{H}\mathcal{D}_{a^+}^{\alpha_1,\beta_1;\varphi} \psi_p \left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi} u \right)(t) - h(t, u(t), {}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi} u(t)) \right\|_{C_\varphi^\mu} < \varepsilon, \qquad (3.6)$$

$\exists v \in C_\varphi^\mu$ *solution of (1.1), i.e.*

$${}^{H}\mathcal{D}_{a^+}^{\alpha_1,\beta_1;\varphi} \psi_p \left({}^{H}\mathcal{D}_{a^+}^{\alpha_2,\beta_2;\varphi} v \right)(t) = h(t, v(t), {}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi} v(t)), \qquad (3.7)$$

such that, the inequality

$$\|u - v\|_{C^{\mu}_{\varphi}} \leq \lambda\varepsilon,$$

holds.

Definition 5. *The equation (1.1) has the Ulam–Hyers stability in the generalized sense if $\exists\ \varphi \in C(J, \mathbb{R}_+)$, such that for each $\varepsilon > 0$, $t \in J$, and for each $u \in C^{\mu}_{\varphi}$ solution of:*

$$\left\| {}^{H}\mathcal{D}^{\alpha_1,\beta_1;\varphi}_{a^+}\psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}u\right)(t) - h(t, u(t), {}^{RL}\mathcal{D}^{\mu;\varphi}_{a^+}u(t))\right\|_{C^{\mu}_{\varphi}} < \varepsilon, \tag{3.8}$$

$\exists v \in C^{\mu}_{\varphi}$ solution of (1.1) that satisfies

$$\|u(t) - v(t)\|_{C^{\mu}_{\varphi}} \leq \varepsilon\varphi(t).$$

In the light of the first definition and using the above existence and uniqueness theorem, we present to the reader the following result.

Theorem 3. *If the assumptions $(\mathcal{H}_2)$ are satisfied, then Eq (1.1) is Ulam–Hyers stable under the condition that $N_1 + N_2 < 1$, where*

$$N_1 = \frac{2(q-1)\Delta^{q-2}M^{\alpha_2}}{\Gamma(\alpha_2+1)}\left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)}\right),$$

and

$$N_2 = \left(\frac{4\Upsilon(q-1)\Delta^{q-2}M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{(q-1)\Delta^{q-2}M^{\rho}}{\Gamma(\rho+1)}\right)\left(\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)} + \frac{\Gamma(\gamma_2)M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}\right).$$

Proof. Let $u \in C^{\mu}_{\varphi}$ be a solution of the inequality (3.6), i.e.

$$\left\| {}^{H}\mathcal{D}^{\alpha_1,\beta_1;\varphi}_{a^+}\psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}u\right)(t) - h(t, u(t), {}^{RL}\mathcal{D}^{\mu;\varphi}_{a^+}u(t))\right\|_{C^{\mu}_{\varphi}} < \varepsilon, \quad \forall t \in J. \tag{3.9}$$

Let $v \in C^{\mu}_{\varphi}$ be a unique solution of:

$${}^{H}\mathcal{D}^{\alpha_1,\beta_1;\varphi}_{a^+}\psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}v\right)(t) = h(t, v(t), {}^{RL}\mathcal{D}^{\mu;\varphi}_{a^+}v(t)), \quad \forall t \in J,$$

and

$$\begin{cases} u(a) = v(a), \ u(b) = v(b) \\ \text{and} \\ \psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}u\right)(a) = \psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}v\right)(a), \\ \psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}u\right)(b) = \psi_p\left({}^{H}\mathcal{D}^{\alpha_2,\beta_2;\varphi}_{a^+}v\right)(b), \end{cases}$$

By using Proof of Lemma 6

$$
\begin{aligned}
v(t) &= \frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)X_v(s,a)ds \\
&\quad -\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X_v(t,a)dt \\
&\quad +\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i),
\end{aligned}
$$

where

$$
X_v(s,a) = \psi_q\left(\frac{1}{\Gamma(\alpha_1)}\int_a^s \varphi'(s)\varphi_{\alpha_1-1}(s,z)h_v(z)dz + \frac{\left(\mathcal{I}_{0^+}^{\rho;\varphi}u(\zeta) - \mathcal{I}_{0^+}^{\alpha_1;\varphi}h_v(b)\right)}{\varphi_{\gamma_1-1}(b,a)}\varphi_{\gamma_1-1}(s,a)\right).
$$

By integration of inequality (3.9), for any $t \in J$, we have

$$
\begin{aligned}
&\left\| u(t) - \frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)X_u(s,a)ds \right. \qquad (3.10)\\
&+\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)X_u(t,a)dt \\
&\left. -\frac{\varphi_{\gamma_2-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\sum_{i=1}^{n}\lambda_i u(\zeta_i)\right\|_C \\
\leq\ & \mathcal{I}_{a^+}^{\alpha_2;\varphi}\psi_q\left(\mathcal{I}_{a^+}^{\alpha_1;\varphi}\varepsilon\right) = \frac{M^{q-1}\varphi_{\alpha_1+\alpha_2}(t,a)}{\Gamma(\alpha_1+\alpha_2+1)}\varepsilon.
\end{aligned}
$$

On the other hand, for any $u, v \in C_\varphi^\mu$, we have the following estimate

$$
\begin{aligned}
&\|u(t) - v(t)\|_C \qquad (3.11)\\
<\ & \frac{M^{q-1}\varphi_{\alpha_1+\alpha_2}(t,a)}{\Gamma(\alpha_1+\alpha_2+1)}\varepsilon \\
&+\sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_2)}\int_a^t \varphi'(s)\varphi_{\alpha_2-1}(t,s)\left(X_u - X_v\right)(s,a)ds\right| \\
&+\sup_{t\in[a,b]}\left|\frac{\varphi_{\gamma_2-1}(t,a)}{\Gamma(\alpha_2)\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)\left(X_u - X_v\right)(t,a)dt\right| \\
<\ & \frac{M^{\alpha_1+\alpha_2+q1}}{\Gamma(\alpha_1+\alpha_2+1)}\varepsilon + \frac{2(q-1)\Delta^{q-2}M^{\alpha_2}}{\Gamma(\alpha_2+1)}\left(\frac{4\Upsilon M^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{M^{\rho}}{\Gamma(\rho+1)}\right)\|u-v\|_{C_\varphi^\mu} \\
<\ & \frac{M^{\alpha_1+\alpha_2+q-1}}{\Gamma(\alpha_1+\alpha_2+1)}\varepsilon + N_1\|u-v\|_{C_\varphi^\mu}.
\end{aligned}
$$

Also, for any $t \in J$, we have

$$
\begin{aligned}
&\left\| {}^{RL}\mathcal{D}_{a^+}^{\mu;\varphi}\left(u(t)-v(t)\right)\right\|_C \qquad (3.12)\\
\leq\ & \frac{M^{q-1}\varphi_{\alpha_1+\alpha_2-\mu}(t,a)}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\varepsilon\\
&+\sup_{t\in[a,b]}\left|\frac{1}{\Gamma(\alpha_2-\mu)}\int_a^t \varphi'(s)\varphi_{\alpha_2-\mu-1}(t,s)\left(X_u-X_v\right)(s,a)ds\right.\\
&\left.+\frac{\Gamma(\gamma_2)}{\Gamma(\alpha_2)\Gamma(\gamma_2-\mu)}\frac{\varphi_{\gamma_2-\mu-1}(t,a)}{\varphi_{\gamma_2-1}(b,a)}\int_a^b \varphi'(t)\varphi_{\alpha_2-1}(b,t)\left(X_u-X_v\right)(t,a)dt\right|\\
\leq\ & \frac{M^{\alpha_1+\alpha_2+q-\mu-1}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\varepsilon+\left(\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)}+\frac{\Gamma(\gamma_2)M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}\right)\sup_{t\in[a,b]}\left|X_u-X_v\right|\\
\leq\ & \frac{M^{\alpha_1+\alpha_2+q-\mu-1}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\varepsilon\\
&+\left(\frac{4\Upsilon(q-1)\Delta^{q-2}M^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{(q-1)\Delta^{q-2}M^{\rho}}{\Gamma(\rho+1)}\right)\left(\frac{M^{\alpha_2-\mu}}{\Gamma(\alpha_2-\mu+1)}+\frac{\Gamma(\gamma_2)M^{\alpha_2-\mu}}{\Gamma(\alpha_2+1)\Gamma(\gamma_2-\mu)}\right)\|u-v\|_{C_\varphi^\mu}\\
\leq\ & \frac{M^{\alpha_1+\alpha_2+q-\mu-1}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\varepsilon+N_2\|u-v\|_{C_\varphi^\mu}.
\end{aligned}
$$

So, by (3.11) and (3.12) we have

$$\|u-v\|_{C_\varphi^\mu}\leq\varepsilon\left(\frac{M^{\alpha_1+\alpha_2+q-1}}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{M^{\alpha_1+\alpha_2+q-\mu-1}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\right)+(N_1+N_2)\|u-v\|_{C_\varphi^\mu}.$$

Therefore, we get

$$\|u-v\|_{C_\varphi^\mu}\leq\lambda\varepsilon,$$

such that

$$\lambda=\frac{1}{1-(N_1+N_2)}\left(\frac{M^{\alpha_1+\alpha_2+q-1}}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{M^{\alpha_1+\alpha_2+q-\mu-1}}{\Gamma(\alpha_1+\alpha_2-\mu+1)}\right),$$

for any $t \in J$. This implies that the Ulam-Hyers stability condition is satisfied. □

3.4. **Illustrative exemple.** Consider the following problem

$${}^{H}\mathcal{D}_{0^+}^{\frac{13}{10},\frac{6}{7};t^2}\psi_p\left(\left({}^{H}\mathcal{D}_{0^+}^{\frac{17}{10},\frac{2}{3};t^2}u\right)\right)(t)=h(t,u(t),{}^{RL}\mathcal{D}_{a^+}^{\frac{1}{2};t^2}u(t)),\ t\in J=[0,2], \qquad (3.13)$$

$$
\begin{aligned}
u(a) &= 0,\ u(2)=\sum_{i=1}^{n}\left(\frac{3i}{11}\right)u\left(\frac{i}{2+i}\right),\\
\psi_p\left(\left({}^{H}\mathcal{D}_{0^+}^{\frac{17}{10},\frac{2}{3};t^2}u\right)\right)(0) &= 0,\ \psi_p\left(\left({}^{H}\mathcal{D}_{0^+}^{\frac{17}{10},\frac{2}{3};t^2}u\right)\right)(2)=\mathcal{I}_{a^+}^{\frac{3}{2};t^2}u\left(\frac{4}{3}\right)\\
f(t,u(t),v(t)) &= \exp\left(\frac{1}{7\left(1+t^2\right)}\right)u(t)+\frac{v(t)}{\left(1+e^t\right)},
\end{aligned}
$$

then assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied with

$$\Upsilon = \pi_2^* = \frac{1}{2}, \pi_1^* = e^{\frac{1}{7}}, \text{ and } M = 4.$$

We conclude that (3.13) has an unique solution.

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