Eur. J. Math. Anal. 6 (2026) 7
doi: 10.28924/ada/ma.6.7

Growth of Solutions of Certain Linear Differential Equations With Dominant Coefficient of Lower $[p, q]$ −Order Near a Singular Point

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Abstract. In this article, we study the growth of solutions to certain linear differential equations with analytic coefficients in $\overline{\mathbb{C}} - \{z_0\}$, where $z_0 \in \mathbb{C}$ is an essential singularity. We derive estimates for the lower bounds of the $[p, q]$-order of these solutions. Our results improve and extend the previous findings of Liu, Long and Zeng, and those of Long and Zeng.

1. Introduction and main results

For $k \geq 2$, we consider the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_j(z)$ $(j = 0, ..., k-1)$ are analytic functions in $\overline{\mathbb{C}} - \{z_0\}$ with $z_0 \in \mathbb{C}$ being a finite singularity. To analyze the growth of solutions of (1.1) we employ the concepts of $[p, q]$-order and $[p, q]$-type, originally introduced by Juneja and co-authors for entire functions (see [11], [12]). These concepts have since been extended to various settings — including entire or meromorphic functions in $\mathbb{C}$, functions analytic in the unit disc (see e.g. [1]- [3], [10], [15]- [17], [20]- [22]), and more recently, functions analytic in $\overline{\mathbb{C}} - \{z_0\}$, as in the works of Dahmani and Belaïdi [5], Long and Zeng [19].

In particular, Long and Zeng modified these concepts to study functions that are analytic except at a finite singularity, and derived several results concerning the growth of solutions of (1.1). Building on their approach, we introduce the lower $[p, q]$-order and lower $[p, q]$-type, defined analogously, and study their applications to solutions of equation (1.1).

The principal tool in our analysis is Nevanlinna theory, and we assume the reader is familiar with its fundamental results and standard notations (see [7], [9], [14], [23]).

Received: 21 Sep 2025.
Key words and phrases. Linear differential equations, Analytic function, Singular point, $[p, q]$-order, Lower $[p, q]$-order, Lower $[p, q]$-type.

We begin by introducing some notations and definitions, following [5], [19]. For any real $r \in (0,\infty)$, define $\exp r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, we also define for all sufficiently large r, $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. We denote $\exp_0 r := \log_0 r = r$, $\exp_1 r := \log_{-1} r$. Also, we denote the logarithmic measure of a set $E \subset (0,1)$ by $m_l(E) = \int_E \frac{dt}{t}$.

In order to develop our study, we firstly recall some related notations. Let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, $z_0 \in \mathbb{C}$ is some essential singularity.

The Nevanlinna fundamentals are the most important step here. For that reason, we dedicated this section to fully developing the theory for a function with singular point z_0 (see, [4], [6], [8]). Define the counting function of f near z_0 by the following formula

$$N_{z_0}(r,f) = -\int_{\infty}^{r} \frac{n(t,f) - n(\infty,f)}{t} dt - n(\infty,f)\log r,$$

where $n(t,f)$ denote the number of poles of f in the region $\{z \in \mathbb{C} : t \le |z - z_0|\} \cup \{\infty\}$ counting its multiplicities, we also define the proximity function near z_0 by

$$m_{z_0}(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(z_0 - re^{i\phi})| d\phi.$$

Summing up together, the characteristic function of f near z_0 will be

$$T_{z_0}(r,f) = m_{z_0}(r,f) + N_{z_0}(r,f).$$

Definition 1.1. ([13]) Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$. The function f is called a transcendental or admissible meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ provided that

$$\lim_{r\to 0} \frac{T_{z_0}(r,f)}{\log \frac{1}{r}} = +\infty$$

and f is a rational function in $\overline{\mathbb{C}} - \{z_0\}$ provided that

$$\liminf_{r\to 0} \frac{T_{z_0}(r,f)}{\log \frac{1}{r}} < +\infty.$$

Now, we introduce the concepts of $[p,q]$ −order and the lower $[p,q]$ −order for functions that are analytic or meromorphic in $\overline{\mathbb{C}} - \{z_0\}$, following the approach in [19].

Definition 1.2. ([5], [19]) Let f be a meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$, p and q be two integers with $p \ge q \ge 1$. The $[p,q]$ −order of growth and the lower $[p,q]$ −order of f are defined respectively by

$$\rho_{[p,q],T}(f,z_0) = \limsup_{r\longrightarrow 0} \frac{\log_p T_{z_0}(r,f)}{\log_q \frac{1}{r}}$$

and

$$\mu_{[p,q],T}(f,z_0) = \liminf_{r\longrightarrow 0} \frac{\log_p T_{z_0}(r,f)}{\log_q \frac{1}{r}}.$$

For an analytic function f in $\overline{\mathbb{C}}-\{z_0\}$, the $[p,q]$ −order and the lower $[p,q]$ −order of growth of f are defined respectively by

$$\rho_{[p,q],M}(f,z_0)=\limsup_{r\longrightarrow 0}\frac{\log_{p+1}M_{z_0}(r,f)}{\log_q\frac{1}{r}}$$

and

$$\mu_{[p,q],M}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_{p+1}M_{z_0}(r,f)}{\log_q\frac{1}{r}},$$

where $M_{z_0}(r,f)=\max\{|f(z)|:|z-z_0|=r\}$.

Definition 1.3. ([19]) Let p,q be two integers such that $p\geq q\geq 1$, and let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho=\rho_{[p,q]}(f,z_0)\in(0,\infty)$. Then, the $[p,q]$−type of f is defined by

$$\tau_{[p,q],T}(f,z_0)=\limsup_{r\to 0}\frac{\log_{p-1}T_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\rho}}.$$

If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho=\rho_{[p,q]}(f,z_0)\in(0,\infty)$, then the $[p,q]$−type of f is defined by

$$\tau_{[p,q],M}(f,z_0)=\limsup_{r\to 0}\frac{\log_{p}M_{z_0}(r,f)}{(\log_{q-1}\frac{1}{r})^{\rho}}.$$

Definition 1.4. ([5]) Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, if $\mu=\mu_{[p,q]}(f,z_0)\in(0,\infty)$, then the lower $[p,q]$−type of f is defined by

$$\underline{\tau}_{[p,q],T}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_{p}T_{z_0}(r,f)}{\left(\log_{q-1}\frac{1}{r}\right)^{\mu}}.$$

If f is an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $\mu=\mu_{[p,q]}(f,z_0)\in(0,\infty)$, then the lower $[p,q]$−type of f is defined by

$$\underline{\tau}_{[p,q],M}(f,z_0)=\liminf_{r\longrightarrow 0}\frac{\log_{p}M_{z_0}(r,f)}{\left(\log_{q-1}\frac{1}{r}\right)^{\mu}}.$$

Remark 1.5. *Let $p\geq q\geq 1$ be integers, and let f be an analytic function in $\overline{\mathbb{C}}-\{z_0\}$ of $[p,q]$-order. Then by using ([6], Lemma 2.2), we get $\rho_{[p,q],T}(f,z_0)=\rho_{[p,q],M}(f,z_0)$ and* $\mu_{[p,q],T}(f,z_0)=\mu_{[p,q],M}(f,z_0)$. Therefore, in the sequel, we denote

$$\rho_{[p,q],T}(f,z_0)=\rho_{[p,q],M}(f,z_0)=\rho_{[p,q]}(f,z_0),$$

$$\mu_{[p,q],T}(f,z_0)=\mu_{[p,q],M}(f,z_0)=\mu_{[p,q]}(f,z_0).$$

Recently in [18], Liu, Long and Zeng investigated the growth of solutions of the second order complex linear differential equation

$$f''+A(z)f'+B(z)f=0, \tag{1.2}$$

where $A(z)$ and $B(z)$ are analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ and obtained the following three results.

Theorem 1.6. *([18]) Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying $\mu(A,z_0) < \mu(B,z_0) < \infty$. Then, every non trivial solution $f(z)$ of (1.2), that is analytic in $\overline{\mathbb{C}}-\{z_0\}$, satisfies $\rho_2(f,z_0) \geq \mu(B,z_0)$.*

Theorem 1.7. *([18]) Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying the following conditions:*
(i) $\mu(A,z_0) = \mu(B,z_0)$;
(ii) $\underline{\tau}(A,z_0) < \underline{\tau}(B,z_0)$.
Then, every non trivial solution $f(z)$ of (1.2), that is analytic in $\overline{\mathbb{C}}-\{z_0\}$, satisfies $\rho_2(f,z_0) \geq \mu(B,z_0)$.

Theorem 1.8. *([18]) Let $A(z)$ and $B(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ satisfying $\mu(B,z_0) < \mu(A,z_0) < \infty$. Then, every non trivial solution $f(z)$ of (1.2), that is analytic in $\overline{\mathbb{C}}-\{z_0\}$, satisfies $\rho(f,z_0) \geq \mu(A,z_0)$.*

In this work, we improve the results of Liu, Long and Zeng for higher-order linear differential equations of the form (1.1) where most of the coefficients are of $[p,q]$-order. Firstly, we investigate the growth of solutions of (1.1) when $A_0(z)$ is a dominant coefficient with concept of lower order.

Theorem 1.9. *Let $k \geq 2$ and $p \geq q \geq 1$ be integers. Suppose that $A_0(z), ..., A_{k-1}(z)$ are analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ and that*

$$\max_{2\leq j\leq k-1}\{\rho_{[p,q]}(A_j,z_0), \mu_{[p,q]}(A_1,z_0)\} < \mu_{[p,q]}(A_0,z_0) < \infty.$$

Then every non trivial solution f that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (1.1) satisfies $\rho_{[p,q]}(f,z_0) = +\infty$ and $\rho_{[p+1,q]}(f,z_0) \geq \mu_{[p,q]}(A_0,z_0)$.

The following result shows that the coefficient $A_0(z)$ is a dominant coefficient in terms of lower $[p,q]$-type.

Theorem 1.10. *Let $k \geq 2$ and $p \geq q \geq 1$ be integers. Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$. Assume that the following three assumptions hold simultaneously:*
(i) $\rho = \max_{2\leq j\leq k-1}\{\rho_{[p,q]}(A_j,z_0), \mu_{[p,q]}(A_1,z_0)\} \leq \mu_{[p,q]}(A_0,z_0) < \infty$, $\mu_{[p,q]}(A_0,z_0) > 0$;
(ii) $\underline{\tau}_{[p,q],M}(A_1,z_0) < \underline{\tau}_{[p,q],M}(A_0,z_0) = \underline{\tau}$ when $\mu_{[p,q]}(A_1,z_0)\} = \mu_{[p,q]}(A_0,z_0)$;
(iii)

$$\begin{aligned}\tau_1 &= \max_{2\leq j\leq k-1}\left\{\tau_{[p,q],M}(A_j,z_0) : \rho_{[p,q]}(A_j,z_0) = \mu_{[p,q]}(A_0,z_0)\right\}\\ &< \underline{\tau}_{[p,q],M}(A_0,z_0) = \underline{\tau}\end{aligned}$$

when $\mu_{[p,q]}(A_0,z_0) = \max_{2\leq j\leq k-1}\left\{\rho_{[p,q]}(A_j,z_0)\right\}$.
Then every non trivial solution f that is analytic in $\overline{\mathbb{C}}-\{z_0\}$ of (1.1) satisfies $\rho_{[p,q]}(f,z_0) = +\infty$ and $\rho_{[p+1,q]}(f,z_0) \geq \mu_{[p,q]}(A_0,z_0)$.

We previously obtained two results concerning the growth of solutions to equation (1.1) when the coefficient $A_0(z)$ is dominant. A natural question arises: what can be said about the growth of solutions when the dominant coefficient is $A_s(z)$, with $s \neq 0$? In what follows, we address this question by considering the case where the dominant coefficient belongs to the set $\{1, 2, ..., k-1\}$, and we establish the following result.

Theorem 1.11. *Let $k \geq 2$ and $p \geq q \geq 1$ be integers. Let $A_0(z), ..., A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} - \{z_0\}$. Suppose that exists an integer $s \in \{1, 2, ..., k-1\}$ such that*

$$\max_{\substack{1 \leq j \leq k-1 \\ j \neq s}} \{\rho_{[p,q]}(A_j, z_0), \mu_{[p,q]}(A_0, z_0)\} < \mu_{[p,q]}(A_s, z_0) < \infty.$$

Then every transcendental solution f that is analytic in $\overline{\mathbb{C}} - \{z_0\}$ of (1.1) satisfies $\rho_{[p,q]}(f, z_0) \geq \mu_{[p,q]}(A_s, z_0)$.

Remark 1.12. The assumption that f is analytic in $\overline{\mathbb{C}} - \{z_0\}$ is necessary. The following example illustrates that there exists a solution f to equation (1.1) which is not analytic in $\overline{\mathbb{C}} - \{z_0\}$, even though all the coefficients $A_j(z)$ $(j = 0, ..., k-1)$ of (1.1) are analytic in $\overline{\mathbb{C}} - \{z_0\}$. For example, consider the equation

$$f''' + \exp_2\left\{\frac{1}{z_0 - z}\right\} f'' + \frac{3}{z_0 - z} \exp_2\left\{\frac{1}{z_0 - z}\right\} f'$$
$$+ \left(\frac{3}{(z_0 - z)^2} \exp_2\left\{\frac{1}{z_0 - z}\right\} + \frac{6}{(z_0 - z)^3}\right) f = 0. \tag{1.3}$$

The function $f(z) = (z_0 - z)^3$ is a solution of (1.3), yet it is not analytic in $\overline{\mathbb{C}} - \{z_0\}$. Therefore, in our results, we always assume that $f(z)$ is analytic in $\overline{\mathbb{C}} - \{z_0\}$.

2. Some auxiliary lemmas

We present some lemmas in this section which are useful in the proofs of our theorems.

Lemma 2.1. *([5]) Let $p \geq q \geq 1$ be integers, and let f be nonconstant analytic function in $\overline{\mathbb{C}} - \{z_0\}$ with $\mu_{[p,q]}(f, z_0) = \mu < \infty$. Then there exists a set $E_1 \subset (0, 1)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E_1$, we have*

$$\mu = \lim_{r \longrightarrow 0} \frac{\log_p T_{z_0}(r, f)}{\log_q \frac{1}{r}} = \lim_{r \longrightarrow 0} \frac{\log_{p+1} M_{z_0}(r, f)}{\log_q \frac{1}{r}},$$

and for any given $\varepsilon > 0$ and all $|z - z_0| = r \in E_1$ the following hold

$$M_{z_0}(r, f) \leq \exp_p\left\{\left(\log_{q-1} \frac{1}{r}\right)^{\mu + \varepsilon}\right\}$$

and

$$T_{z_0}(r, f) \leq \exp_{p-1}\left\{\left(\log_{q-1} \frac{1}{r}\right)^{\mu + \varepsilon}\right\}.$$

Lemma 2.2. *([6]) Let f be a nonconstant meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$, let $\gamma>1$, $\varepsilon>0$ be given real constants and $k\in\mathbb{N}$. Then there exist a set $E_2\subset(0,r_0]$, $(r_0\in(0,1))$ having finite logarithmic measure and a constant $\lambda>0$ that depends on γ and k such that for all $|z-z_0|=r\in(0,r_0]\setminus E_2$, we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq\lambda\left[\frac{1}{r^2}T_{z_0}\left(\frac{1}{\gamma}r,f\right)\log T_{z_0}(r,f)\right]^k.$$

Lemma 2.3. *([5]) Let $p\geq q\geq1$ be integers, and let f be nonconstant analytic function in $\overline{\mathbb{C}}-\{z_0\}$ with $0<\mu_{[p,q]}(f,z_0)=\mu<\infty$ and $0<\underline{\tau}_{[p,q]}(f,z_0)=\underline{\tau}<\infty$. Then there exists a set $E_3\subset(0,1)$ having infinite logarithmic measure such that for all $|z-z_0|=r\in E_3$, we have*

$$M_{z_0}(r,f)<\exp_p\left\{(\underline{\tau}+\varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu}\right\}.$$

Lemma 2.4. *([19]) Let f be a meromorphic function in $\overline{\mathbb{C}}-\{z_0\}$. Then the following statements hold:*
(i) $T_{z_0}\left(r,\frac{1}{f}\right)=T_{z_0}(r,f)+O(1)$;
(ii) $T_{z_0}(r,f')<O\left(T_{z_0}(r,f)+\log\frac{1}{r}\right)$, $r\in(0,r_1]\setminus E_4$, *where* $E_4\subset(0,r_1]$ *with* $m_l(E_4)<+\infty$.

Lemma 2.5. *Let $p\geq q\geq1$ be integers, and let $f_1(z)$ and $f_2(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ such that*

$$\mu_{[p,q]}(f_1,z_0)=\mu_1>0,\ \ \rho_{[p,q]}(f_2,z_0)=\rho_2<\infty,\ \ \rho_2<\mu_1.$$

Then there exists $r_2\in(0,1)$, such that for all $|z-z_0|=r\in(0,r_2)$ the following holds

$$\lim_{r\to0}\frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)}=0.$$

Proof. By $\mu_{[p,q]}(f_1,z_0)=\mu_1$, $\rho_{[p,q]}(f_2,z_0)=\rho_2$ and the Definition 1.2, for any given ε with $0<\varepsilon<\frac{\mu_1-\rho_2}{2}$, there exists $r_2\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_2)$ the following hold

$$T_{z_0}(r,f_1)\geq\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}\tag{2.1}$$

and

$$T_{z_0}(r,f_2)\leq\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_2+\varepsilon}\right\}.\tag{2.2}$$

Combining (2.1) and (2.2), it follows that for all $|z-z_0|=r\in(0,r_2)$, we have

$$0\leq\frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)}\leq\frac{\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_2+\varepsilon}\right\}}{\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}}$$

$$=\exp\left\{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_2+\varepsilon}\right\}-\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}\right\}$$

$$= \exp\left\{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}\left(\frac{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho_2+\varepsilon}\right\}}{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}}-1\right)\right\}\to 0,\ r\to 0.$$

This implies the conclusion holds. □

Lemma 2.6. *Let $p \geq q \geq 1$ be integers, and let $f_1(z)$ and $f_2(z)$ be analytic functions in $\overline{\mathbb{C}}-\{z_0\}$ such that*

$$\mu_{[p,q]}(f_1,z_0)=\mu_1>0,\ \ \mu_{[p,q]}(f_2,z_0)=\mu_2<\infty,\ \ \mu_2<\mu_1.$$

Then there exists a set $E_5\subset(0,1)$ having infinite logarithmic measure such that, for all $|z-z_0|=r\in E_5$, we have

$$\lim_{r\to 0}\frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)}=0.$$

Proof. By $\mu_{[p,q]}(f_1,z_0)=\mu_1$ and the Definition 1.2, for any given ε with $0<\varepsilon<\frac{\mu_1-\mu_2}{2}$, there exists $r_3\in(0,1)$ such that for all $|z-z_0|=r\in(0,r_3)$ the following holds

$$T_{z_0}(r,f_1)\geq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}. \tag{2.3}$$

Concerning the Lemma 2.1, by $\mu_{[p,q]}(f_2,z_0)=\mu_2$ we deduce the existence of a set $E_5\subset(0,r_3)$ that has infinite logarithmic measure such that for all $|z-z_0|=r\in E_5$, we have

$$T_{z_0}(r,f_2)\leq \exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_2+\varepsilon}\right\}. \tag{2.4}$$

Combining (2.3) and (2.4), we deduce that for all $|z-z_0|=r\in E_5\cap(0,r_3)=E_5$, which is a set of infinite logarithmic measure

$$0\leq\frac{T_{z_0}(r,f_2)}{T_{z_0}(r,f_1)}\leq\frac{\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_2+\varepsilon}\right\}}{\exp_{p-1}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}}$$

$$=\exp\left\{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_2+\varepsilon}\right\}-\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}\right\}$$

$$= \exp\left\{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}\left(\frac{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_2+\varepsilon}\right\}}{\exp_{p-2}\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_1-\varepsilon}\right\}}-1\right)\right\}\to 0,\ r\to 0.$$

Hence the conclusion holds. □

Lemma 2.7. *Let $p\geq q\geq 1$ be integers, and let f, g be non-constant meromorphic functions in $\overline{\mathbb{C}}-\{z_0\}$ with $\rho_{[p,q]}(f,z_0)$ as $[p,q]$-order and $\mu_{[p,q]}(g,z_0)$ as lower $[p,q]$-order. Then we have*

$$\mu_{[p,q]}(f+g,z_0)\leq\max\left\{\rho_{[p,q]}(f,z_0),\mu_{[p,q]}(g,z_0)\right\}$$

and

$$\mu_{[p,q]}(fg,z_0)\leq\max\left\{\rho_{[p,q]}(f,z_0),\mu_{[p,q]}(g,z_0)\right\}.$$

Furthermore, if $\mu_{[p,q]}(g,z_0) > \rho_{[p,q]}(f,z_0)$, *then we obtain*

$$\mu_{[p,q]}(f+g,z_0) = \mu_{[p,q]}(fg,z_0) = \mu_{[p,q]}(g,z_0).$$

Proof. Without loss of generality, we may assume that $\rho_{[p,q]}(f,z_0) < +\infty$ and $\mu_{[p,q]}(g,z_0) < +\infty$. By definition of the lower $[p,q]$ −order of g, there exists a sequence $r_n \longrightarrow 0$ $(n \longrightarrow +\infty)$ such that

$$\lim_{n\longrightarrow+\infty} \frac{\log_p T_{z_0}(r_n,g)}{\log_q \frac{1}{r_n}} = \mu_{[p,q]}(g,z_0).$$

Hence, for any given $\varepsilon > 0$, there exists a positive integer N_1 such that for $n > N_1$

$$T_{z_0}(r_n,g) \leq \exp_p\left\{(\mu_{[p,q]}(g,z_0)+\varepsilon)\log_q \frac{1}{r_n}\right\}.$$

Similarly, by the definition of the $[p,q]$ −order of f, for any given $\varepsilon > 0$, there exists $R > 0$ such that for $0 < r \leq R$

$$T_{z_0}(r,f) \leq \exp_p\left\{(\rho_{[p,q]}(f,z_0)+\varepsilon)\log_q \frac{1}{r}\right\}.$$

Since $r_n \longrightarrow 0$ $(n \longrightarrow +\infty)$, there exists a positive integer N_2 such that $r_n < R$, and thus

$$T_{z_0}(r_n,f) \leq \exp_p\left\{(\rho_{[p,q]}(f,z_0)+\varepsilon)\log_q \frac{1}{r_n}\right\}$$

holds for $n > N_2$. Using the standard inequalities

$$T_{z_0}(r,f+g) \leq T_{z_0}(r,f) + T_{z_0}(r,g) + \ln 2$$

and

$$T_{z_0}(r,fg) \leq T_{z_0}(r,f) + T_{z_0}(r,g).$$

Then, for any given $\varepsilon > 0$, we have for $n > \max\{N_1,N_2\}$

$$\begin{aligned} T_{z_0}(r_n,f+g) &\leq T_{z_0}(r_n,f) + T_{z_0}(r_n,g) + \ln 2 \\ &\leq \exp_p\left\{(\rho_{[p,q]}(f,z_0)+\varepsilon)\log_q \frac{1}{r_n}\right\} + \exp_p\left\{(\mu_{[p,q]}(g,z_0)+\varepsilon)\log_q \frac{1}{r_n}\right\} + \ln 2 \\ &\leq 2\exp_p\left\{\left(\max\left\{\rho_{[p,q]}(f,z_0),\mu_{[p,q]}(g,z_0)\right\}+\varepsilon\right)\log_q \frac{1}{r_n}\right\} + \ln 2 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} T_{z_0}(r_n,fg) &\leq T_{z_0}(r_n,f) + T_{z_0}(r_n,g) \\ &\leq 2\exp_p\left\{\left(\max\left\{\rho_{[p,q]}(f,z_0),\mu_{[p,q]}(g,z_0)\right\}+\varepsilon\right)\log_q \frac{1}{r_n}\right\}. \end{aligned} \tag{2.6}$$

Since $\varepsilon > 0$ is arbitrary, from (2.5) and (2.6), we easily obtain

$$\mu_{[p,q]}(f+g,z_0) \leq \max\left\{\rho_{[p,q]}(f,z_0),\mu_{[p,q]}(g,z_0)\right\} \tag{2.7}$$

and

$$\mu_{[p,q]}(fg,z_0) \leq \max\left\{\rho_{[p,q]}(f,z_0),\mu_{[p,q]}(g,z_0)\right\}. \tag{2.8}$$

Assume now that $\mu_{[p,q]}(g,z_0) > \rho_{[p,q]}(f,z_0)$. Considering that

$$T_{z_0}(r,g) = T_{z_0}(r,f+g-f) \leq T_{z_0}(r,f+g) + T_{z_0}(r,f) + \ln 2 \tag{2.9}$$

and

$$T_{z_0}(r,g) = T_{z_0}\left(r, \frac{fg}{f}\right) \leq T_{z_0}(r, fg) + T_{z_0}\left(r, \frac{1}{f}\right)$$
$$= T_{z_0}(r, fg) + T_{z_0}(r, f) + O(1). \tag{2.10}$$

By (2.9), (2.10) and the same method as above we obtain that

$$\mu_{[p,q]}(g, z_0) \leq \max\left\{\mu_{[p,q]}(f+g, z_0), \rho_{[p,q]}(f, z_0)\right\} = \mu_{[p,q]}(f+g, z_0), \tag{2.11}$$

$$\mu_{[p,q]}(g, z_0) \leq \max\left\{\mu_{[p,q]}(fg, z_0), \rho_{[p,q]}(f, z_0)\right\} = \mu_{[p,q]}(fg, z_0). \tag{2.12}$$

Combining the inequalities (2.7) and (2.11), we conclude that

$$\mu_{[p,q]}(f+g, z_0) = \mu_{[p,q]}(g, z_0)$$

and similarly, using (2.8) and (2.12), we obtain

$$\mu_{[p,q]}(fg, z_0) = \mu_{[p,q]}(g, z_0).$$

□

Lemma 2.8. *([19]) Let $g : (0,1) \to \mathbb{R}$, $h : (0,1) \to \mathbb{R}$ be monotone decreasing functions such that $g(r) \geq h(r)$ possibly outside an exceptional set $E_6 \subset (0,1)$ that has finite logarithmic measure. Then, for any given $\beta > 1$, there exists a constant $0 < r_4 < 1$ such that for all $r \in (0, r_4)$, we have $g(r^\beta) \geq h(r)$.*

3. Proof of Theorems 1.9- 1.11

3.1. Proof of Theorem 1.9.

Proof. Suppose that f ($\not\equiv 0$) is a solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. Set

$$\rho = \max_{2\leq j\leq k-1}\{\rho_{[p,q]}(A_j, z_0), \mu_{[p,q]}(A_1, z_0)\} < \mu_{[p,q]}(A_0, z_0) = \mu_0.$$

For any given ε $(0 < \varepsilon < \frac{\mu_0 - \rho}{2})$, there exists $r_5 \in (0,1)$, such that for all $|z - z_0| = r \in (0, r_5)$

$$|A_j(z)| \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho+\varepsilon}\right\},\ j = 2, .., k-1 \tag{3.1}$$

and

$$|A_0(z)| \geq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_0-\varepsilon}\right\}. \tag{3.2}$$

By Lemma 2.1, there exists a set $E_1 \subset (0, r_5)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E_1$,

$$|A_1(z)| \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_1,z_0)+\varepsilon}\right\} \leq \exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\rho+\varepsilon}\right\}. \tag{3.3}$$

We rewrite (1.1) as

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \left|\frac{f^{(k-1)}(z)}{f(z)}\right| |A_{k-1}(z)| + \cdots + \left|\frac{f'(z)}{f(z)}\right| |A_1(z)|. \tag{3.4}$$

By Lemma 2.2, there exist a set $E_2 \subset (0, r_5]$ that has a finite logarithmic measure and a constant $\lambda > 0$ that depends on $\alpha > 1$ and $j = 1, 2, ..., k$ such that for all $r = |z - z_0|$ satisfying $r \in (0, r_5] \setminus E_2$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \lambda \left[\frac{1}{r^2} T_{z_0}\left(\frac{1}{\alpha} r, f\right) \log T_{z_0}\left(\frac{1}{\alpha} r, f\right)\right]^j, (j = 1, 2, ..., k). \tag{3.5}$$

Substituting (3.1), (3.2), (3.3) and (3.5) into (3.4), we obtain that for every

$$|z - z_0| = r \in (0, r_5) \cap E_1 \cap (0, r_5] \setminus E_2 = E_1 \setminus E_2,$$

which is a set of infinite logarithmic measure, the following inequality holds

$$\exp_p\left\{\left(\log_{q-1} \frac{1}{r}\right)^{\mu_0 - \varepsilon}\right\} \leq \lambda k \left[\frac{1}{r^2} T_{z_0}(\frac{1}{\alpha} r, f) \log T_{z_0}(\frac{1}{\alpha} r, f)\right]^k \exp_p\left\{\left(\log_{q-1} \frac{1}{r}\right)^{\rho + \varepsilon}\right\}. \tag{3.6}$$

Since ε is chosen such that $0 < \varepsilon < \frac{\mu_0 - \rho}{2}$, then from (3.6), for all $|z - z_0| = r \in E_1 \setminus E_2$, we obtain

$$\exp\left\{(1 - o(1)) \exp_{p-1}\left\{\left(\log_{q-1} \frac{1}{r}\right)^{\mu_0 - \varepsilon}\right\}\right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\frac{1}{\alpha} r, f)\right]^{2k}.$$

Hence, by applying Lemma 2.8, we conclude that $\mu_{[p,q]}(f, z_0) = +\infty$ and $\rho_{[p+1,q]}(f, z_0) \geq \mu_{[p,q]}(A_0, z_0)$. □

3.2. **Proof of Theorem 1.10.**

Proof. Suppose that f $(\not\equiv 0)$ is a solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. In the following, we divide the proof into four cases:

(i) We suppose that $\rho < \mu_{[p,q]}(A_0, z_0)$. Then, by Theorem 1.9, we obtain $\rho_{[p,q]}(f, z_0) = +\infty$ and $\rho_{[p+1,q]}(f, z_0) \geq \mu_{[p,q]}(A_0, z_0)$.

(ii) We suppose that $\max\limits_{2 \leq j \leq k-1}\{\rho_{[p,q]}(A_j, z_0)\} = \alpha < \mu_{[p,q]}(A_1, z_0) = \mu_{[p,q]}(A_0, z_0)$ and $\underline{\tau}_{[p,q],M}(A_1, z_0) < \underline{\tau}_{[p,q],M}(A_0, z_0) = \underline{\tau}$. By the definition of $\rho_{[p,q]}(A_j, z_0)$ for any given $\varepsilon (> 0)$, there exists $r_6 \in (0, 1)$ such that for all $|z - z_0| = r \in (0, r_6)$, we have

$$|A_j(z)| \leq \exp_p\left\{\left(\log_{q-1} \frac{1}{r}\right)^{\alpha + \varepsilon}\right\}, \ j = 2, .., k - 1. \tag{3.7}$$

By the definition of $\underline{\tau}_{[p,q],M}(A_0, z_0)$, for sufficiently small $\varepsilon > 0$, there exists $r_7 \in (0, r_6)$ such that for all $|z - z_0| = r \in (0, r_7)$, we have

$$|A_0(z)| \geq \exp_p\left\{(\underline{\tau} - \varepsilon)\left(\log_{q-1} \frac{1}{r}\right)^{\mu_{[p,q]}(A_0, z_0)}\right\}. \tag{3.8}$$

By the definition of $\underline{\tau}_{[p,q],M}(A_1, z_0)$ and Lemma 2.3, there exists a set $E_3 \subset (0, r_7)$ having infinite logarithmic measure such that for all $|z - z_0| = r \in E_3$, we have

$$\begin{aligned} |A_1(z)| &\leq \exp_p\left\{(\underline{\tau}_{[p,q],M}(A_1, z_0) + \varepsilon)\left(\log_{q-1} \frac{1}{r}\right)^{\mu_{[p,q]}(A_1, z_0)}\right\} \\ &= \exp_p\left\{(\underline{\tau}_{[p,q],M}(A_1, z_0) + \varepsilon)\left(\log_{q-1} \frac{1}{r}\right)^{\mu_{[p,q]}(A_0, z_0)}\right\}. \end{aligned} \tag{3.9}$$

By Lemma 2.2, there exist a set $E_2 \subset (0, r_6]$ of finite logarithmic measure and a constant $\lambda > 0$, depending on $\alpha > 1$ and $j = 1, 2, ..., k$ such that the inequality (3.5) holds for all $r = |z - z_0|$ satisfying $r \in (0, r_6] \setminus E_2$. By substituting (3.5) and (3.7)-(3.9) into (3.4), we obtain that for every

$$|z - z_0| = r \in (0, r_6) \cap (0, r_7) \cap E_3 \cap (0, r_6] \setminus E_2 = E_3 \setminus E_2,$$

which is a set of infinite logarithmic measure, the following inequality holds

$$\exp_p \left\{ (\underline{\tau} - \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \right\} \leq \left[1 + (k-2) \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\alpha + \varepsilon} \right\} \right.$$
$$\left. + \exp_p \left\{ (\underline{\tau}_{[p,q],M}(A_1, z_0) + \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \right\} \right] \lambda \left[\frac{1}{r} T_{z_0}(\frac{1}{\alpha} r, f) \right]^{2k}. \tag{3.10}$$

Now, we may choose a sufficiently small $\varepsilon, 0 < 2\varepsilon < \min\{\mu_{[p,q]}(A_0, z_0) - \alpha, \underline{\tau} - \underline{\tau}_{[p,q],M}(A_1, z_0)\}$. Then, from (3.10) for all $|z - z_0| = r \in E_3 \setminus E_2$, we obtain

$$\exp \left\{ (1 - o(1)) \exp_{p-1} \left\{ (\underline{\tau} - \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \right\} \right\} \leq k\lambda \left[\frac{1}{r} T_{z_0}(\frac{1}{\alpha} r, f) \right]^{2k}.$$

Hence, by applying Lemma 2.8, we conclude that $\rho_{[p,q]}(f, z_0) = +\infty$ and $\rho_{[p+1,q]}(f, z_0) \geq \mu_{[p,q]}(A_0, z_0)$.

(iii) We suppose that $\mu_{[p,q]}(A_1, z_0) < \max_{2 \leq j \leq k-1} \{\rho_{[p,q]}(A_j, z_0)\} = \mu_{[p,q]}(A_0, z_0)$ and $\tau_1 = \max_{2 \leq j \leq k-1} \{\tau_{[p,q],M}(A_j, z_0) : \rho_{[p,q]}(A_j, z_0) = \mu_{[p,q]}(A_0, z_0)\} < \underline{\tau}_{[p,q],M}(A_0, z_0) = \underline{\tau}$. By the definitions of $\rho_{[p,q]}(A_j, z_0)$ and $\tau_{[p,q],M}(A_j, z_0)$, for any given $\varepsilon > 0$, there exists $r_8 \in (0, r_7)$ such that for all $|z - z_0| = r \in (0, r_8)$, we have

$$|A_j(z)| \leq \begin{cases} \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\rho_{[p,q]}(A_j, z_0) + \varepsilon} \right\} \leq \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0) - \varepsilon} \right\}, \\ \qquad \text{if } \rho_{[p,q]}(A_j, z_0) < \mu_{[p,q]}(A_0, z_0), \ j \neq 0, 1, \\ \exp_p \left\{ (\tau_1 + \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \right\}, \\ \qquad \text{if } \rho_{[p,q]}(A_j, z_0) = \mu_{[p,q]}(A_0, z_0), \ j \neq 0, 1. \end{cases} \tag{3.11}$$

By Lemma 2.1, there exists a set $E_1 \subset (0, r_7)$ of infinite logarithmic measure such that the inequality (3.3) holds for all $|z - z_0| = r \in E_1$. Substituting (3.3), (3.5), (3.8) and (3.11) into (3.4), we obtain that for all

$$|z - z_0| = r \in E_1 \cap (0, r_8) \cap (0, r_7) \cap (0, r_6] \setminus E_2 = E_1 \setminus E_2,$$

which is still a set of infinite logarithmic measure, the following inequality holds

$$\exp_p \left\{ (\underline{\tau} - \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \right\} \leq \left[1 + O \left(\exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0) - \varepsilon} \right\} \right) \right.$$
$$\left. + O \left(\exp_p \left\{ (\tau_1 + \varepsilon) \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_0, z_0)} \right\} \right) + \exp_p \left\{ \left(\log_{q-1} \frac{1}{r} \right)^{\mu_{[p,q]}(A_1, z_0) + \varepsilon} \right\} \right]$$
$$\times \lambda \left[\frac{1}{r} T_{z_0}(\frac{1}{\alpha} r, f) \right]^{2k}. \tag{3.12}$$

Now, we may choose a sufficiently small ε satisfying

$$0 < 2\varepsilon < \min\{\mu_{[p,q]}(A_0, z_0) - \mu_{[p,q]}(A_1, z_0), \underline{\tau} - \tau_1\}.$$

Then, from (3.12) for $|z - z_0| = r \in E_1 \setminus E_2$, we obtain

$$\exp\left\{(1 - o(1)) \exp_{p-1}\left\{(\underline{\tau} - \varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\right\} \leq k\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r, f)\right]^{2k}.$$

Applying Lemma 2.8, it follows that $\mu_{[p,q]}(f, z_0) = +\infty$ and $\rho_{[p+1,q]}(f, z_0) \geq \mu_{[p,q]}(A_0, z_0)$.
(iv) We suppose that $\max_{2\leq j\leq k-1}\{\rho_{[p,q]}(A_j, z_0)\} = \mu_{[p,q]}(A_1, z_0) = \mu_{[p,q]}(A_0, z_0)$ and

$$\max_{2\leq j\leq k-1}\left\{\underline{\tau}_{[p,q],M}(A_1, z_0), \tau_{[p,q],M}(A_j, z_0) : \rho_{[p,q]}(A_j, z_0) = \mu_{[p,q]}(A_0, z_0)\right\}$$

$$= \tau_2 < \underline{\tau}_{[p,q],M}(A_0, z_0) = \underline{\tau}.$$

Substituting (3.5), (3.8), (3.9) and (3.11) into (3.4), we obtain that for every

$$|z - z_0| = r \in E_3 \cap (0, r_7) \cap (0, r_8) \cap (0, r_6] \setminus E_2 = E_3 \setminus E_2,$$

which is a set of infinite logarithmic measure, the following inequality holds

$$\exp_p\left\{(\underline{\tau} - \varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\} \leq [1+$$

$$O\left(\exp_p\left\{(\tau_2 + \varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\right)$$

$$+O\left(\exp_p\left\{\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)-\varepsilon}\right\}\right)$$

$$+ \exp_p\left\{(\underline{\tau}_{[p,q],M}(A_1, z_0) + \varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\Bigg]\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r, f)\right]^{2k}. \tag{3.13}$$

Now, we may choose a sufficiently small ε satisfying $0 < 2\varepsilon < \underline{\tau} - \tau_2$. Then from (3.13) for $|z - z_0| = r \in E_3 \setminus E_2$, we obtain

$$\exp\left\{(1 - o(1)) \exp_{p-1}\left\{(\underline{\tau} - \varepsilon)\left(\log_{q-1}\frac{1}{r}\right)^{\mu_{[p,q]}(A_0,z_0)}\right\}\right\} \leq k\lambda\left[\frac{1}{r}T_{z_0}(\frac{1}{\alpha}r, f)\right]^{2k}.$$

By applying Lemma 2.8, this implies $\mu_{[p,q]}(f, z_0) = +\infty$ and

$$\rho_{[p+1,q]}(f, z_0) \geq \mu_{[p,q]}(A_0, z_0).$$

□

3.3. **Proof of Theorem 1.11.**

Proof. Suppose there exists $s \in \{1, ..., k-1\}$ such that

$$\max_{\substack{1\leq j\leq k-1\\ j\neq s}}\{\rho_{[p,q]}(A_j, z_0), \mu_{[p,q]}(A_0, z_0)\} < \mu_{[p,q]}(A_s, z_0).$$

Assume that $f(z) \not\equiv 0$ is a rational solution of (1.1) which is analytic in $\overline{\mathbb{C}}-\{z_0\}$. Then $f^{(s)}(z) \not\equiv 0$ and by (1.1), we have

$$A_s(z)\, f^{(s)}(z) = -f^{(k)}(z) - \sum_{\substack{j=0\\ j\neq s}}^{k-1} A_j(z)\, f^{(j)}(z)\,.$$

By Lemma 2.7, it follows that

$$\begin{aligned}\mu(A_s, z_0) = \mu(A_s f^{(s)}, z_0) &= \mu\left(-f^{(k)} - \sum_{j=0,\ j\neq s}^{k-1} A_j f^{(j)}, z_0\right)\\ &\leq \max_{\substack{1\leq j\leq k-1\\ j\neq s}}\{\rho_{[p,q]}(A_j, z_0), \mu_{[p,q]}(A_0, z_0)\},\end{aligned}$$

which is a contradiction. Hence, f must be a transcendental.

Suppose that f is a transcendental solution of equation (1.1) which is analytic in $\overline{\mathbb{C}} - \{z_0\}$. The equation (1.1) yields

$$m_{z_0}(r, A_s) \leq \sum_{j=0,\ j\neq s}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{j=0,\ j\neq s}^{k-1} m_{z_0}(r, A_j) + \log k. \tag{3.14}$$

By Lemma 2.4, there exists a set $E_4 \subset (0, r_1]$ for a fixed $r_1 \in (0, 1)$ of finite logarithmic measure such that for all $|z - z_0| = r \in (0, r_1] \setminus E_4$, we have

$$T_{z_0}(r, f') < O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right).$$

Consequently

$$T_{z_0}(r, f^{(j)}) < O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right).$$

Then, it follows

$$\sum_{j=0,\ j\neq s}^{k} m_{z_0}\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \leq O\left(T_{z_0}(r, f) + \log\frac{1}{r}\right). \tag{3.15}$$

By Lemma 2.5, there exists $r_2 \in (0, r_1)$ such that for all $|z - z_0| = r \in (0, r_2)$ the following holds

$$\lim_{r\to 0}\frac{T_{z_0}(r, A_j)}{T_{z_0}(r, A_s)} = 0,\ \ j = 1, ..., k-1, j \neq s,$$

so for any given $\varepsilon \in \left(0, \frac{1}{2(k-1)}\right)$

$$m_{z_0}(r, A_j) \leq \varepsilon m_{z_0}(r, A_s),\ \ j = 1, ..., k-1, j \neq s. \tag{3.16}$$

Also, by applying Lemma 2.6, for any $\varepsilon \in (0, \frac{1}{2(k-1)})$, there exists a set $E_5 \subset (0, r_2)$ with infinite logarithmic measure such that for all $|z - z_0| = r \in E_5$

$$m_{z_0}(r, A_0) \leq \varepsilon m_{z_0}(r, A_s). \tag{3.17}$$

Combining (3.14), (3.15), (3.16) and (3.17), we obtain that for every

$$|z - z_0| = r \in E_5 \cap (0, r_2) \cap (0, r_1] \setminus E_4 = E_5 \setminus E_4,$$

which is a set of infinite logarithmic measure, the following holds

$$m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right) + \sum_{j=0,\ j \neq s}^{k-1} \varepsilon m_{z_0}(r, A_s) + \log k$$

$$= O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right) + \varepsilon\, (k-1)\, m_{z_0}(r, A_s) + \log k.$$

So

$$(1 - (k-1)\,\varepsilon)\, m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right). \tag{3.18}$$

Since $\varepsilon \in \left(0, \frac{1}{2(k-1)}\right)$, we have

$$1 - (k-1)\,\varepsilon > 1 - (k-1)\,\frac{1}{2\,(k-1)} = \frac{1}{2}.$$

Then, from (3.18) and using the fact that f is transcendental, for all $|z - z_0| = r \in E_5 \setminus E_4$, we obtain

$$\frac{1}{2} m_{z_0}(r, A_s) \leq O\left(T_{z_0}(r, f) + \log \frac{1}{r}\right) = O\left(T_{z_0}(r, f)\right).$$

Hence, by applying Lemma 2.8, we conclude that $\rho_{[p,q]}(f, z_0) \geq \mu_{[p,q]}(f, z_0) \geq \mu_{[p,q]}(A_s, z_0)$. □

4. Examples

Here, we provide some examples that illustrate all what we did before.

Example 4.1. Consider the third-order linear differential equation equation

$$f''' + A_2(z) f'' + A_1(z) f' + A_0(z) f = 0,$$

with coefficients

$$A_0(z) = \left(\frac{51}{z^6} + \frac{99}{z^9} + \frac{27}{z^{12}}\right) \exp\left\{\frac{1}{z^3}\right\}$$
$$+ \left(\frac{108}{z^9} + \frac{81}{z^{12}}\right) \exp\left\{\frac{2}{z^3}\right\} + \frac{27}{z^{12}} \exp\left\{\frac{3}{z^3}\right\}$$
$$A_1(z) = \frac{1}{z^2},\ \ A_2(z) = \frac{1}{z}.$$

A straightforward computation shows that

$$\mu_{[1,1]}(A_0, 0) = 3,\ \ \mu_{[1,1]}(A_1, 0) = 0,\ \ \rho_{[1,1]}(A_2, 0) = 0,$$

so that

$$\max\left\{\rho_{[1,1]}(A_2, 0), \mu_{[1,1]}(A_1, 0)\right\} = 0 < \mu_{[1,1]}(A_0, 0) = 3.$$

Thus, $A_0\,(z)$ is the dominant coefficient, and all assumptions of Theorem 1.9 are satisfied. Moreover, the nontrivial solution

$$f(z) = \exp_2\left\{\frac{1}{z^3}\right\},$$

is analytic in $\overline{\mathbb{C}} \setminus \{0\}$, and satisfies

$$\rho_{[1,1]}(f,0) = +\infty$$

and

$$\rho_{[2,1]}(f,0) = 3 \geq \mu_{[1,1]}(A_0,0) = 3.$$

Example 4.2. Let $f(z) = \frac{1}{z^2}\exp_2\left\{\frac{1}{z^3}\right\}$, analytic in $\overline{\mathbb{C}} \setminus \{0\}$. Then f satisfies the third-order linear differential equation

$$f''' + A_2(z)f'' + A_1(z)f' + A_0(z)f = 0,$$

where the coefficients $A_j(z)$ $(j = 0,1,2)$ are given by

$$A_0(z) = \frac{18}{z^{12}}\exp\left\{\frac{3}{z^3}\right\} + \left(\frac{72}{z^{12}} + \frac{111}{z^9}\right)\exp\left\{\frac{2}{z^3}\right\}$$
$$+ \left(\frac{27}{z^{12}} + \frac{162}{z^9} + \frac{162}{z^6}\right)\exp\left\{\frac{1}{z^3}\right\} + \frac{24}{z^3},$$
$$A_1(z) = -\frac{9}{z^5}\exp\left\{\frac{1}{z^3}\right\}$$

and

$$A_2(z) = \frac{1}{z^4}\exp\left\{\frac{1}{z^3}\right\}.$$

A simple computation of the $[1,1]$-order and lower $[1,1]$-order, $[1,1]$-type and lower $[1,1]$-type shows that

$$\max\left\{\rho_{[1,1]}(A_2,0), \mu_{[1,1]}(A_1,0)\right\} = \max\{3,3\}$$
$$= 3 = \mu_{[1,1]}(A_0,0)$$

and

$$\max\left\{\tau_{[1,1]}(A_2,0), \underline{\tau}_{[1,1]}(A_1,0)\right\} = 1 < \underline{\tau}_{[1,1]}(A_0,0) = 3.$$

Hence, the assumptions of Theorem 1.10 are satisfied. Moreover, we observe that

$$\rho_{[1,1]}(f,0) = +\infty$$

and

$$\rho_{[2,1]}(f,0) = 3 \geq \mu_{[1,1]}(A_0,0) = 3.$$

Example 4.3. Consider the third-order linear differential equation

$$f''' - 2e^{-\frac{1}{z}}f'' + \left(\frac{3}{z} - \frac{4}{z^2} - \frac{5}{z^3}\right)f' + \left(\frac{1}{z^3} + \frac{2}{z^4}\right)f = 0.$$

This equation admits the analytic solution $f(z) = e^{\frac{1}{z}} + 2$ in $\overline{\mathbb{C}} \setminus \{0\}$. A direct computation shows that the coefficients satisfy

$$\mu_{[1,1]}(A_2,0) = 1, \rho_{[1,1]}(A_1,0) = \mu_{[1,1]}(A_0,0) = 0,$$

so that

$$\max\left\{\rho_{[1,1]}(A_1,0), \mu_{[1,1]}(A_0,0)\right\} = 0 < \mu_{[1,1]}(A_2,0) = 1,$$

indicating that A_2 is the dominant coefficient. By Theorem 1.11, it follows that

$$\rho_{[1,1]}(f,0) = 1 \geq \mu_{[1,1]}(A_2,0) = 1.$$

Competing interests: The authors declare that there is no conflict of interest regarding the publication of this paper.

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