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Inverse Sturm-Liouville Problems on Bounded Time Scales

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Abstract. In this study, we introduce some properties of the Sturm-Liouville boundary value problem on bounded time scales. Then, to give the solution of inverse problems, uniqueness theorems are provided for two characteristic data: the Weyl function and a set of j-th eigenvalues of countably infinite Sturm-Liouville boundary value problems that are obtained by changing boundary conditions.

1. Introduction

A time scale is an arbitrary closed subset of the $\mathbb{R}$. Time scale theory was first considered by Hilger S. [1]. The theory unifies discrete and continuous analysis. The direct problem for Sturm-Liouville boundary value problems on time scales was studied in some publications (see [2–5] and references therein). However, there are only a few works on the inverse problem on time scales. Some of them are as follows: Yurko [6] considered the time scale $\mathbb{T} = \bigcup_{k=1}^{N}[a_k, b_k], N \geq 2, b_{k-1} < a_k \leq b_k < a_{k+1}, a_1 < b_1, a_N < b_N, a_k = b_k, k = \overline{2, N-1}$ and proved a uniqueness theorem for the inverse problem using its Weyl-type function and its spectra [7]. Then, he gave an algorithm for the solution to the problem. Kuznetsova [8] considered the problem with some special boundary conditions on the time scale $\mathbb{T} = \bigcup_{l=1}^{N+M}[a_l, b_l], N < \infty, M < \infty, a_{l-1} \leq b_{l-1} < a_l \leq b_l, l = \overline{2, N+M}, a_l < b_l \quad iff \quad l \in \{l_k\}_{k=1}^{N}$. She proved a theorem that included the uniqueness for three different data (Weyl function, two spectra, and one spectrum with its corresponding weight numbers). Also, She gave asymptotic formulae for eigenvalues and weight numbers for her problem. Adalar and Özkan [9] studied Half-inverse Sturm-Liouville problem on a form of time scales.

In this paper, we consider the inverse Sturm-Liouville problems $L[q, h, H]$ of the form

$$\ell y := -y^{\triangle\triangle}(t) + q(t)y(\sigma(t)) = \lambda y(\sigma(t)) \quad ; \quad t \in \mathbb{T}^{0^2} \quad , \tag{1}$$

with the separable boundary conditions

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$$U(y) := y^{\triangle}(\sigma(a_0^0)) - hy(\sigma(a_0^0)) = 0, \tag{2}$$

$$V(y) := y^{\triangle}(\sigma_-(a_n^{s_n})) + Hy(\sigma_-(a_n^{s_n})) = 0, \tag{3}$$

where $q \in C_{rd}(\mathbb{T})$ is a reel function, λ is a spectral parameter, h and H are real numbers, and $\mathbb{T}$ is a bounded time scale of the form

$$\mathbb{T} := \left(\bigcup_{i=0}^{n} S_i\right) \cup \left(\bigcup_{i=1}^{n} [a_i, b_i]\right),$$

where $S_i = \{a_i^0, a_i^1, a_i^2, ..., a_i^{s_i}\}$, $a_i < b_i$, $a_i^{j_i} < a_i^{j_i+1}$ for $-1 \leq j_i \leq s_i$ in which $a_i^{s_i+1} = a_{i+1}, a_i^{-1} = b_i$. For the sake of simplicity, we set $a = \sigma(a_0^0), b = \sigma_-(a_n^{s_n})$.

Properties of the general problem with separable boundary conditions on bounded time scales are given. Then, using these properties, a uniqueness theorem is obtained using the Weyl function data under the given condition. Finally, a second theorem is presented using data not previously studied on time scales. The paper is organized as follows: Section 2 introduces some fundamental concepts about the time scale theory and gives some basic properties of the problem. Section 3 proves some related lemmas. Then, the uniqueness theorems are presented.

2. Preliminaries

We provide some fundamental concepts about the time scale theory from our references and therein as follows:

Let $\mathbb{T}$ be an arbitrary time scale. Forward and backward jump operators on the $\mathbb{T}$ are defined as follows:

$$\sigma(t) = \begin{cases} \inf\{s \in \mathbb{T} : s > t\} & , t \neq \max \mathbb{T}, \\ \max \mathbb{T} & , t = \max \mathbb{T}, \end{cases},$$

$$\sigma_-(t) = \begin{cases} \sup\{s \in \mathbb{T} : s < t\} & , t \neq \min \mathbb{T}, \\ \min \mathbb{T} & , t = \min \mathbb{T}. \end{cases}$$

A point $t \in \mathbb{T}$ is called right-dense, right-isolated, left-dense, left-isolated, isolated, and dense if $\sigma(t) = t, \sigma(t) > t, \sigma_-(t) = t, \sigma_-(t) < t, \sigma_-(t) < t < \sigma(t)$ and $\sigma_-(t) = t = \sigma(t)$, respectively. A function f on $\mathbb{T}$ is called rd-continuous if it is continuous at all right-dense points and has left-sided limits at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. Denote

$$\mathbb{T}^0 = \begin{cases} \mathbb{T} - \{\sup \mathbb{T}\} & , \text{ if } \sup \mathbb{T} \text{ is left-isolated}, \\ \mathbb{T} & , \text{ otherwise}, \end{cases} ; \mathbb{T}^{0^n} = (\mathbb{T}^{0^{n-1}})^0.$$

Definition 2.1. ([1]) A function f on the $\mathbb{T}$ is called delta-differentiable at $t \in \mathbb{T}$ if, for any $\varepsilon > 0$, there exists a neighborhood $U = (t-\delta, t+\delta) \cap \mathbb{T}$ such that $|(f(\sigma(t)) - f(s)) - f^{\triangle}(t)(\sigma(t) - s)| < \varepsilon$ for all $s \in U$. $f^{\triangle}(t)$ is called the delta-derivative of the function f at the point t. The n-th order delta-derivative of f is defined by $f^{\triangle^n} := (f^{\triangle^{n-1}})^{\triangle}$ on $\mathbb{T}^{k^n}$. The set of rd-continuously n-th order delta-differentiable functions is denoted by $C_{rd}^n(\mathbb{T})$.

Let t_1 and t_2 be right-isolated and right-dense points on $\mathbb{T}$ respectively. If f is a $\triangle$-differentiable function on these points, then

$$f^{\triangle}(t_1) = \frac{f(\sigma(t_1)) - f(t_1)}{\sigma(t_1) - t_1} \quad , \quad f^{\triangle}(t_2) = \lim_{s \to t_2, s > t_2} \frac{f(t_2) - f(s)}{t_2 - s}.$$

In particular, if $t \in \mathbb{T}$ is a dense point and f is a delta-differentiable function at t, then f is differentiable at t and $f^{\triangle}(t) = f'(t)$.

Definition 2.2. ([1]) If $F^{\triangle}(t) = f(t)$, then for any points a and b, the formula $\int_a^b f(t)\triangle t := F(b) - F(a)$, is called $\triangle$-integral of f on $[a, b] \cap \mathbb{T}$.

2.1. **Some properties.** Before presenting our main results, we need to give some properties of Sturm-Liouville problems on time scales.

Definition 2.3. ([11]) Let $\mu(t) = \sigma(t) - t$. The equation

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t) \tag{4}$$

is called regressive provided $p, q, f \in C_{rd}$ such that the condition

$$1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0 \quad for \quad all \quad t \in \mathbb{T}^0$$

holds.

Theorem 2.4. ([11]) *Assume that the Eqn-(4) is regressive. If $t_0 \in \mathbb{T}^k$, then the initial value problem*

$$y^{\triangle\triangle} + p(t)y^{\triangle} + q(t)y = f(t), \quad y(t_0) = y_0, \quad y^{\triangle}(t_0) = y_0^{\triangle},$$

where y_0 and y_1 are the given constant, has a unique solution, which is defined on the whole the time scale $\mathbb{T}$.

Consider the equation

$$y^{\triangle\triangle}(t) + p(t)y^{\triangle}(\sigma(t)) + q(t)y(\sigma(t)) = 0. \tag{5}$$

(5) is regressive if $p, q \in C_{rd}$ and

$$1 + \mu(t)p(t) \neq 0 \quad for \quad all \quad \mathbb{T}.$$

If (5) is regressive, then it is equivalent to a homogenous regressive equation of the form (4).

Theorem 2.5. ([2]) *Consider Eqn-(1) with the boundary conditions*

$$\alpha_1 y(\sigma_-(a)) + \gamma_1 y^{\triangle}(\sigma_-(a)) = 0, \quad \alpha_2 y(b) + \gamma_2 y^{\triangle}(b) = 0,$$

and

$$\alpha_1 y(\sigma_-(a)) + \gamma_1 y^{\triangle}(\sigma_-(a)) = 0, \qquad y(b) = 0,$$

with $(\alpha_1^2 + \gamma_1^2)(\alpha_1^2 + \gamma_1^2) \neq 0$. Let λ_j and $\tilde{\lambda}_j$ be j-th eigenvalues of these two boundary value problems. If $j \in \mathbb{N}$ with $\tilde{\lambda}_j < \infty$, then

$$\lambda_j \leq \tilde{\lambda}_j < \lambda_{j+1}.$$

3. Results

From the definition of $\triangle$-derivative and Eqn-(1), it is seen for $i=\overline{0,n}$ and $-1\le j\le s_i$, that

$$\begin{pmatrix} y(a_i^{j+1}) \\ y^{\triangle}(a_i^{j+1}) \end{pmatrix} = \begin{pmatrix} \alpha_{11}^{i,j} & \alpha_{12}^{i,j} \\ \alpha_{21}^{i,j} & \alpha_{22}^{i,j} \end{pmatrix} \begin{pmatrix} y(a_i^{j}) \\ y^{\triangle}(a_i^{j}) \end{pmatrix}, \tag{6}$$

where $\alpha_{k,l}^{i,j}=(A_i^j)^{k+l-2}(q_i^j-\lambda)^{k-1}+\frac{1+(-1)^{\min k,l}}{2}$, $A_i^j=a_i^{j+1}-a_i^j$, and $q_i^j=q(a_i^j)$.Taking this and the boundary condition (2) into account and moving from the a to the right on the $\mathbb{T}$, for each $i=\overline{0,n-1}$, we have the following boundary values problems respectively:

$$\begin{aligned} y''(x)+(\lambda-q(x))y &= 0, \qquad x\in(a_{i+1},b_{i+1}), \\ y'(a_{i+1})P_1^{s_i}(\lambda)-y(a_{i+1})P_0^{s_i}(\lambda) &= 0, \\ y'(b_{i+1})R_1^{s_{i+1}}(\lambda)+y(b_{i+1})R_0^{s_{i+1}}(\lambda) &= 0, \end{aligned} \tag{7}$$

where

$$\begin{pmatrix} P_1^{s_i} \\ P_0^{s_i} \end{pmatrix} = \beta_{s_i}\begin{pmatrix} y(b_i) \\ y^{\triangle}(b_i) \end{pmatrix} \quad , \quad \begin{pmatrix} R_1^{s_{i+1}} \\ R_0^{s_{i+1}} \end{pmatrix} = {\beta_{s_{i+1}}}^{-1}\begin{pmatrix} y(a_{i+2}) \\ y'(a_{i+2}) \end{pmatrix}, \tag{8}$$

$$\beta_{s_0}:=\prod_{j=1}^{s_0}\begin{pmatrix} \alpha_{11}^{0,s_0+1-j} & \alpha_{12}^{0,s_0+1-j} \\ \alpha_{21}^{0,s_0+1-j} & \alpha_{22}^{0,s_0+1-j} \end{pmatrix}, \beta_{s_i}:=\prod_{j=-1}^{s_i}\begin{pmatrix} \alpha_{11}^{i,s_i-1-j} & \alpha_{12}^{i,s_i-1-j} \\ \alpha_{21}^{i,s_i-1-j} & \alpha_{22}^{i,s_i-1-j} \end{pmatrix},$$

$$\beta_{s_n}:=\prod_{j=-1}^{s_n-2}\begin{pmatrix} \alpha_{11}^{n,s_n-3-j} & \alpha_{12}^{n,s_n-3-j} \\ \alpha_{21}^{n,s_n-3-j} & \alpha_{22}^{n,s_n-3-j} \end{pmatrix}.$$

It can be seen that $\beta_{s_i}^{-1}$ exists since

$$\begin{vmatrix} \alpha_{11}^{i,j} & \alpha_{12}^{i,j} \\ \alpha_{21}^{i,j} & \alpha_{22}^{i,j} \end{vmatrix} = (A_i^j)^2(q(a_i^j-\lambda)+1-(A_i^j)^2(q(a_i^j-\lambda)=1\neq 0.$$

Lemma 3.1. *For $i=\overline{0,n}$, the following formulas hold.*

$$\begin{aligned} \beta_{s_0}[k,l] &= B_{s_0}^{k,l}\left[\lambda^{s_0+k-2}-C_{s_0}^{k,l}\lambda^{s_0+k-3}+O(\lambda^{s_0+k-4})\right], \\ \beta_{s_i}[k,l] &= B_{s_i}^{k,l}\left[\lambda^{s_i+k}-C_{s_i}^{k,l}\lambda^{s_i+k-1}+O(\lambda^{s_i+k-2})\right], \\ \beta_{s_n}[k,l] &= B_{s_n}^{k,l}\left[\lambda^{s_n+k-2}-C_{s_n}^{k,l}\lambda^{s_n+k-3}+O(\lambda^{s_n+k-4})\right], \end{aligned}$$

where

$$\begin{aligned} B_{s_0}^{k,l} &= (-1)^{s_0+k-2}(A_0^1)^{l-2}(A_0^{s_0})^{k-2}\prod_{j=1}^{s_0}(A_0^j)^2, \\ C_{s_0}^{k,l} &= \sum_{j=1}^{s_0-1}(A_0^jA_0^{j+1})^{-1}+\sum_{j=1}^{s_0+k-2}q(a_0^j)+\sum_{j=3-l}^{s_0+k-2}(A_0^j)^{-2}, \end{aligned}$$

$$\begin{aligned}
B_{s_i}^{k,l} &= (-1)^{s_i+k-2}(A_i^{-1})^{l-2}(A_i^{s_i})^{k-2}\prod_{j=-1}^{s_i}(A_i^j)^2,\\
C_{s_i}^{k,l} &= \sum_{j=-1}^{s_i-1}(A_i^jA_i^{j+1})^{-1}+\sum_{j=-1}^{s_i+k-2}q(a_i^j)+\sum_{j=1-l}^{s_i+k-2}(A_i^j)^{-2},\\
B_{s_n}^{k,l} &= (-1)^{s_n+k-2}(A_n^{-1})^{l-2}(A_n^{s_n-2})^{k-2}\prod_{j=-1}^{s_n-2}(A_n^j)^2,\\
C_{s_n}^{k,l} &= \sum_{j=-1}^{s_n-3}(A_n^jA_n^{j+1})^{-1}+\sum_{j=-1}^{s_n+k-4}q(a_n^j)+\sum_{j=1-l}^{s_n+k-4}(A_n^j)^{-2}.
\end{aligned}$$

Proof. By direct calculation, it is seen that the formulae are satisfied when $s_i = 0$. Now, let us assume that they also hold for $s_i = p$ and check it for $s_i = p+1$ as follows:

$$\beta_{p+1} = \begin{pmatrix}\alpha_{11}^{i,p+1} & \alpha_{12}^{i,p+1}\\ \alpha_{21}^{i,p+1} & \alpha_{22}^{i,p+1}\end{pmatrix}\begin{pmatrix}\beta_p[1,1] & \beta_p[1,2]\\ \beta_p[2,1] & \beta_p[2,2]\end{pmatrix}$$

or equivalently

$$\beta_{p+1}[k,l] = \sum_{j=1}^{2}\alpha_{k,j}^{p+1}\beta_p[j,l].$$

Calculating this for the first element of β_{m+1} gives that

$$\begin{aligned}
\beta_{p+1}[1,1] &= \beta_p[1,1] + A_i^{p+1}\beta_p[2,1]\\
&= A_i^{p+1}B_p^{2,1}\left[\lambda^{p+2}-\left(C_p^{2,1}-\frac{B_p^{1,1}}{A_i^{p+1}B_p^{2,1}}\right)\lambda^{p+1}+O(\lambda^p)\right]\\
&= B_{p+1}^{1,1}\left[\lambda^{p+2}-C_{p+1}^{1,1}\lambda^{p+1}+O(\lambda^p)\right],
\end{aligned}$$

since

$$\begin{aligned}
A_i^{p+1}\quad B_p^{2,1} &= (-1)^{(p+1)+1-2}(A_i^0)^{-1}(A_i^{p+1})^{-1}\prod_{j=-1}^{p+1}(A_i^j)^2 = B_{p+1}^{1,1},\\
C_p^{2,1} &- \frac{B_p^{1,1}}{A_i^{p+1}B_p^{2,1}} = C_p^{2,1} - \frac{B_p^{1,1}}{B_{p+1}^{1,1}}\\
&= \sum_{j=-1}^{(p+1)-1}\left(A_i^jA_i^{j+1}\right)^{-1}+\sum_{j=0}^{(p+1)+1-2}q_i^j+\sum_{j=1-1}^{(p+1)+1-2}\left(A_i^j\right)^{-2}\\
&= C_{p+1}^{1,1}.
\end{aligned}$$

Other parts of the matrix can be calculated similarly. □

From (8) and Lemma (3.1), for $i=\overline{1,n-1}$

$$\begin{aligned}
P_1^{s_i}(\lambda) &= \beta_{s_i}[1,1]y(b_i)+\beta_{s_i}[1,2]y^{\triangle}(b_i)\\
&= (B_{s_i}^{1,1}y(b_i)+B_{s_i}^{1,2}y'(b_i))\left[\lambda^{s_i+1}+O(\lambda^{s_i})\right],
\end{aligned}\tag{9}$$

and similarly

$$\begin{aligned} P_0^{s_i}(\lambda) &= (B_{s_i}^{2,1}y(b_i) + B_{s_i}^{2,2}y'(b_i))\left[\lambda^{s_i+2} + O(\lambda^{s_i+1})\right] \\ &= -A_i^{s_i}(B_{s_i}^{1,1}y(b_i) + B_{s_i}^{1,2}y'(b_i))\left[\lambda^{s_i+2} + O(\lambda^{s_i+1})\right]. \end{aligned} \tag{10}$$

In particular,

$$\begin{aligned} P_1^{s_0}(\lambda) &= (B_{s_0}^{1,1} + hB_{s_0}^{1,2})\left[\lambda^{s_0-1} + O(\lambda^{s_0-2})\right], \\ P_0^{s_0}(\lambda) &= (B_{s_0}^{2,1} + hB_{s_0}^{2,2})\left[\lambda^{s_0} + O(\lambda^{s_0-1})\right]. \end{aligned}$$

We show that these functions do not vanish together at the same time.

Lemma 3.2. *The functions $P_j^{s_i}, j = 0, 1$ have no common zeros for each $i = \overline{0, n-1}$.*

Proof. By computing directly from (8), we see for $i = \overline{0, n-1}$ that

$$P_1^{s_i}(\lambda) = P_1^{s_i-1}(\lambda) + A_i^{s_i}P_0^{s_i-1}(\lambda), \tag{11}$$

$$\begin{aligned} P_0^{s_i}(\lambda) &= A_i^{s_i}(q_i^{s_i} - \lambda)P_1^{s_i-1}(\lambda) + ((A_i^{s_i})^2(q_i^{s_i} - \lambda) + 1)P_0^{s_i-1}(\lambda) \\ &= A_i^{s_i}(q_i^{s_i} - \lambda)(P_1^{s_i-1}(\lambda) + A_i^{s_i}P_0^{s_i-1}(\lambda)) + P_0^{s_i-1}(\lambda) \\ &= A_i^{s_i}(q_i^{s_i} - \lambda)P_1^{s_i}(\lambda) + P_0^{s_i-1}(\lambda), \end{aligned} \tag{12}$$

where $P_1^{s_i-1}(\lambda) := y(b_i)$ and $P_0^{s_i-1}(\lambda) := y^{\triangle}(b_i)$ when $S_i = \emptyset$. Assume that $P_1^{s_i}(\lambda)$ and $P_0^{s_i}(\lambda)$ have at least one common zero μ_0. In the other words,

$$P_1^{s_i}(\mu_0) = P_0^{s_i}(\mu_0) = 0.$$

Thus, it is seen from (11) and (12) that the μ_0 must satisfy the following equalities:

$$\begin{aligned} P_1^{s_i-1}(\mu_0) &= P_1^{s_i-2}(\mu_0) = \cdots = P_1^{-1}(\mu_0), \\ P_0^{s_i-1}(\mu_0) &= P_0^{s_i-2}(\mu_0) = \cdots = P_0^{-1}(\mu_0), \end{aligned}$$

which contradicts the uniqueness. □

One can show that $R_1^{s_i}(\lambda)$ and $R_0^{s_i}(\lambda)$ are have no common zeros for each i.
Let the functions $C_i(x, \lambda), S_i(x, \lambda)$, and $\phi_i(x, \lambda)$ for $i = \overline{1, n}$ be solutions to

$$y''(x) + (q(x) - \lambda)y(x) = 0 \quad , \quad x \in (a_i, b_i)$$

with the following initial conditions:

$$\begin{aligned} C_i(a_i, \lambda) - 1 &= C_i'(a_i, \lambda) = S_i'(a_i, \lambda) - 1 = S_i(a_i, \lambda) \\ &= \phi_i(a_i, \lambda) - P_1^{s_{i-1}}(\lambda) = \phi_i'(a_i, \lambda) - P_0^{s_{i-1}}(\lambda) = 0. \end{aligned}$$

It is well-known from the literature that C_i and S_i satisfy the following asymptotic formulas independently of $x \in (a_i, b_i)$:

$$\begin{aligned} C_i(x,\lambda) &= \cos\rho(x-a_i) + O\left[|\rho|^{-1} exp(\tau(x-a_i))\right], \\ S_i(x,\lambda) &= \frac{\sin\rho(x-a_i)}{\rho} + O\left[|\rho|^{-2} exp(\tau(x-a_i))\right], \\ C_i'(x,\lambda) &= -\rho\sin\rho(x-a_i) + O\left[exp(\tau(x-a_i))\right], \\ S_i'(x,\lambda) &= \cos\rho(x-a_i) + O\left[|\rho|^{-1} exp(\tau(x-a_i))\right], \end{aligned}$$

where $\lambda = \rho^2, \tau = |Im\rho|$. We can obtain the function φ as

$$\varphi(x,\lambda) = \varphi_i(x,\lambda) \quad , \quad x \in (a_i, b_i]$$

by solving successive initial value problems for $i = \overline{1,n}$. For $x \in S_i \cup \{a_{i+1}\}$, the function $\varphi(x,\lambda)$ can be determined by using φ_i and (6).

Lemma 3.3. *The following asymptotic formulaes for $|\rho| \to \infty$ hold:*

$$\begin{aligned} \phi_i(x,\lambda) &= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)\prod_{k=1}^{i-1}\cos\rho(b_k-a_k)B_{s_k}^{2,2}\rho^{u_i+2i-5}\sin\rho(x-a_i) \\ &+ O\left[|\rho|^{u_i+2i-6} exp(\tau[(x-a_i) + \sum_{k=1}^{i-1}(b_k-a_k)])\right], \\ \phi_i'(x,\lambda) &= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)\prod_{k=1}^{i-1}\cos\rho(b_k-a_k)B_{s_k}^{2,2}\rho^{u_i+2i-4}\cos\rho(x-a_i) \\ &+ O\left[|\rho|^{u_i+2i-5} exp(\tau[(x-a_i) + \sum_{k=1}^{i-1}(b_k-a_k)])\right], \end{aligned}$$

where $u_i = 2\sum_{k=0}^{i-1}(s_k+1)$ and $i = \overline{1,n}$. We do not consider the part $\prod_{k=1}^{i-1}\cos\rho(b_k-a_k)B_{s_k}^{2,2}$ if i=1.

Proof. By using fundamental solutions for $i = 1$, (9), and (10), we have

$$\begin{aligned} \phi_1(x,\lambda) &= P_1^{s_0}(\lambda)C_1(x,\lambda) + P_0^{s_0}(\lambda)S_1(x,\lambda) \\ &= (B_{s_0}^{1,1} + B_{s_0}^{1,2}h)(\lambda^{s_0-1} + O(\lambda^{s_0-2}))\cos\rho(x-a_1) \\ &+ (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)(\lambda^{s_0} + O(\lambda^{s_0-1}))\frac{\sin\rho(x-a_1)}{\rho} \\ &+ O[|\rho|^{2s_0-2} exp(\tau(x-a_1))] \\ &= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)\rho^{2s_0-1}\sin\rho(x-a_1) + O\left[|\rho|^{2s_0-2} exp(\tau(x-a_1))\right] \\ &= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)\rho^{u_1-3}\sin\rho(x-a_1) + O\left[|\rho|^{u_1-4} exp(\tau(x-a_1))\right]. \end{aligned}$$

and similarly

$$
\begin{aligned}
\phi_1'(x,\lambda) &= P_1^{s_0}(\lambda)C_1'(x,\lambda) + P_0^{s_0}(\lambda)S_1'(x,\lambda) \\
&= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)\rho^{2s_0}\cos\rho(x-a_1) + O\left[|\rho|^{2s_0-1}exp(\tau(x-a_1))\right] \\
&= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)\rho^{u_1-2}\cos\rho(x-a_1) + O\left[|\rho|^{u_1-3}exp(\tau(x-a_1))\right].
\end{aligned}
$$

Assume that the formulaes are hold for $i = n-1$. We show that they also are satisfied for $i = n$.

$$
\begin{aligned}
\phi_n(x,\lambda) &= P_1^{s_{n-1}}(\lambda)C_n(x,\lambda) + P_0^{s_{n-1}}(\lambda)S_n(x,\lambda) \\
&= (B_{s_{n-1}}^{1,1}\phi_{n-1}(b_{n-1},\lambda) + B_{s_{n-1}}^{1,2}\phi_{n-1}'(b_{n-1},\lambda)) \\
&\times (\lambda^{s_{n-1}+1} + O(\lambda^{s_{n-1}}))(\cos\rho(x-a_n) + O[|\rho|^{-1}exp(\tau(x-a_n))]) \\
&+ (B_{s_{n-1}}^{2,1}\phi_{n-1}(b_{n-1},\lambda) + B_{s_{n-1}}^{2,2}\phi_{n-1}'(b_{n-1},\lambda)) \\
&\times (\lambda^{s_{n-1}+2} + O(\lambda^{s_{n-1}+1}))(\frac{\sin\rho(x-a_n)}{\rho} + O[|\rho|^{-2}exp(\tau(x-a_n))]) \\
&= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)(\prod_{k=1}^{n-1}\cos\rho(b_k-a_k)B_{s_k}^{2,2})\rho^{u_n+2n-5}\sin\rho(x-a_n) \\
&+ O\left[|\rho|^{u_n+2n-6}exp(\tau[(x-a_n) + \sum_{k=1}^{n-1}(b_k-a_k)])\right].
\end{aligned}
$$

The formula for $\phi_n'(x,\lambda)$ can be derived by following similar steps. □

From the above calculations, it can also be seen that the following formulas are valid for $i = \overline{0, n-1}$.

$$
\begin{aligned}
P_1^{s_i} &= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)(-A_i^{s_i})^{-1}(\prod_{k=1}^{i}\cos\rho(b_k-a_k)B_{s_k}^{2,2})\rho^{u_{i+1}+2i-4} \\
&+ O\left[|\rho|^{u_{i+1}+2i-5}exp(\tau\sum_{k=1}^{i}(b_k-a_k))\right], \\
P_0^{s_i} &= (B_{s_0}^{2,1} + B_{s_0}^{2,2}h)(\prod_{k=1}^{i}\cos\rho(b_k-a_k)B_{s_k}^{2,2})\rho^{u_{i+1}+2i-2} \\
&+ O\left[|\rho|^{u_{i+1}+2i-3}exp(\tau\sum_{k=1}^{i}(b_k-a_k))\right].
\end{aligned}
$$

Theorem 3.4. *Assume that for each i the functions $R_j^{s_i}(\lambda), j = 0,1$ are polynomials. Then, the Weyl function of (1)-(3), h, and the information about $q(x)$ on isolated points determine H and the potential $q(x)$ on the $\mathbb{T}$ uniquely.*

Proof. Let $M(\lambda)$ be the Weyl functions of the problem (1)-(3). For $i = 0$, $P_0^{s_0}(\lambda)$ and $P_1^{s_0}(\lambda)$ are polynomials which can be obtained directly from (2)-(6) by using the information about $q(a_0^i)$,

$i = \overline{0, s_0}$ and h. So, we see from the reference [10] that the hypotheses are enough to determine $R_0^{s_1}(\lambda)$, $R_1^{s_1}(\lambda)$ and $q(x)$ on the interval (a_1, b_1) uniquely. Taking (8) into account, we have

$$\begin{pmatrix} P_1^{s_1} \\ P_0^{s_1} \end{pmatrix} = \beta_{s_1} \begin{pmatrix} R_1^{s_1} \\ R_0^{s_1} \end{pmatrix}.$$

Thus, $P_1^{s_1}$ and $P_0^{s_1}$ are polynomials and determined uniquely. The Weyl function carries information about the potential on the entire time-scale. By applying these steps successively, we conclude that the potential and H are determined uniquely. □

The above theorem can be easily generalized to the problem involving polynomials of the spectral parameter in the boundary condition.
We give also another classic uniqueness theorem for bounded time scales.

Theorem 3.5. *Let $\{\lambda_n(q_i, \beta)\}, i = 1, 2$ be eigenvalues of problems (1)-(3) with $h = \beta$ and $H = \infty$ on $\mathbb{T}$. Let us fix j as a positive integer and suppose that β_k for $k = 1, 2, ...$ are distinct real numbers. If*

$$\lambda_j(q_1, \beta_k) = \lambda_j(q_2, \beta_k),$$

then $q_1 = q_2$ on $\mathbb{T}$.

Proof. Consider the function

$$F(\lambda) := \psi_1(a_0^0, \lambda)\psi_2^{\triangle}(a_0^0, \lambda) - \psi_2(a_0^0, \lambda)\psi_1^{\triangle}(a_0^0, \lambda).$$

where $\psi_i(x, \lambda)$, i=1,2 are the solutions of Eq.1 for the potentials q_1 and q_2 under the initial conditions $\psi_i(a_n^{s_n}, \lambda) = 0$, $\psi_i^{\triangle}(a_n^{s_n}, \lambda) = 1$. By adding the terms $\beta_k\psi_1(a_0^0)\psi_2(a_0^0) - \beta_k\psi_1(a_0^0\psi_2(a_0^0)$ to the function F it can be seen $F(\lambda_j(q_i, \beta_k)) = 0$. Moreover, under assumption of the theorem we have $F(\lambda) = 0$. The result is obtained from the uniqueness for regressive equations and the references ([8], [12]). □

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