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Donoho-Stark Theorem for the Second Hankel-Clifford Transform

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Abstract. In this work, we obtain an analog of Donoho-Stark theorem for the second Hankel-Clifford transform for functions in $f \in L^1_\mu \cap L^2_\mu$, using the properties of this transform.

1. Introduction and Preliminaries

The uncertainty principle is a fundamental principle of quantum mechanics which states that the more precisely the position of a particle is known, the less precisely its momentum can be known, and vice versa. This principle has far-reaching implications in the physical sciences and beyond, and is a cornerstone of quantum mechanics.

The uncertainty principle was first formulated by German physicist Werner Heisenberg in 1927. He was trying to understand the behavior of particles on the atomic scale and found a mathematical equation that described the uncertainty of position and momentum of a particle. Heisenberg's equation would later be known as the uncertainty principle.

The uncertainty principle can be stated as follows: it is impossible to simultaneously measure the exact position and momentum of a particle. This means that the more accurately a particle's position is known, the less accurately its momentum can be known, and vice versa.

The uncertainty principle has far-reaching implications in the physical sciences and beyond. It is used in the study of chaotic systems, where a small uncertainty in the initial conditions of a system can lead to large uncertainties in its future behavior. It also has implications in the fields of cryptography and quantum computing.

In summary, the uncertainty principle is a fundamental principle of quantum mechanics which states that the more precisely the position of a particle is known, the less precisely its momentum can be known, and vice versa. It was first formulated by German physicist Werner Heisenberg in 1927 and is closely related to wave-particle duality. The uncertainty principle has far-reaching implications in the physical sciences and beyond, and is a cornerstone of quantum mechanics.

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In mathematics, the uncertainty principle is a fundamental principle and also plays an important role in signal processing. It states that a function f and its Fourier transform $\mathcal{F}(f)$ cannot be simultaneously well concentrated, i.e if the supports of a function $f \in L^1(\mathbb{R}^d)$ and its Fourier transform { are contained in bounded rectangles, then f vanishes almost everywhere.
We say that f is Δ-concentrated on a measurable set E if:

$$\|f - \chi_E f\| < \Delta.$$

In [11], Donoho and Stark that if f of unit L^2-norm is Δ_A concentrated on a measurable set A and its Fourier transform $\mathcal{F}$ is Δ_B concentrated on a measurable set B, then:

$$|A||B| \geq (1 - \Delta_A - \Delta_B)^2.$$

Here, $|A|$ denote the Lebesque measure of the set A.
There are various mathematical formulations for this principle as well as extensions to other transforms, see for example [1], [2], [12], [14] and [15], we refer also to the book [16] and the surveys [9], [13] for further references.
In this paper ,our objective will be to continue the work carried out by our research laboratory, ie to prove that the theorem established by Donoho and Stark remains valid for other integral transformation. More precisely, we will study the Donoho-Stark theorem for the second Hankel-Clifford transform.
Let us we briefly collect the pertinent definitions and facts relevant for the second Hankel-Clifford analysis, which can be founded in [7], [8], [17] and [18].
Let L^p_μ denotes the classe of measurable functions of f such that the integral

$$\int_0^{+\infty} |f(t)|^p t^\mu dt,$$

is finite, with $\mu \geq 0$. Also let L^∞_μ be the collection of almost every finite.
Hence, endowwed L^p_μ with the norm

$$\|f\|_{L^p_\mu} = \left(\int_0^{+\infty} |f(t)|^p t^\mu dt \right)^{\frac{1}{p}},$$

if $1 \leq p < +\infty$ and

$$\|f\|_{L^\infty_\mu} = ess \sup |f(t)|,$$

if $p = \infty$.
Betancor [4] and Méndez [5] introduced the first and the second Hankel-Clifford transformations of the function $f \in L^1_\mu$ of order μ by

$$\mathcal{H}_{1,\mu}(\lambda) = \lambda^\mu \int_0^{+\infty} C_\mu(t\lambda) f(t) dt,$$

and

$$\mathcal{H}_{2,\mu}(\lambda) = \int_0^{+\infty} C_\mu(t\lambda) g(t) t^\mu dt.$$

such that $\lambda \in \mathbb{R}^+$ and C_μ defined by

$$C_\mu(t) = t^{\frac{\mu}{2}} J_\mu(2\sqrt{t}).$$

Where J_μ is the Bessel function of the first kind of order μ. If $h_{1,\mu}$ and $h_{2,\mu}$ in L^1_μ, then the first and second inverse Hankel-Clifford transformations are given by

$$f(t) = t^\mu \int_0^{+\infty} C_\mu(t\lambda)\mathcal{H}_{1,\mu}(f)(\lambda)d\lambda,$$

and

$$g(t) = \int_0^{+\infty} C_\mu(t\lambda)\mathcal{H}_{2,\mu}(f)(\lambda)\lambda^\mu d\lambda.$$

Theorem 1.1. *If $f, g \in L^2_\mu$ then we have the Parseval's formula*

$$\int_0^{+\infty} f(y)g(y)y^\mu dy = \int_0^{+\infty} \mathcal{H}_{2,\mu}(f)(t))\mathcal{H}_{2,\mu}(f)(t))t^\mu dt.$$

If $f = g$ we get

$$\int_0^{+\infty} |f(y)|^2 y^\mu dy = \int_0^{+\infty} |\mathcal{H}_{2,\mu}(f)(t)|^2 t^\mu dt \tag{1.1}$$

We have also the following relation which connect the Bessel-Clifford function and the normalized spherical Bessel function:

$$C_\mu(y) = \frac{1}{\Gamma(\mu+1)} j_\mu(2\sqrt{y}),$$

such that:

$$j_\mu(y) = \Gamma(\mu+1)\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(k+\mu+1)}(\frac{y}{2})^{2k},$$

for every $y \in \mathbb{C}$.
From [6] , we have the following lemma:

Lemma 1.2. *Let $\mu \geq -\frac{1}{2}$. The following inequalities are fulfilled:*

(1) $|J_\mu(y)| \leq 1$ *for every* $y \in \mathbb{R}$*;*

(2) $1 - j_\mu(y) = O(x^2)$ *if* $0 \leq y \leq 1$*;*

(3) $1 - j_\mu(y) = O(1)$ *if* $y \geq 1$.

We have the following relation which connect the Bessel-Clifford function and the normalized spherical Bessel function:

$$C_\mu(y) = \frac{1}{\Gamma(\mu+1)} j_\mu(2\sqrt{y}).$$

2. Auxiliary results

We will start by this definition.

Definition 2.1. *Let $f \in L^p_\mu$, we say that f is Δ_E-concentrated on a measurable set E if there is a function g vanishing outside E such that*

$$\|f - g\|_{L^p_\mu} \leq \Delta_E \|f\|_{L^p_\mu}.$$

We defined the orthogonal projection operator as follow:

$$P_E(f(y)) = \begin{cases} f(y), & y \in E, \\ 0, & otherwise. \end{cases}$$

We defined the frequency-limiting operator as follow:

$$\mathcal{H}_{2,\mu}(Q_E f) = P_E \mathcal{H}_{2,\mu}(f).$$

Now, we are going to state and prove two results that will characterize the mapping Δ_E-concentrated on a measurable set E.

Lemma 2.2. *Let $f \in L^2_\mu$ and E is a mesurable set of $\mathbb{R}^+$. Then f is Δ_E-concentrated on E if and only if*

$$\|f - P_E f\|_{L^p_\mu} \leq \Delta_E \|f\|_{L^p_\mu}.$$

Proof. If we have that

$$\|f - P_E f\|_{L^p_\mu} \leq \Delta_E \|f\|_{L^p_\mu},$$

and since that $P_E f$ vanishing outside E we get that f is Δ_E-concentrated on E.
Conversely, if f is Δ_E-concentrated on E, then there is a function g vanishing outside E such that:

$$\|f - g\|_{L^p_\mu} \leq \Delta_E \|f\|_{L^p_\mu}$$

Then:

$$\begin{aligned}
\|f - P_E f\|_{L^p_\mu} &= \left(\int_{\{\mathbb{R}^+\}-E} |f(y)|^p y^\mu dy \right)^{\frac{1}{p}} \\
&= \left(\int_{\{\mathbb{R}^+\}-E} |f(y) - g(y)|^p y^\mu dy \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\mathbb{R}^+} |f(y) - g(y)|^p y^\mu dy \right)^{\frac{1}{p}} \\
&= \|f - g\|_{L^p_\mu} \\
&\leq \Delta_E \|f\|_{L^p_\mu}.
\end{aligned}$$

■

Lemma 2.3. *Let $f \in L^2_\mu$ and F is a mesurable set of $\mathbb{R}^+$. Then $\mathcal{H}_{2,\mu} f$ is Δ_F-concentrated on F if and only if*

$$\|\mathcal{H}_{2,\mu}(f) - \mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} \leq \Delta_F \|f\|_{L^2_\mu}.$$

Proof. If we have that:

$$\|\mathcal{H}_{2,\mu}(f) - \mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} \leq \Delta_F \|f\|_{L^2_\mu},$$

by Parseval identity (1.1) we get

$$\|\mathcal{H}_{2,\mu}(f) - \mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} \leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu},$$

and we have that:

$$\mathcal{H}_{2,\mu}(Q_F f) = P_F \mathcal{H}_{2,\mu}(f).$$

So $\mathcal{H}_{2,\mu}(Q_F f)$ is vanishing outside F, which show that $\mathcal{H}_{2,\mu}(f)$ is Δ_F-concentrated on F. Now if $\mathcal{H}_{2,\mu}(f)$ is Δ_F-concentrated on F, there is a function g vanishing outside F such that:

$$\|\mathcal{H}_{2,\mu}(f) - g\|_{L^2_\mu} \leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

Then

$$\begin{aligned}
\|\mathcal{H}_{2,\mu}(f) - \mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} &= \|\mathcal{H}_{2,\mu}(f) - P_F \mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \\
&= \left(\int_{\{\mathbb{R}^+\}-F} \left|\mathcal{H}_{2,\mu}(f)(t)\right|^2 t^\mu dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\{\mathbb{R}^+\}-F} |\mathcal{H}_{2,\mu}(f)(t) - g(t)|^2 t^\mu dt \right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^+} |\mathcal{H}_{2,\mu}(f)(t) - g(t)|^2 t^\mu dt \right)^{\frac{1}{2}} \\
&= \|\mathcal{H}_{2,\mu}(f) - g\|_{L^2_\mu} \\
&\leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.
\end{aligned}$$

By Parseval relation (1.1) we get

$$\|\mathcal{H}_{2,\mu}(f) - \mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} \leq \Delta_F \|f\|_{L^2_\mu}.$$

■

Now we will move on to see a result that will be useful later.

Lemma 2.4. *Let E and F be a measurable subset of $\mathbb{R}^+$. If $f \in L^2_\mu$ then:*

$$\|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq A^{\frac{\mu}{4}} M^{\frac{\mu}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \|f\|_{L^2_\mu}$$

such that M and A is a constant to be defined.

Proof. We have that:

$$\mathcal{H}_{2,\mu}(Q_F P_E f) = P_F \mathcal{H}_{2,\mu}(P_E f).$$

So

$$\|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} = \left(\int_F |\mathcal{H}_{2,\mu}(P_E f)(\lambda)|^2 \lambda^\mu dt \right)^{\frac{1}{2}}.$$

We have that:

$$\mathcal{H}_{2,\mu}(P_E f)(\lambda) = \int_E C_\mu(t\lambda) f(t) t^\mu dt.$$

Then

$$\mathcal{H}_{2,\mu}(P_E f)(\lambda) = \frac{1}{\Gamma(\mu+1)} \int_E j_u(2\sqrt{t\lambda}) f(t) t^\mu dt.$$

So

$$|\mathcal{H}_{2,\mu}(P_E f)(\lambda)| \leq \frac{1}{|\Gamma(\mu+1)|} \int_E t^\mu f(t) dt.$$

Then by using Hölder's inequality we get:

$$|\mathcal{H}_{2,\mu}(P_E f)(\lambda)| \leq \frac{1}{|\Gamma(\mu+1)|} \left(\int_E \left(t^{\frac{\mu}{2}} f(t)\right)^2 dt \right)^{\frac{1}{2}} \left(\int_E t^{\frac{\mu}{2}} dt \right)^{\frac{1}{2}}.$$

Since E is measurable on $\mathbb{R}^+$, then there exists $A > 0$ such that for every $t \in E$ we have that $t \leq A$ So

$$|\mathcal{H}_{2,\mu}(P_E f)(\lambda)| \leq \frac{1}{|\Gamma(\mu+1)|} A^{\frac{\mu}{4}} \|f\|_{L^2_\mu} |E|^{\frac{1}{2}}.$$

Then

$$\|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq \left(\int_F \left(\frac{1}{|\Gamma(\mu+1)|} A^{\frac{\mu}{4}} \|f\|_{L^2_\mu} |E|^{\frac{1}{2}} \right)^2 \lambda^\mu d\lambda \right)^{\frac{1}{2}}.$$

So

$$\|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq \frac{1}{|\Gamma(\mu+1)|} A^{\frac{\mu}{4}} \|f\|_{L^2_\mu} |E|^{\frac{1}{2}} \left(\int_F \lambda^\mu d\lambda \right)^{\frac{1}{2}}.$$

Since F is mesurable on $\mathbb{R}^+$, then F is bounded. So, there exists $M > 0$ such that

$$\forall \lambda \in F : \lambda \leq M.$$

Which shows that

$$\|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq M^{\frac{\mu}{2}} A^{\frac{\mu}{4}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \|f\|_{L^2_\mu}.$$

■

3. Main results

We now proceed to establish analogous results of the Donoho-Stark theorem for the second Hankel-Clifford transformation.

Theorem 3.1. *Let E and F be a measurable subset of $\mathbb{R}^+$ and $f \in L^2_\mu$. If f is Δ_E-concentrated on E in L^2_μ and $h_{2,\mu}(f)$ is Δ_F-concentrated on F in L^2_μ, then:*

$$\Big(1-(\Delta_E+\Delta_F)\Big)^2 \leq A^{\frac{\mu}{2}} M^{\mu}|E||F|.$$

Such that M and A is defined in Lemma 2.4.

Proof. Let $f \in L^2_\mu$, using the triangle inequality we have:

$$\|\mathcal{H}_{2,\mu}(f)-\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq \|\mathcal{H}_{2,\mu}(f)-\mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} + \|\mathcal{H}_{2,\mu}(Q_F f)-\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu}.$$

Since $\mathcal{H}_{2,\mu}(f)$ is Δ_F-concentrated on F in L^2_μ, then by Lemma 2.3

$$\|\mathcal{H}_{2,\mu}(f)-\mathcal{H}_{2,\mu}(Q_F f)\|_{L^2_\mu} \leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu},$$

and we have that:

$$\begin{aligned}
\|\mathcal{H}_{2,\mu}(Q_F f)-\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} &= \|\mathcal{H}_{2,\mu}\Big(Q_F(f-P_E f)\Big)\|_{L^2_\mu}\\
&= \Big(\int_F \Big|\mathcal{H}_{2,\mu}(f-P_E f)(t)\Big|^2 t^\mu dt\Big)^{\frac{1}{2}}\\
&\leq \Big(\int_{\mathbb{R}^+} \Big|\mathcal{H}_{2,\mu}(f-P_E f)(t)\Big|^2 t^\mu dt\Big)^{\frac{1}{2}}\\
&\leq \|\mathcal{H}_{2,\mu}(f)-\mathcal{H}_{2,\mu}(P_E f)\|_{L^2_\mu}.
\end{aligned}$$

By using Parseval ineaquality (1.1) we get that:

$$\|\mathcal{H}_{2,\mu}(Q_F f)-\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq \|f-P_E f\|_{L^2_\mu}.$$

Since f is Δ_E-concentrated on E in L^2_μ, then

$$\|f-P_E f\|_{L^2_\mu} \leq \Delta_E \|f\|_{L^2_\mu}.$$

So by using Parseval inequality (1.1) we get

$$\|\mathcal{H}_{2,\mu}(f)-\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq \Big(\Delta_E+\Delta_F\Big)\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

On the other hand by the triangular inequality

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq \|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} + \|\mathcal{H}_{2,\mu}(f)-\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu}.$$

By using Lemma 2.4 we get:

$$\|\mathcal{H}_{2,\mu}(Q_F P_E f)\|_{L^2_\mu} \leq A^{\frac{\mu}{4}} M^{\frac{\mu}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \|f\|_{L^2_\mu}.$$

So:

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq A^{\frac{\mu}{4}} M^{\frac{\mu}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \|f\|_{L^2_\mu} + \Big(\Delta_E + \Delta_F\Big) \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

By using Parseval equality we get

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq A^{\frac{\mu}{4}} M^{\frac{\mu}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} + \Big(\Delta_E + \Delta_F\Big) \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

Then:

$$1 \leq A^{\frac{\mu}{4}} M^{\frac{\mu}{2}} |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} + \Big(\Delta_E + \Delta_F\Big).$$

Which shows that:

$$\Big(1 - (\Delta_E + \Delta_F)\Big)^2 \leq A^{\frac{\mu}{2}} M^{\mu} |E||F|.$$

■

Theorem 3.2. *Let E and F be a measurable subset of $\mathbb{R}^+$and $f \in L^1_\mu \cap L^2_\mu$. If f is Δ_E-concentrated on E in L^1_μ and $\mathcal{H}_{2,\mu}(f)$ is Δ_F-concentrated on F in L^2_μ norm, then:*

$$(1 - \Delta_E)^2 (1 - \Delta_F)^2 \leq \frac{A^\mu M^\mu}{|\Gamma(\mu+1)|^2} |E||F|.$$

Such that A and M is defined in Lemma 2.4.

Proof. Using the triangular inequality we get:

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq \|\mathcal{H}_{2,\mu}(f) - P_F \mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} + \|P_F \mathcal{H}_{2,\mu}(f)\|_{L^2_\mu},$$

and we have:

$$\|P_F \mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} = \left(\int_F |\mathcal{H}_{2,\mu}(f)(t)|^2 t^\mu dt \right)^{\frac{1}{2}}.$$

We have that $\mathcal{H}_{2,\mu}(f)$ is Δ_F-concentrated on F in L^2_μ norm, then:

$$\|\mathcal{H}_{2,\mu}(f) - P_F \mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

So

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} + M^{\frac{\mu}{2}} |F|^{\frac{1}{2}} \|\mathcal{H}_{2,\mu}(f)\|_\infty.$$

We have that:

$$\|\mathcal{H}_{2,\mu}(f)\|_\infty = \sup_{\omega \in \mathbb{R}^+} \left| \int_0^{+\infty} f(t) C_\mu(t\omega) t^\mu dt \right|.$$

So

$$\left| \int_0^{+\infty} f(t) C_\mu(t\omega) t^\mu dt \right| \leq \int_0^{+\infty} |f(t)||C_\mu(t\omega)| t^\mu dt.$$

Then:

$$\left| \int_0^{+\infty} f(t) C_\mu(t\omega) t^\mu dt \right| \leq \frac{1}{|\Gamma(\mu+1)|} \|f\|_{L^1_\mu}.$$

It follows that:

$$\|\mathcal{H}_{2,\mu}(f)\|_\infty \leq \frac{1}{|\Gamma(\mu+1)|} \|f\|_{L^1_\mu}.$$

Which shows that:

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq \Delta_F \|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} + \frac{M^{\frac{\mu}{2}}}{|\Gamma(\mu+1)|} |F|^{\frac{1}{2}} \|f\|_{L^1_\mu}.$$

Then:

$$\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu} \leq \frac{M^{\frac{\mu}{2}}}{(1-\Delta_F)|\Gamma(\mu+1)|}|F|^{\frac{1}{2}}\|f\|_{L^1_\mu}.$$

On the other hand by using the triangular inequality we get:

$$\|f\|_{L^1_\mu} \leq \|f - P_E f\|_{L^1_\mu} + \|P_E f\|_{L^1_\mu}.$$

We have that

$$\|P_E f\|_{L^1_\mu} = \int_E |f(y)|y^\mu dy.$$

Since f is Δ_E-concentrated on E in L^1_μ norm we have that:

$$\|f - P_E f\|_{L^1_\mu} \leq \Delta_E \|f\|_{L^1_\mu}.$$

By using Hölder inequality we get

$$\|P_E f\|_{L^1_\mu} \leq \left(\int_E y dy\right)^{\frac{1}{2}} \left(\int_E |f(y)|^2 y^\mu dy\right)^{\frac{1}{2}}.$$

It follows that

$$\|P_E f\|_{L^1_\mu} \leq |E|^{\frac{1}{2}} A^{\frac{\mu}{2}} \|f\|_{L^2_\mu}.$$

Then

$$\|f\|_{L^1_\mu} \leq \Delta_E \|f\|_{L^1_\mu} + A^{\frac{\mu}{2}}|E|^{\frac{1}{2}}\|f\|_{L^2_\mu}.$$

So

$$\|f\|_{L^1_\mu} \leq \Delta_E \|f\|_{L^1_\mu} + A^{\frac{\mu}{2}}|E|^{\frac{1}{2}}\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

Then

$$\|f\|_{L^1_\mu} \leq \frac{A^{\frac{\mu}{2}}}{(1-\Delta_E)}|E|^{\frac{1}{2}}\|\mathcal{H}_{2,\mu}(f)\|_{L^2_\mu}.$$

By using this inequality , we get

$$\|f\|_{L^1_\mu} \leq \frac{A^{\frac{\mu}{2}}|E|^{\frac{1}{2}}}{(1-\Delta_E)} \frac{M^{\frac{\mu}{2}}}{(1-\Delta_F)|\Gamma(\mu+1)|}|F|^{\frac{1}{2}}\|f\|_{L^1_\mu}.$$

So

$$(1-\Delta_E)(1-\Delta_F) \leq \frac{A^{\frac{\mu}{2}}M^{\frac{\mu}{2}}}{|\Gamma(\mu+1)|}|E|^{\frac{1}{2}}|F|^{\frac{1}{2}}.$$

Which show that:

$$(1-\Delta_E)^2(1-\Delta_F)^2 \leq \frac{A^\mu M^\mu}{|\Gamma(\mu+1)|^2}|E||F|.$$

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Conflict of interest

The authors declare that they have no confict of interest.

Data Availibility

The manuscript has no associated data.

References

[1] W.O. Amrein, A.M. Berthier, On support properties of L^p-functions and their Fourier transforms, J. Funct. Anal. 24 (1977), 258–267. http://doi.org/10.1016/0022-1236(77)90056-8.

[2] A. Abouelaz, A. Achak, R. Daher, Donoho-Stark's uncertainty principle for the quaternion Fourier transform, Bol. Soc. Mat. Mex. (2019). http://doi.org/10.1007/s40590-019-00251-5.

[3] F.G. Mehler, Ueber eine mit den kugel-und cylinderfunctionen verwandte function und ihre anwendung in der theorie elektricitatsvertheilung, Math. Ann. 18 (1881), 161–194.

[4] J.J. Betancor, Two complex variants of Hankel type transformation of generalized functions, Port. Math. 46 (1989), 229–243. http://eudml.org/doc/115667.

[5] J.M.R. Méndez Pérez, M.M. Socas Robayna, A pair of generalized Hankel-Clifford transformations and their applications, J. Math. Anal. Appl. 154 (1991), 543–557. http://doi.org/10.1016/0022-247X(91)90057-7.

[6] V.A. Abilov, F.V. Abilova, Approximation of functions by Fourier-Bessel sums, Izv. Vyssh. Uchebn. Zaved. Mat. (2001), 3–9. https://www.mathnet.ru/eng/ivm917.

[7] A. Prasad, V.K. Singh, M.M. Dixit, Pseudo-differential operators involving Hankel-Clifford transformation, Asian-Eur. J. Math. 5 (2012), 15 pp. http://doi.org/10.1142/S1793557112500404.

[8] A. Prasad, M. Kumar, Continuity of pseudo-differential operators $h_{1,\mu}$ involving Hankel translation and Hankel convolution on some Gevrey spaces, Integral Transforms Spec. Funct. 21 (2010), 465–477. http://doi.org/10.1080/10652469.2010.508801.

[9] A. Bonami, B. Demange, A survey on uncertainty principles related to quadratic forms, Collect. Math. (2006), 1–36. http://eudml.org/doc/41791.

[10] L.P. Castro, L.T. Minh, N.M. Tuan, New convolutions for quadratic-phase Fourier integral operators and their applications, Mediterr. J. Math. 15 (2018), 13. http://doi.org/10.1007/s00009-017-1063-y.

[11] D.L. Donoho, P.B. Stark, Uncertainty principles and signal recovery, SIAM J. Appl. Math. 49 (1989), 906–931. http://doi.org/10.1137/0149056.

[12] A. Achak, R. Daher, N. Safouane, Titchmarsh theorems and K-functionals for the two-sided quaternion Fourier transform, Int. J. Eng. Appl. Phys. 1 (2021), 26–37. https://ijeap.org/ijeap/article/view/2.

[13] H.J. Landau, W.L. Miranker, The recovery of distorted band-limited signal, J. Math. Anal. Appl. 2 (1961), 97–104. http://doi.org/10.1016/0022-247X(61)90047-6.

[14] F. Soltani, L^p-Donoho-Stark uncertainty principles for the Dunkl transform on $\mathbb{R}^d$, J. Phys. Math. 5 (2014). http://doi.org/10.4172/2090-0902.1000127.

[15] F. Soltani, J. Ghazwani, A variation of the L^p uncertainty principles for the Fourier transform, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 42 (2016), 10–24. https://proc.imm.az/volumes/42-1/42-01-02.pdf.

[16] V. Havin, B. Joricke, The uncertainty principle in harmonic analysis, Springer-Verlag, Berlin, 1994. http://doi.org/10.1007/978-1-4612-4253-4.

[17] R.S. Pathak, P.K. Pandey, A class of pseudo-differential operator associated with Bessel operators, J. Math. Anal. Appl. 196 (1995), 736–747. http://doi.org/10.1006/jmaa.1995.1177.

[18] A.H. Zemanian, Generalized integral transformations, Interscience Publishers, New York, 1968. https://www.worldcat.org/title/2803925.