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Schwarz Boundary Value Problem for Higher-Order Complex Partial Differential Equations on a Triangle

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Abstract. In this paper, we consider the Schwarz boundary value problem for higher-order complex partial differential equations on a triangle. We first introduce the poly-Schwarz operator and the T-type operator for the triangle, and then investigate the boundary behavior of these operators. In addition, we discuss the solvability conditions and present an explicit solution to the Schwarz problem for the inhomoheneous polyanalytic equation on the triangle.

1. Introduction

Boundary value problems for partial differential equations play a central role in both pure and applied mathematics. They arise naturally in the mathematical modeling of a wide range of physical, engineering, and biological phenomena, such as heat conduction, fluid flow, elasticity, and electromagnetism. A boundary value problem seeks a solution to a partial differential equation that satisfies specific conditions on the boundary of the domain, and understanding the existence, uniqueness, and regularity of such solutions is one of the fundamental questions in mathematical analysis [1-4].

From a theoretical standpoint, the study of boundary value problems is closely linked with several branches of analysis, such as functional analysis, operator theory, and the theory of Sobolev spaces. These analytical frameworks provide rigorous tools for formulating partial differential equations, solving them, and analyzing the qualitative properties of their solutions. Moreover,

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boundary value problems also appear in various applied contexts, which further highlights their mathematical significance.

One of the important higher-order differential operators is the polyanalytic operator $\partial_{\bar{z}}^n$, which serves as a natural generalization of the classical Cauchy–Riemann operator $\partial_{\bar{z}}$. While the Cauchy–Riemann operator plays a central role in complex analysis and the theory of analytic functions, the polyanalytic operator extends these ideas to a broader class of functions and equations, often referred to as generalized analytic or biharmonic functions. This operator appears in various boundary value problems and has significant applications in complex analysis, potential theory, and mathematical physics.

In recent years, boundary value problems for complex partial differential equations have attracted substantial attention, and numerous results have been established (see [4–24]). For example, in [5], the harmonic Green and Neumann functions were explicitly constructed for a specific triangle in the complex plane. The Schwarz boundary value problem for the inhomogeneous Cauchy–Riemann equation on a triangle was studied in [6]. The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc was investigated in [7], while in [8], Y. Wang examined the Schwarz problem for the polyanalytic equation in the half unit disc. Furthermore, the Schwarz problem for the inhomogeneous polyanalytic equation in a sector ring was analyzed by Z. Du, Y. Wang, and M. Ku in [9]. In 2025, we further extended this line of research by studying the Schwarz boundary value problem for the polyanalytic equation in a half-lens domain [10].

In the present paper, we extend the boundary conditions and investigate the Schwarz boundary value problem for the polyanalytic equation on a triangle. In addition to description of the domain, we introduce the poly-Schwarz operator and the T-type operator associated with the triangle. We then investigate the fundamental properties of these operators. Finally, by employing these properties, we study the Schwarz problem for the inhomogeneous polyanalytic equation and derive an explicit solution.

Let $\mathbb{T}$ be a triangle in the complex plane $\mathbb{C}$ with vertices at $0, 1$ and i. The boundary of the domain $\mathbb{T}$ is denoted by $\partial\mathbb{T}$ and consists of three sides, oriented counter-clockwise. The oriented segment from 1 to i, denoted as $[1, i]$, is parameterized by

$$\gamma_1 : t \mapsto 1 - t + it, \quad t \in [0, 1].$$

Similarly, the oriented segment from i to 0, denoted $[i, 0]$, is parameterized by

$$\gamma_2 : t \mapsto i(1 - t), \quad t \in [0, 1],$$

and the oriented segment from 0 to 1, denoted $[0, 1]$, is parameterized by the identity map

$$\gamma_3 : t \mapsto t, \quad t \in [0, 1].$$

The purpose of the present paper is to provide explicit solution and solvability conditions for a basic boundary value problem for the polyanalytic equation in $\mathbb{T}$.

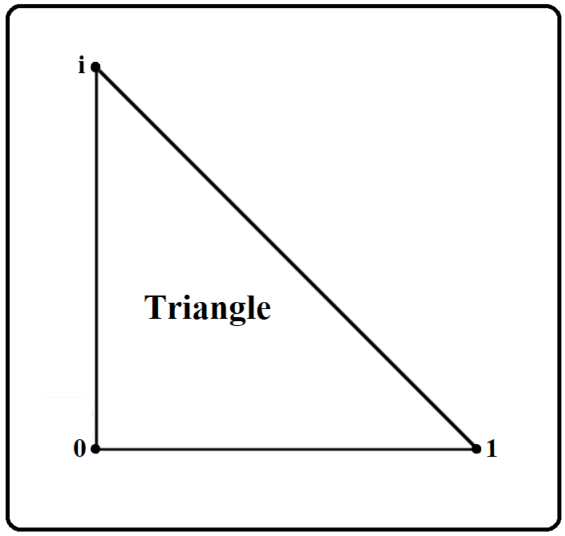

Figure 1. Triangle $\mathbb{T}$

The Schwarz problem is a basic boundary value prolem in complex analysis. It concerns the determination of an analytic function in a given domain by specifying the values of its real and imaginary parts on the boundary. The importance of the Schwarz problem lies in its wide range of applications and its deep connections with various branches of mathematical analysis. The methods developed for solving the Schwarz problem—such as integral representations, singular integral operators, and conformal mappings—have had a profound impact on the modern theory of elliptic equations and on the study of generalized analytic functions. In applied mathematics, these techniques are used in fields such as fluid dynamics, electrostatics, and elasticity, where boundary value problems naturally arise.

Integral operators play a central role in solving basic boundary value problems for the Cauchy-Riemann equations. Next, we introduce integral operators on the triangle.

The Schwarz-type operator on the boundary of the triangle domain is defined as

$$S_\alpha[\gamma](z) = \frac{1}{\pi i}\int_{\partial\mathbb{T}} \gamma(\zeta)\sum_m\sum_n \left(g_{m,n}(\zeta,z) - \frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2}\right)d\zeta, \quad z\in\mathbb{T} \tag{1}$$

where

$$g_{m,n}(\zeta,z) = \frac{1}{\zeta - z - 2m - 2ni} + \frac{1}{\zeta + iz - (2m+1) - (2n+1)i} + \frac{1}{\zeta - iz - (2m+1) - (2n-1)i} + \frac{1}{\zeta + z - (2m+2) - 2ni}$$

and $\alpha \in \mathbb{C}$ is a fixed constant, $\gamma \in C(\partial\mathbb{T};\mathbb{R})$. The Schwarz-type operator $S[\gamma](z)$ is analytic and satisfies [6]

$$\operatorname{Re} S[\gamma](\zeta) = \gamma(\zeta), \qquad \zeta \in \partial\mathbb{T}. \tag{2}$$

The Pompeiu-type operator on the triangle is defined by

$$T_\alpha[f](z) = -\frac{1}{\pi}\iint_{\mathbb{T}} \left(f(\zeta)G_\alpha(\zeta,z) - \overline{f(\zeta)}G_\alpha(\bar{\zeta},z)\right) d\xi\, d\eta, \qquad z \in \mathbb{T}. \tag{3}$$

where

$$G_\alpha(\zeta,z) = \sum_m \sum_n \Big(g_{m,n}(\zeta,z) - g_{m,n}(\zeta,\alpha)\Big)$$

and $f \in L_p(\mathbb{T},\mathbb{C})$, $p > 2$, $\alpha \in \mathbb{C}$ is a fixed constant, $\zeta = \xi + i\eta$. The operator $T_\alpha[f](z)$ possesses the following differentiability and boundary properties [6]

$$\partial_{\bar{z}} T_\alpha[f](z) = f, \tag{4}$$

$$\lim_{z\to\zeta} \operatorname{Re} T_\alpha[f](z) = 0, \qquad \zeta \in \partial\mathbb{T}. \tag{5}$$

In [6], Wang Yufeng and Wang Yanjin studied the Schwarz boundary value problem for the inhomogeneous Cauchy-Riemann equation on the triangle. The result is as follows.

Theorem 1. ([6]) *The Schwarz problem for the inhomogeneous equation in the triangle domain*

$$\begin{cases} (\partial_{\bar{z}}\omega)(z) = f(z), \qquad z \in \mathbb{T}, \quad f \in L_p(\mathbb{T};\mathbb{C}),\ p > 2, \\ \operatorname{Re}(\partial_{\bar{z}}\omega)(\zeta) = \gamma(\zeta), \quad \zeta \in \partial\mathbb{T}, \quad \gamma \in C(\partial\mathbb{T};\mathbb{R}). \end{cases} \tag{6}$$

is solvable and its solution can be expressed as

$$\omega(z) = S_\alpha[\gamma](z) + ic + T_\alpha[f](z),$$

where $c = \operatorname{Im}\omega \in \mathbb{R}$, S_α *is the Schwarz-type operator, and* T_α *is the Pompeiu-type operator.*

2. Poly-Schwarz and T-Type Operators on the Triangle

Poly-Schwarz operators play an important role in solving boundary value problems for generalized Cauchy-Riemann equations. In the following, we introduce the poly-Schwarz operator for the triangle domain and examine its key properties.

Let $\gamma_0, \gamma_1, \ldots, \gamma_n \in C(\partial\mathbb{T};\mathbb{R})$. For $n \in \mathbb{N}$, we define the poly-Schwarz operator S_n on $\mathbb{T}$ as follows:

$$S_n[\gamma_0,\gamma_1,\ldots,\gamma_{n-1}](z) =$$

$$\sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\left(\frac{1}{\pi i}\int_{\partial\mathbb{T}} (t - z + \overline{t-z})^k \gamma_k(\zeta) \sum_m \sum_n \left(g_{m,n}(\zeta,z) - \frac{g_{m,n}(\zeta,\alpha) + g_{m,n}(\zeta,\bar{\alpha})}{2}\right) d\zeta\right). \tag{7}$$

It is straightforward to verify that $S_1 = S_\alpha$ where S_α is the Schwarz-type operator defined by (1). Furthermore, S_n can be expressed as:

$$S_n[\gamma_0,\dots,\gamma_{n-1}](z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \Bigg\{ \frac{1}{\pi i} \int_{\partial\mathbb{T}} \sum_{t=0}^{k} \binom{k}{t} \left(\zeta+\overline{\zeta}\right)^t (-z-\overline{z})^{k-t} \gamma_k(\zeta) \sum_m \sum_n \left(g_{m,n}(\zeta,z) - \frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2} d\zeta \right) \Bigg\}. \tag{8}$$

Hence,

$$S_n[\gamma_0,\gamma_1,...,\gamma_{n-1}](z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{t=0}^{k} \binom{k}{t} (-z-\overline{z})^{k-t} S[\hat{\gamma}_{k,t}](z), \qquad z\in\mathbb{T}, \tag{9}$$

with $\hat{\gamma}_{k,t}(\zeta) = \left(\zeta+\overline{\zeta}\right)^t \gamma_k(\zeta), \quad \zeta\in\partial\mathbb{T}$ for $k=0,1,2,...,n-1$ and $t=0,1,2,...,k$.

Next, we check the boundary behavior of the poly-Schwarz operator.

Theorem 2. *Let $\gamma_0,\gamma_1,\dots,\gamma_{n-1}\in C(\partial\mathbb{T},\mathbb{R})$. Then for each $t=0,1,\dots,n-1$, we have*

$$\lim_{z\to\zeta} \left\{ \operatorname{Re} \partial_{\bar{z}}^t S_n[\gamma_0,\gamma_1,\dots,\gamma_{n-1}](z) \right\} = \gamma_t(\zeta), \qquad \zeta\in\partial\mathbb{T}. \tag{10}$$

Proof. For the case $t=0$, using equations (2) and (9), we get

$$\lim_{z\to\zeta} \{\operatorname{Re} S_n[\gamma_0,\gamma_1,\dots,\gamma_{n-1}](z)\} = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{t=0}^{k} \binom{k}{t} (-\zeta-\overline{\zeta})^{k-t} \hat{\gamma}_{k,t}(\zeta) = \gamma_0(\zeta), \qquad \zeta\in\partial\mathbb{T}, \tag{11}$$

which implies that (10) is true.

Now, suppose $t>0$. Then the t-th derivative of $S_n[\gamma_0,\dots,\gamma_{n-1}](z)$ with respect to $\bar{z}$ is given by

$$\begin{aligned}
\partial_{\bar{z}}^t S_n[\gamma_0,\gamma_1,\dots,\gamma_{n-1}](z) &= \sum_{k=t}^{n-1} \frac{(-1)^{k-t}}{(k-t)!} \Bigg(\frac{1}{\pi i} \int_{\partial\mathbb{T}} \left(\zeta - z + \overline{\zeta - z}\right)^{k-t} \gamma_k(\zeta) \\
&\qquad \times \sum_m \sum_n \left[g_{m,n}(\zeta,z) - \frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2} \right] d\zeta \Bigg) \\
&= \sum_{k=t}^{n-1} \frac{(-1)^{k-t}}{(k-t)!} \Bigg(\frac{1}{\pi i} \int_{\partial\mathbb{T}} \sum_{s=0}^{k-t} \binom{k-t}{s} \left(\zeta+\overline{\zeta}\right)^s (-z-\overline{z})^{k-t-s} \gamma_k(\zeta) \\
&\qquad \times \sum_m \sum_n \left[g_{m,n}(\zeta,z) - \frac{g_{m,n}(\zeta,\alpha)+g_{m,n}(\zeta,\bar{\alpha})}{2} \right] d\zeta \Bigg) \\
&= \sum_{k=t}^{n-1} \frac{(-1)^{k-t}}{(k-t)!} \left(\sum_{s=0}^{k-t} \binom{k-t}{s} (-1)^{k-t-s} (-z-\overline{z})^{k-t-s} S_\alpha[\hat{\gamma}_{k,s}](\zeta) \right)
\end{aligned}$$

Therefore, one similarly obtains

$$\begin{aligned}&\lim_{z\to\zeta}\left\{\partial_{\bar z}^{t}S_n[\gamma_0,\gamma_1,\ldots,\gamma_{n-1}](z)\right\}\\&=\sum_{k=t}^{n-1}\frac{(-1)^{k-t}}{(k-t)!}\left(\sum_{s=0}^{k-t}\binom{k-t}{s}(-1)^{k-t-s}\left(\zeta-\bar\zeta\right)^{k-t-s}\left(\zeta-\bar\zeta\right)^{s}\gamma_k(\zeta)\right)\\&=\gamma_t(\zeta).\end{aligned}$$

This completes the proof. □

The next theorem establishes a differentiability property of the poly-Schwarz operator.

Theorem 3. *If the functions $\gamma_0,\gamma_1,...,\gamma_{n-1}\in C(\partial\mathbb{T},\mathbb{R})$, it follows that*

$$\partial_{\bar z}^{n}S_n[\gamma_0,\gamma_1,...,\gamma_{n-1}](z)=0,\quad z\in\mathbb{T}. \tag{12}$$

Proof. Since the Schwarz-type operator is analytic in the triangle domain, equation (9) implies that

$$\partial_{\bar z}^{n}S_n[\gamma_0,\gamma_1,...,\gamma_{n-1}](z)=\sum_{k=0}^{n-1}\frac{(-1)^k}{k!}\sum_{t=0}^{k}\binom{k}{t}\left(\partial_{\bar z}^{n}(-z-\bar z)^{k-t}\right)S[\hat\gamma_{k,t}](z)=0. \tag{13}$$

Hence, the proof is finished □

Next, we introduce the T-type operator on the triangle. This operator plays a key role in solving boundary value problems associated with inhomogeneous polyanalytic equations.

We define the T-type operator as

$$T_t[f](z)=\frac{(-1)^t}{\pi(t-1)!}\iint_{\mathbb{T}}\left(\zeta-z+\overline{\zeta-z}\right)^{t-1}\left(f(\zeta)\,G_\alpha(\zeta,z)-\overline{f(\zeta)}\,G_\alpha(\bar\zeta,z)\right)d\xi\,d\eta, \tag{14}$$

where $z\in\mathbb{T}$, $t=1,2,3,\ldots$, $p>2$, $f\in L_p(\mathbb{T};\mathbb{C})$, and $\alpha\in\mathbb{T}$. It is straightforward to verify that when $t=1$, the operator reduces to

$$T_\alpha[f](z)=-\frac{1}{\pi}\iint_{\mathbb{T}}\left(f(\zeta)G_\alpha(\zeta,z)-\overline{f(\zeta)}G_\alpha(\bar\zeta,z)\right)d\xi\,d\eta,\qquad z\in\mathbb{T},$$

and it satisfies

$$\partial_{\bar z}T_1[f](z)=\partial_{\bar z}T_\alpha[f](z)=f(z). \tag{15}$$

Let $T_0[f](z)=f(z)$, $z\in\mathbb{T}$, then (15) is equivalent to

$$\partial_{\bar z}T_1[f](z)=T_0[f](z),\ \ z\in\mathbb{T}. \tag{16}$$

We now consider the following result, which generalizes (4).

Theorem 4. *If $f\in L_p(M;\mathbb{C})$, $p>2$, then*

$$\partial_{\bar z}T_t[f](z)=T_{t-1}[f](z),\quad z\in\mathbb{T},\ t=1,2,.... \tag{17}$$

Proof. If we assume that $t = 1$, then

$$\partial_{\bar{z}} T_1[f](z) = T_0[f](z). \tag{18}$$

For $t > 1$, we have

$$\begin{aligned} T_t[f](z) &= \frac{(-1)^t}{\pi(t-1)!} \iint_{\mathbb{T}} \left(\zeta - z + \overline{\zeta - z}\right)^{t-1} \left(f(\zeta)\, G_\alpha(\zeta, z) - \overline{f(\zeta)}\, G_\alpha(\bar{\zeta}, z)\right) \, d\xi \, d\eta, \\ &= \frac{(-1)^{t-1}}{(t-1)!} \sum_{k=0}^{t-1} \binom{t-1}{k} (-z - \overline{z})^{t-k-1}\, T_1[f_k](z), \quad z \in \mathbb{T}, \end{aligned} \tag{19}$$

where

$$f_k(\zeta) = \left(\zeta + \overline{\zeta}\right)^k f(\zeta), \quad k = 0, 1, 2, ..., t-1.$$

Therefore

$$\begin{aligned} \partial_{\bar{z}} T_t[f](z) &= \frac{(-1)^{t-1}}{(t-1)!} \sum_{k=0}^{t-1} \binom{t-1}{k} \left\{ \partial_{\bar{z}} \, (-z - \overline{z})^{t-k-1}\, T_1[f_k](z) + (-z - \overline{z})^{t-k-1}\, \partial_{\bar{z}} T_1[f_k](z) \right\} \\ &= \frac{(-1)^{t-2}}{(t-2)!} \sum_{k=0}^{t-2} \binom{t-2}{k} (-z - \overline{z})^{t-k-2}\, T_1[f_k](z) \\ &= T_{t-1}[f](z). \end{aligned}$$

Hence, the proof is complete. □

Next, we examine the boundary behavior of the T-type operator on $\mathbb{T}$.

Theorem 5. *If $f \in L_p(\mathbb{T}; \mathbb{C})$, $p > 2$, then*

$$\lim_{z \to \zeta,\, z \in \mathbb{T}} \operatorname{Re} T_t[f](z) = 0, \quad \zeta \in \partial\mathbb{T}, \quad t = 1, 2, \ldots \tag{20}$$

Proof. From (19), we have

$$\operatorname{Re}\left\{T_t[f](z)\right\} = \frac{(-1)^{t-1}}{(t-1)!} \sum_{k=0}^{t-1} \binom{t-1}{k} (-z - \overline{z})^{t-k-1} \operatorname{Re}\left\{T_1[f_k](z)\right\}, \quad z \in \mathbb{T}. \tag{21}$$

According to (5), we have

$$\lim_{z \to \zeta,\, z \in \mathbb{T}} \operatorname{Re} T_1[f_k](z) = 0, \quad \zeta \in \partial\mathbb{T}. \tag{22}$$

Combining (21) and (22), it follows that

$$\lim_{z \to \zeta,\, z \in \mathbb{T}} \operatorname{Re} T_t[f](z) = 0, \quad \zeta \in \partial\mathbb{T}, \quad t = 1, 2, \ldots. \tag{23}$$

This completes the proof. □

3. Schwarz Problem on the Triangle

In this section, we investigate the Schwarz boundary value problem for inhomogeneous poly-analytic equations in $\mathbb{T}$. By employing the properties of poly-Schwarz and T-type operators, we study the Schwarz problem rigorously and derive an explicit solution.

First, we consider the Schwarz problem for a polyanalytic function ω satisfying

$$\begin{cases} \partial_{\bar{z}}^n \omega(z) = 0, & z \in \mathbb{T}, \\ \operatorname{Re}\left(\partial_{\bar{z}}^k \omega\right)(\zeta) = 0, & \zeta \in \partial\mathbb{T}, \quad k = 0, 1, \ldots, n-1, \end{cases} \tag{24}$$

and we show that the problem is solvable and admits an explicit representation of the solution. Since

$$\partial_{\bar{z}}^n \omega(z) = 0, \qquad z \in \mathbb{T}, \tag{25}$$

the function ω is polyanalytic of order n. Therefore, it possesses the standard decomposition (see, for example, [1,2,12])

$$\omega(z) = \sum_{k=0}^{n-1} \frac{(z+\bar{z})^k}{k!} \varphi_k(z), \qquad z \in \mathbb{T}, \tag{26}$$

where each φ_k is analytic in $\mathbb{T}$. Substituting this representation into the boundary conditions (24) leads to the equivalent system

$$\operatorname{Re}\, \varphi_k(\zeta) = 0, \qquad \zeta \in \partial\mathbb{T}, \quad k = 0, 1, \ldots, n-1. \tag{27}$$

According to Theorem 1, each of these boundary conditions implies that

$$\varphi_k(z) = ic_k, \qquad c_k \in \mathbb{R}, \quad k = 0, 1, \ldots, n-1. \tag{28}$$

Inserting these expressions into the decomposition (26), we obtain the explicit form of the solution:

$$\omega(z) = \sum_{k=0}^{n-1} \frac{(z+\bar{z})^k}{k!} ic_k, \tag{29}$$

where the constants $c_k \in \mathbb{R}$ are arbitrary.

In the following, we study the Schwarz problem for the inhomogeneous polyanalytic equation.

Theorem 6. *The Schwarz boundary value problem for the inhomogeneous polyanalytic equation on the triangle*

$$\begin{cases} \left(\partial_{\bar{z}}^n \omega\right)(z) = f(z), & z \in \mathbb{T}, \quad f \in L_p(\mathbb{T}; \mathbb{C}), \quad p > 2, \\ \operatorname{Re}\left(\partial_{\bar{z}}^k \omega\right)(\zeta) = \gamma_k(\zeta), & \zeta \in \partial\mathbb{T}, \quad \gamma_k \in C(\partial\mathbb{T}; \mathbb{R}), \\ \operatorname{Im} \partial_{\bar{z}}^k \omega(\zeta) = c_k, & \zeta \in \partial\mathbb{T}, \quad k = 1, 2, \ldots, n-1, \end{cases} \tag{30}$$

is solvable and its solution can be written as

$$\omega(z) = S_n[\gamma_0, \gamma_1, ..., \gamma_{n-1}](z) + T_n[f](z) + \sum_{k=0}^{n-1} \frac{(z+\bar{z})^k}{k!} ic_k, \tag{31}$$

where $c_k \in \mathbb{R}$, *for* $k = 0, 1, 2, ..., n-1$ *and the operators* S_n, T_n *are defined by (7) and (14), respectively.*

Proof. Let us define

$$\psi(z) = S_n[\gamma_0, \ldots, \gamma_{n-1}](z) + T_n[f](z), \quad z \in \mathbb{T},$$

where S_n and T_n are the operators given in (7) and (14). By Theorems 2–5, ψ satisfies the inhomogeneous polyanalytic equation

$$\partial_{\bar{z}}^n \psi = f \quad \text{in } \mathbb{T}$$

and the boundary conditions

$$\operatorname{Re} \partial_{\bar{z}}^k \psi = \gamma_k \quad \text{on } \partial\mathbb{T}, \quad k = 0, \ldots, n-1.$$

Consequently, ψ serves as a particular solution of the Schwarz boundary value problem.

Now let

$$\omega(z) = \psi(z) + \varphi(z). \tag{32}$$

Substituting $\omega(z)$ from (32) into (30) yields

$$\begin{cases} \left(\partial_{\bar{z}}^n \varphi\right)(z) = 0, & in \quad \mathbb{T}, \\ \left\{\operatorname{Re}\left(\partial_{\bar{z}}^k \varphi\right)\right\}(\zeta) = 0, & on \quad \partial\mathbb{T},\ k = 0, 1, 2, ..., n-1. \end{cases} \tag{33}$$

which is just the Schwarz problem for the polyanalytic function. Therefore,

$$\phi(z) = \sum_{k=0}^{n-1} \frac{(z + \bar{z})^k}{k!} i c_k, \quad c_k \in \mathbb{R}.$$

Hence, the Schwarz boundary value problem for the polyanalytic equation (30) is solvable by (31), which completes the proof. □

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