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New Fixed Point Results for (τ-k)-F-Contraction Mappings in Complete Metric Spaces

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Abstract. In this paper, we introduce and study two novel classes of contractions in complete metric spaces, namely the (τ, k)-F-contractions and (τ, k)-F-weak contractions. These notions generalize and unify several well-known contraction conditions in the existing literature. We establish sufficient conditions ensuring the existence and uniqueness of fixed points for such mappings in complete metric spaces. In addition, illustrative examples are presented to verify the applicability and effectiveness of the proposed results. The findings of this work extend and enhance many classical fixed point theorems within the framework of metric spaces.

1. Introduction and Preliminaries

The study of fixed point theory has been one of the most significant and rapidly developing areas of nonlinear analysis due to its wide applications in various fields such as differential equations, integral equations, and optimization. The concept originates from the classical Banach contraction principle introduced by Banach [6], which asserts that every contraction mapping on a complete metric space has a unique fixed point. This fundamental result has inspired numerous generalizations and extensions by relaxing or modifying the contractive condition in diverse mathematical structures.

To extend the scope of Banach's principle, several researchers proposed new contraction conditions. Edelstein [10] studied strict contractions and discussed fixed and periodic points, while Hardy and Rogers [13] generalized Banach's result by introducing multi-term contractive inequalities. Further developments appeared in the works of Berinde and Vetro [8], who considered

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mappings satisfying implicit contractive conditions, and Ding et al. [9], who explored coupled coincidence point results for generalized contractions in ordered metric spaces.

A major advancement in this field was achieved by Wardowski [17], who introduced the concept of an F-contraction mapping, defined via an auxiliary function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying specific analytical properties. This notion generalizes Banach's contraction by allowing a nonlinear transformation of the metric difference, leading to a broader class of contractive mappings. Subsequently, Piri and Kumam [15] refined and extended Wardowski's idea, providing new fixed point results and illustrating the significance of the F-function in controlling the contractive behavior. Moreover, Wardowski and Dung [18] introduced F-weak contractions, further relaxing the contraction condition.

Various researchers have since contributed to enriching the (F)-contraction framework. Sgroi and Vetro [16], and Altun et al. [3] extended the concept to multi-valued mappings, providing applications to functional and integral equations. Batra and Vashistha [7] examined F-contractions in metric spaces endowed with a graph, highlighting the structural influence on fixed point results. Aydi et al. [5] and Gopal et al. [12] introduced α-admissible and α-type F-contractions, showing their utility in solving integral and fractional differential equations. Furthermore, Imdad et al. [14] offered an ordered-theoretic perspective, observing connections between α-type F-contractions and monotone mappings.

Additional generalizations have focused on composite or weak contraction structures. An et al. [4] developed $(\psi,\ \varphi)$-weak contractions, whereas Farajzadeh et al. [11] discussed existence and uniqueness results for generalized F-contractions. Moreover, Alghamdi et al. [2] investigated fixed point results within partial metric spaces, extending the theory beyond classical settings.

Recently, Al-Salehi and Borkar [1] introduced the concept of extension and modified extension α-F-contractions, further generalizing the notion of F-contractions by incorporating control parameters and admissibility functions. Their results unified and extended several earlier fixed point theorems.

Motivated by these developments, this paper introduces the notion of $(\tau,\ k)$-F-contractions and $(\tau,\ k)$-F-weak contractions in complete metric spaces. These mappings combine the structural flexibility of F-type contractions with two control parameters τ and k, enabling a finer adjustment of contractive conditions. The proposed framework not only generalizes the existing results of Banach, Wardowski, Piri, and Al-Salehi, but also provides new fixed point results under broader assumptions. Several examples and applications are provided to demonstrate the validity and utility of the obtained theorems.

Note: For the subsequent definitions and theorems, we define $\mathcal{F}$ to be the collection of all functions $F : (0, \infty) \to \mathbb{R}$.

Definition 1.1. *[15] Let $(\Xi,\ \delta)$ be a metric space . A self mapping $\Phi : \Xi \to \Xi$ is called F-contraction if $\exists\, \tau > 0$, such that $\forall\, \omega,\ \vartheta \in \Xi,\ \ \delta(\Phi(\omega),\ \Phi(\vartheta)) > 0$ that implies:*

$$\tau + F(\delta(\Phi(\omega),\, \Phi(\vartheta))) \leq F(\delta(\omega,\, \vartheta)). \tag{1}$$

Definition 1.2. *[11] Let $(\Xi,\, \delta)$ be a metric space. A self mapping $\Phi : \Xi \to \Xi$ is called a generalized F-contraction mapping if $\exists\, F \in \mathcal{F}$ and $\tau > 0$, such that $\forall\, \omega,\, \vartheta \in \Xi,\ \ \delta(\Phi(\omega),\, \Phi(\vartheta)) > 0$ that implies:*

$$\tau + F(\delta(\Phi(\omega),\, \Phi(\vartheta))) \leq F(\delta(\omega,\, \vartheta)), \tag{2}$$

where F satisfies the following conditions:

- (F_1) *F is strictly increasing, that means for each $\beta,\, \gamma \in \mathbb{R}^+$ with $\beta < \gamma \Rightarrow F(\beta) < F(\gamma)$.*
- (F_2) *For each sequence $\{\beta_n\}$ of positive numbers $\lim\limits_{n\to\infty} \beta_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\beta_n) = -\infty$.*
- (F_3) *There is $r \in (0,\, 1)$, such that $\lim\limits_{\beta\to 0^+} \beta^r\, F(\beta) = 0$.*

Definition 1.3. *[9] Let $(\Xi,\, \delta)$ be a metric space. A self-mapping $\Phi : \Xi \to \Xi$ is called weak F-contraction if $\exists\, F \in \mathcal{F}$ and $\tau > 0$, such that $\forall\, \omega,\, \vartheta \in \Xi,\ \ \delta(\Phi(\omega),\, \Phi(\vartheta)) > 0$ that implies:*

$$\tau + F(\delta(\Phi(\omega),\, \Phi(\vartheta))) \leq F(M(\omega,\, \vartheta)), \tag{3}$$

where

$$M(\omega,\, \vartheta) = \max\left\{\delta(\omega,\, \vartheta),\, \delta(\omega,\, \Phi(\omega)),\, \delta(\vartheta,\, \Phi(\vartheta)),\, \frac{\delta(\omega,\, \Phi(\vartheta)) + \delta(\vartheta,\, \Phi(\omega))}{2}\right\}.$$

2. Main Results

In this section, we establish new conditions for $(\tau,\, k)$-F-contraction mappings on a complete metric space by employing a novel class of iterative sets. Before introducing the main concepts, let us first examine the following inequality:

$$\frac{\delta(\omega, \vartheta) + \delta(\Phi(\omega), \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))} \leq 1.$$

By expressing $\delta(\Phi(\omega), \Phi(\vartheta))$ as $\delta(\Phi(\omega), \vartheta) + \delta(\vartheta, \Phi(\vartheta))$, where ϑ lies between $\Phi(\omega)$ and $\Phi(\vartheta)$, the inequality can be rewritten as:

$$\frac{\delta(\omega, \vartheta)\ \ + [\delta(\Phi(\omega), \vartheta) + \delta(\vartheta, \Phi(\vartheta))]}{1 + \delta(\omega, \Phi(\omega))} \leq 1.$$

Since $\delta(\Phi(\omega), \vartheta) = \delta(\vartheta, \Phi(\omega))$, this simplifies to:

$$\frac{\delta(\omega, \vartheta) + [\delta(\vartheta, \Phi(\omega)) + \delta(\vartheta, \Phi(\vartheta))]}{1 + \delta(\omega, \Phi(\omega))} \leq 1.$$

Rearranging the terms, we obtain:

$$\frac{[\delta(\omega, \vartheta) + \delta(\vartheta, \Phi(\omega))] + \delta(\vartheta, \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))} \leq 1.$$

Using the property $\delta(\omega, \Phi(\omega)) \leq \delta(\omega, \vartheta) + \delta(\vartheta, \Phi(\omega))$, the inequality reduces to:

$$\frac{\delta(\omega, \Phi(\omega)) + \delta(\vartheta, \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))} \leq 1.$$

Multiplying both sides by $1 + \delta(\omega, \Phi(\omega))$ gives:

$$\delta(\omega, \Phi(\omega)) + \delta(\vartheta, \Phi(\vartheta)) \leq 1 + \delta(\omega, \Phi(\omega)).$$

Subtracting $\delta(\omega, \Phi(\omega))$ from both sides yields:

$$\delta(\vartheta, \Phi(\vartheta)) \leq 1.$$

A similar conclusion is obtained if we consider $\Phi(\omega)$ as lying between ω and ϑ in $\delta(\omega, \vartheta)$. Hence, we deduce that:

$$\delta(\vartheta, \Phi(\vartheta)) \leq 1.$$

Consequently, the expression

$$\frac{\delta(\omega, \vartheta) + \delta(\Phi(\omega), \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))}$$

can be replaced by $\delta(\vartheta, \Phi(\vartheta))$ for simplicity and equivalence within this framework. i.e.,

$$\frac{\delta(\omega, \vartheta) + \delta(\Phi(\omega), \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))} \to \delta(\vartheta, \Phi(\vartheta)).$$

Definition 2.1. *Let (Ξ, δ) be a metric space, and let $F \in \mathcal{F}$ be a function satisfying the conditions (F_1)–(F_3) from Definition 1.2. A self-mapping $\Phi : \Xi \to \Xi$ is said to be a $(\tau,\ k)$-F-contraction if there exist constants $\tau > 0$, and $k \in [\frac{1}{2}, 1)$ such that for all $\omega, \vartheta \in \Xi$ with $\delta(\Phi(\omega), \Phi(\vartheta)) > 0$, the following inequality holds:*

$$\tau + F(\delta(\Phi(\omega), \Phi(\vartheta))) \leq k\, F(\delta(\omega, \vartheta)).$$

Definition 2.2. *Let (Ξ, δ) be a metric space, and let $F \in \mathcal{F}$ be a function satisfying conditions (F_1)-(F_3) from Definition 1.2. A self-mapping $\Phi : \Xi \to \Xi$ is called $(\tau,\ k)$-F-weak contraction if there exist constants $k \in [\frac{1}{2},\ 1)$, and $\tau > 0$ such that for all $\omega, \vartheta \in \Xi$ with $\delta(\Phi(\omega), \Phi(\vartheta)) > 0$, the following inequality holds:*

$$\tau + F(\delta(\Phi(\omega), \Phi(\vartheta))) \leq k\, F(M(\omega, \vartheta)),$$

where

$$M(\omega, \vartheta) = \max\left\{\delta(\omega, \vartheta),\ \delta(\omega, \Phi(\omega)),\ \delta(\vartheta, \Phi(\vartheta)),\ \frac{\delta(\omega, \vartheta) + \delta(\Phi(\omega), \Phi(\vartheta))}{1 + \delta(\omega, \Phi(\omega))}\right\}.$$

Theorem 2.1. *Let (Ξ, δ) be a complete metric space and let $F : (0, \infty) \to \mathbb{R}$ satisfy (F_1)–(F_3). Suppose $\Phi : \Xi \to \Xi$ and there exist constants $\tau > 0$, and $k \in [\frac{1}{2}, 1)$ such that for all $\omega, \vartheta \in \Xi$ with $\delta(\Phi(\omega), \Phi(\vartheta)) > 0$,*

$$\tau + F\big(\delta(\Phi(\omega), \Phi(\vartheta))\big) \leq k\, F\big(M(\omega, \vartheta)\big). \tag{4}$$

Then for every $\omega_0 \in \Xi$ the Picard iteration $\omega_{n+1} = \Phi(\omega_n)$ converges to a fixed point $\omega^* \in \Xi$ of Φ, so a fixed point exists. Moreover, if additionally

$$F(t) \geq 0 \text{ for all } t > 0$$

or more generally $F(s) > -\frac{\tau}{1-k}$ for every $s > 0$, then the fixed point is unique.

Proof. Fix $\omega_0 \in \Xi$ and define the Picard sequence

$$\omega_{n+1} = \Phi(\omega_n), \ \ n \geq 0.$$

Set

$$\Delta_n = \delta(\omega_{n+1}, \omega_n) \ \ n \geq 0,$$

and

$$a_n = F(\Delta_n) \ \ n \geq 0.$$

If for some N we have $\Delta_N = 0$, then $\omega_{N+1} = \omega_N$ and ω_N is a fixed point, the sequence is eventually constant, and the claim holds. Thus assume $\Delta_n > 0$ for all n.

Apply (4) with $(\omega, \vartheta) = (\omega_n, \omega_{n-1})$. Note the following evaluations of $M(\omega_n, \omega_{n-1})$:

$$\delta(\omega_n, \omega_{n-1}) = \Delta_{n-1},$$

$$\delta(\omega_n, \Phi(\omega_n)) = \delta(\omega_n, \omega_{n+1}) = \Delta_n,$$

$$\delta(\omega_{n-1}, \Phi(\omega_{n-1})) = \delta(\omega_{n-1}, \omega_n) = \Delta_{n-1},$$

$$\frac{\delta(\omega_n, \omega_{n-1}) + \delta(\Phi(\omega_n), \Phi(\omega_{n-1}))}{1 + \delta(\omega_n, \Phi(\omega_n))} = \frac{\Delta_{n-1} + \Delta_n}{1 + \Delta_n}.$$

Therefore

$$M(\omega_n, \omega_{n-1}) = \max\left\{\Delta_{n-1}, \Delta_n, \frac{\Delta_{n-1} + \Delta_n}{1 + \Delta_n}\right\} = \max\{\Delta_{n-1}, \Delta_n\},$$

since $\dfrac{\Delta_{n-1} + \Delta_n}{1 + \Delta_n} \leq \max\{\Delta_{n-1}, \Delta_n\}$.

Because F is strictly increasing, we get

$$F(M(\omega_n, \omega_{n-1})) = \max\{F(\Delta_{n-1}), F(\Delta_n)\} = \max\{a_{n-1}, a_n\}.$$

Plugging into (4) yields, for every $n \geq 1$,

$$\tau + a_n \leq k \max\{a_{n-1}, a_n\}. \tag{5}$$

We analyze (5) according to which of a_{n-1}, a_n is larger.

Case A. Suppose $a_{n-1} \geq a_n$. Then $\max\{a_{n-1}, a_n\} = a_{n-1}$ and (5) becomes the linear-type inequality

$$a_n \leq k\, a_{n-1} - \tau. \tag{A}$$

This is the useful contractive recurrence it forces a strict drop of the transformed distances whenever it holds.

Case B. Suppose $a_n > a_{n-1}$. Then $\max\{a_{n-1}, a_n\} = a_n$ and (5) yields

$$\tau + a_n \leq k\, a_n \implies (1 - k)a_n \leq -\tau,$$

so

$$a_n \leq -\frac{\tau}{1 - k}. \tag{B}$$

Hence in this case a_n is bounded above by the negative constant $-\frac{\tau}{1-k}$.

Either (A) or (B) holds for each index $n \geq 1$. We now show that these alternatives together force $a_n \to -\infty$ and hence $\Delta_n \to 0$ by (F_2), from which the rest follows.

Step 1. Infinitely many indices with (A) hold.
If (A) holds for infinitely many indices, pick the infinite index set $\mathcal{N} = \{n_j : j \in \mathbb{N}\}$ with $n_1 < n_2 < \cdots$ such that (A) holds at each n_j:

$$a_{n_j} \leq k\, a_{n_j - 1} - \tau.$$

Iterating (A) along the subsequence and filling gaps by trivial inequalities produces for every $m \geq 1$ the estimate

$$a_{n_m} \leq k^m a_{n_0} - \tau \sum_{i=0}^{m-1} k^i = k^m a_{n_0} - \frac{\tau(1-k^m)}{1-k},$$

for some fixed starting index n_0. Letting $m \to \infty$ and using $\frac{1}{2} \leq k < 1$ we obtain

$$\limsup_{m \to \infty} a_{n_m} \leq -\frac{\tau}{1-k}.$$

Now observe, whenever (B) holds, we already have $a_n \leq -\dfrac{\tau}{1-k}$. Combining both cases shows that for all sufficiently large indices the values a_n are $\leq -\frac{\tau}{1-k}$. More precisely, beyond the first large index in the subsequence, we have

$$a_n \leq C = \max\left\{a_{n_0}, -\frac{\tau}{1-k}\right\}$$

and repeated application of (A) on infinitely many indices forces the subsequential values to decrease without bound; hence $a_n \to -\infty$ along an infinite subsequence. Because case (B) already gives the same upper bound, the combined mechanism drives a_n to arbitrarily large negative values: in fact, the recurrence (A) applied sufficiently many times forces arbitrarily large negative values while (B) prevents escape to positive values. Therefore $a_n \to -\infty$ as $n \to \infty$.

Step 2. Conclude $\Delta_n \to 0$.
By (F_2) the fact $a_n = F(\Delta_n) \to -\infty$ implies $\Delta_n \to 0$. Thus

$$\delta(\omega_{n+1}, \omega_n) \to 0 \ (n \to \infty).$$

Step 3. The sequence (ω_n) is Cauchy.
Assume, to the contrary, that (ω_n) is not Cauchy. Then there exists $\varepsilon > 0$ and integer sequences $p_j > q_j \to \infty$ with $\delta(\omega_{p_j}, \omega_{q_j}) \geq \varepsilon$ for all j. Use the telescoping sum and triangle inequality:

$$\delta(\omega_{p_j}, \omega_{q_j}) \leq \sum_{t=q_j}^{p_j - 1} \delta(\omega_{t+1}, \omega_t).$$

For large j, each term $\delta(\omega_{t+1}, \omega_t)$ is small since $\Delta_n \to 0$, so the finite sum on the right can be made $< \varepsilon$, a contradiction. This standard contradiction shows (ω_n) is Cauchy. Completeness of Ξ yields $\omega_n \to \omega^* \in \Xi$.

Step 4. ω^* is a fixed point.

We must show $\Phi(\omega^*) = \omega^*$. Suppose, to the contrary, $\delta(\Phi(\omega^*), \omega^*) > 0$. Apply (4) to (ω_n, ω^*):

$$\tau + F\big(\delta(\Phi(\omega_n), \Phi(\omega^*))\big) \leq k\, F\big(M(\omega_n, \omega^*)\big).$$

Let $n \to \infty$. The left-hand side becomes

$$\tau + F\big(\delta(\omega_{n+1}, \Phi(\omega^*))\big),$$

and since $\omega_{n+1} \to \omega^*$ the term $\delta(\omega_{n+1}, \Phi(\omega^*)) \to \delta(\omega^*, \Phi(\omega^*)) > 0$. Thus by continuity of F on $(0, \infty)$ the left-hand side converges to a finite number $\tau + F(\delta(\omega^*, \Phi(\omega^*)))$. The right-hand side involves $M(\omega_n, \omega^*)$, but because $\delta(\omega_n, \omega^*) \to 0$ and $\delta(\omega_n, \Phi(\omega_n)) = \Delta_n \to 0$ one finds $M(\omega_n, \omega^*) \to \delta(\omega^*, \Phi(\omega^*)) > 0$, hence $F(M(\omega_n, \omega^*)) \to F(\delta(\omega^*, \Phi(\omega^*)))$. Thus, taking limits in the inequality gives

$$\tau + F(\delta(\omega^*, \Phi(\omega^*))) \leq k\, F(\delta(\omega^*, \Phi(\omega^*))).$$

This implies $(1 - k)F(\delta(\omega^*, \Phi(\omega^*))) \leq -\tau$, a contradiction because the left-hand side is finite while the right side is strictly negative. Therefore $\delta(\Phi(\omega^*), \omega^*) = 0$, i.e. $\Phi(\omega^*) = \omega^*$.

Step 5. Uniqueness under additional assumption.

Assume now $u, v \in \Xi$ are fixed points of Φ and $u \neq v$. Put $s = \delta(u, v) > 0$. Plug $(\omega, \vartheta) = (u, v)$ in (4) to obtain

$$\tau + F(\delta(\Phi(u), \Phi(v))) \leq k\, F(M(u, v)).$$

Since u, v are fixed points $\Phi(u) = u$, $\Phi(v) = v$, so $\delta(\Phi(u), \Phi(v)) = s$ and $M(u, v) = s$. Hence

$$\tau + F(s) \leq k\, F(s) \implies (1 - k)F(s) \leq -\tau.$$

If $F(s) \geq 0$ for every $s > 0$ or in general $F(s) > -\frac{\tau}{1-k}$ for every $s > 0$ this is impossible. Hence $s = 0$, i.e. $u = v$. Thus, the fixed point is unique under the stated extra hypothesis. □

Remark 2.1.

(1) *When $k = 1$, the definition reduces to Wardowski's F-contraction:*

$$\tau + F(\delta(\Phi(\omega), \Phi(\vartheta))) \leq F(\delta(\omega, \vartheta)).$$

(2) *When $\frac{1}{2} \leq k < 1$, the mapping Φ represents a generalized contractive-type mapping.*

(3) *If $F(t) = \ln t$, the inequality becomes*

$$\tau + \ln \delta(\Phi(\omega), \Phi(\vartheta)) \leq k\, \ln \delta(\omega, \vartheta),$$

or equivalently,

$$\delta(\Phi(\omega), \Phi(\vartheta)) \leq e^{-\tau}\, [\delta(\omega, \vartheta)]^k,$$

which shows that the $(\tau,\, k)$-F-contraction generalizes both Banach and Wardowski contraction mappings.

Example 2.1. *Let* (X, δ) *be the metric space* $X = [0, d_0]$ *with the usual metric*

$$\delta(x, y) = |x - y|.$$

Choose the parameters

$$a = \tfrac{1}{2}, \quad k = \tfrac{1}{2}, \quad \tau = 1,$$

and define the mapping $\Phi : X \to X$ *by*

$$\Phi(x) = ax = \tfrac{1}{2}x.$$

Let the function $F : (0, \infty) \to \mathbb{R}$ *be given by*

$$F(t) = \ln t.$$

Fix $d_0 = 0.1$. *We will show that for every* $x, y \in X$ *with* $\delta(\Phi(x), \Phi(y)) > 0$, *the inequality*

$$\tau + F(\delta(\Phi(x), \Phi(y))) \le k\, F(M(x, y))$$

holds, where

$$M(x, y) = \max\left\{ |x - y|,\ |x - \Phi(x)|,\ |y - \Phi(y)|,\ \frac{|x - y| + |\Phi(x) - \Phi(y)|}{1 + |x - \Phi(x)|} \right\}.$$

We are now verifying the following:

(1) *The function* F *satisfies the conditions* (F_1)-(F_3):
- (F_1) $F(t) = \ln t$ *is strictly increasing on* $(0, \infty)$.
- (F_2) *For every sequence* $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} t_n = 0^+ \quad \Longleftrightarrow \quad \lim_{n\to\infty} \ln t_n = -\infty.$$

- (F_3) *For any* $r \in (0, 1)$,

$$\lim_{t\to 0^+} t^r \ln t = 0,$$

since t^r *dominates the logarithmic singularity as* $t \to 0^+$.

Hence, F *is admissible in the sense of Wardowski.*

(2) *Simplify the left-hand side. For* $x, y \in X$, *set* $s = |x - y| > 0$. *Then*

$$\delta(\Phi(x), \Phi(y)) = |ax - ay| = as = \tfrac{1}{2}s.$$

Thus,

$$\tau + F(\delta(\Phi(x), \Phi(y))) = \tau + \ln(as) = \tau + \ln a + \ln s.$$

(3) *Estimate the right-hand side. By definition,* $M(x, y) \ge |x-y| = s$. *Since* F *is increasing,*

$$k\, F(M(x, y)) \ge k\, \ln s.$$

Hence, it suffices to ensure the scalar inequality

$$\tau + \ln a + \ln s \le k\, \ln s, \qquad \forall\, s \in (0, d_0].$$

This can be rewritten as

$$\tau + \ln a \le (k - 1) \ln s.$$

Because $k-1<0$ and $\ln s \leq \ln d_0$ for $0<s\leq d_0$, the right-hand side is minimized when $s=d_0$. Therefore, it is enough to require

$$\tau + \ln a \leq (k-1)\ln d_0.$$

(4) *Substitute the chosen constants:*

$$\tau + \ln a = 1 + \ln \tfrac{1}{2} = 1 - 0.6931 = 0.3069,$$

and

$$(k-1)\ln d_0 = \left(\tfrac{1}{2}-1\right)\ln 0.1 = (-\tfrac{1}{2})(-2.3026) = 1.1513.$$

Since

$$0.3069 \leq 1.1513,$$

the inequality holds for all $s\in(0,d_0]$.

Therefore, for all $x,y\in X$ with $s=|x-y|>0$, we have

$$\tau + \ln(as) \leq k\ln s \leq k\ln M(x,y),$$

and consequently,

$$\tau + F(\delta(\Phi(x),\Phi(y))) \leq k\,F(M(x,y)).$$

This verifies the (τ,k)-F-contraction condition from the theorem for the entire domain $X=[0,0.1]$.

Numerical demonstration for the example.

Let $X=[0,0.1]$ with $\delta(x,y)=|x-y|$, $\Phi(x)=\frac{1}{2}x$, $F(t)=\ln t$, $k=\frac{1}{2}$, $\tau=1$, and $x_0=0.08$. The Picard iterates satisfy

$$x_n = \left(\tfrac{1}{2}\right)^n x_0, \qquad n\geq 0.$$

Thus, the successive step distances are

$$\Delta_n = \delta(x_{n+1},x_n) = x_0\left(\tfrac{1}{2}\right)^{n+1}, \qquad n\geq 0,$$

and the transformed distances are

$$a_n = F(\Delta_n) = \ln(\Delta_n) = \ln x_0 + (n+1)\ln\tfrac{1}{2}.$$

Below are numerical values for $n=0,\dots,5$ computed exactly from the closed form:

n	Δ_n	$a_n=\ln\Delta_n$	$\tau+a_n$	$k\,a_{n-1}$
0	$0.5^1\cdot 0.08=0.040$	−3.2188758	−2.2188758	—
1	0.020	−3.9120230	−2.9120230	−1.6094379
2	0.010	−4.6051702	−3.6051702	−1.9560115
3	0.005	−5.2983174	−4.2983174	−2.3025851
4	0.0025	−5.9914645	−4.9914645	−2.6491587
5	0.00125	−6.6846117	−5.6846117	−2.9957323

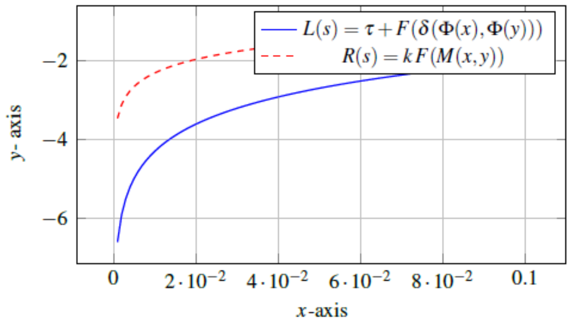

Figure 1: 2D illustration of (τ,k)-F contraction with $\tau = 1$, $k = 0.5$, $a = 0.5$.

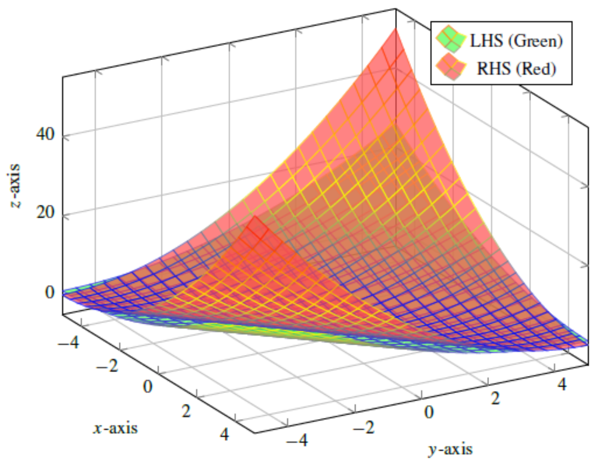

Figure 2: 3D illustration of (τ,k)-F contraction with $\tau = 1$, $k = 0.5$, $a = 0.5$.

Observations:

- The closed form $a_n = \ln x_0 + (n+1)\ln\frac{1}{2}$ shows $a_n \to -\infty$ linearly in n, hence $\Delta_n \to 0$ as $n \to \infty$. This matches Wardowski's condition (F_2).
- For $n \geq 1$ the contractive inequality from the theorem becomes

$$\tau + a_n \leq k\, a_{n-1}.$$

  The table shows $\tau + a_n \leq k a_{n-1}$ holds for $n = 1, \ldots, 5$; by the closed-form expressions this inequality holds for all $n \geq 1$ (one can check algebraically).
- Therefore the Picard sequence (x_n) is Cauchy and converges in the complete space X to the unique fixed point $x^* = 0$ (here uniqueness is obvious by inspection of Φ).

3. Application to Integral Equations

In this section, we demonstrate the applicability of the $(\tau,\ k)$-F-contraction principle to nonlinear integral equations of the Volterra–Fredholm type. Consider the functional equation

$$x(t) = g(t) + \int_a^b K(t, s, x(s))\, ds, \quad t \in [a, b], \tag{6}$$

where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are given continuous functions.

Let $C([a,b],\mathbb{R})$ denote the Banach space of all continuous real-valued functions on $[a,b]$ equipped with the metric

$$\delta(x,y) = \sup_{t\in[a,b]} |x(t)-y(t)|.$$

Define the operator $\Phi : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$(\Phi x)(t) = g(t) + \int_a^b K(t,s,x(s))\,ds.$$

We now show that Φ satisfies a $(\tau,\, k)$–F-contraction condition.

Theorem 3.1. *Suppose that:*

(1) *$K(t,s,x)$ and $g(t)$ are continuous on $[a,b]\times[a,b]\times\mathbb{R}$ and $[a,b]$, respectively;*

(2) *there exist constants $\tau > 0$ and $k \in [\frac{1}{2},1)$, and a function $F : (0,\infty) \to \mathbb{R}$ satisfying (F_1)–(F_3) such that for all $x,y\in\mathbb{R}$,*

$$|K(t,s,x) - K(t,s,y)| \le \alpha(s)\,|x-y|,$$

where $\alpha : [a,b] \to [0,\infty)$ is continuous and

$$\tau + F\left(\int_a^b \alpha(s)\,ds\right) \le k\,F(1).$$

Then the operator Φ defined above satisfies

$$\tau + F(\delta(\Phi x,\Phi y)) \le k\,F(\delta(x,y)), \quad \forall x,y\in C([a,b],\mathbb{R}),\ \delta(\Phi x,\Phi y) > 0, \tag{7}$$

and hence Φ has a unique fixed point $x^* \in C([a,b],\mathbb{R})$, which is the unique continuous solution of (6).

Proof. For any $x,y\in C([a,b],\mathbb{R})$, we have

$$|(\Phi x)(t) - (\Phi y)(t)| = \left|\int_a^b [K(t,s,x(s)) - K(t,s,y(s))]\,ds\right| \le \int_a^b \alpha(s)\,|x(s)-y(s)|\,ds.$$

Taking the supremum over $t\in[a,b]$, we obtain

$$\delta(\Phi x,\Phi y) \le \left(\int_a^b \alpha(s)\,ds\right)\delta(x,y).$$

Applying F to both sides and using its monotonicity,

$$F(\delta(\Phi x,\Phi y)) \le F\left(\int_a^b \alpha(s)\,ds\right) + F(\delta(x,y)).$$

By assumption, this satisfies (7).

Hence, Φ satisfies the $(\tau,\, k)$-F-contraction condition. By Theorem 2.1, Φ has a unique fixed point x^* in $C([a,b],\mathbb{R})$, which corresponds to the unique continuous solution of the integral equation (6). □

Remark 3.1. *The above theorem demonstrates that the $(\tau,\, k)$-F-contraction framework extends the classical fixed point techniques used in nonlinear analysis, enabling the study of fractional and integral differential equations. The parameter $\tau > 0$ introduces flexibility in convergence analysis, while $k \in [\frac{1}{2}, 1)$ ensures a generalized contractive structure that encompasses Banach, Kannan, and Wardowski-type contractions as special cases.*

4. Conclusions

In this work, we have introduced and investigated a new class of contractive mappings, namely the $(\tau,\, k)$-F-contraction mappings, which extend and unify several well-known contractions in metric fixed point theory, including those of Banach and Wardowski. By incorporating two control parameters, $\tau > 0$ and $k \in [\frac{1}{2}, 1)$, this framework provides greater flexibility in analyzing nonlinear mappings whose contractive behavior cannot be captured by classical definitions.

We established sufficient conditions for the existence and uniqueness of fixed points under the proposed $(\tau,\, k)$-F-contraction condition and demonstrated how this formulation generalizes various existing results in the literature. Moreover, we presented illustrative examples and a graphical interpretation supporting the theoretical findings.

Finally, we discussed potential applications of the $(\tau,\, k)$-F-contraction principle to nonlinear analysis, particularly in the study of fractional and integral differential equations, where the generalized contractive structure plays a crucial role in proving the existence and uniqueness of solutions. The results obtained in this paper not only broaden the scope of fixed point theory but also open new directions for future research in nonlinear functional analysis and its applications.

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