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Convergence and Stability of New Approximation Algorithms for Certain Contractive-Type Mappings

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Abstract. We present new fixed points algorithms called multistep H-iterative scheme and multistep SH-iterative scheme. Under certain contractive-type condition, convergence and stability results were established without any imposition of the 'sum conditions', which to a large extent make some existing iterative schemes so far studied by other authors in this direction practically inefficient. Our results complement and improve some recent results in literature.

1. Introduction

There is an intimate connection existing between nonlinear problems and fixed point problems of related contractive-type operators. As a result, researchers have focused more attention on finding approximate fixed points of different contractive-type mappings in recent times; see, for example, [5], [6], [7], [8], [9], [10], [11], [18], [25], [28], etc. and the reference contained in them. Let X be a normed linear space and $\Gamma : X \longrightarrow X$ a given of X. We represent the set of fixed points of Γ by $F(\Gamma) = \{q \in X : q = \Gamma(q)\}$.

For the past forty years or so, some investigation of fixed points via iterative schemes have been a flourishing area of research for many mathematicians. Mann [21], Ishikawa [22] and Noor [19] iterative schemes, with their modifications, have been studied by different authors and different interesting results were obtained. However, to meet up with the demand of the modern fixed point theory, researchers have continually renewed their efforts toward constructing more efficient iterative schemes. In this direction, following Kirk's introduction of his remarkable iterative scheme in 1971, the results below have found thier place in the current literature.

Let X and Γ be as earlier stated.

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(a) For arbitrarily $y_0 \in X$, let the sequence $\{y_n\}_{n=0}^{\infty}$ be defined iteratively as follows:

$$y_{n+1} = \sum_{j=0}^{\ell} \alpha_j \Gamma^j y_n, \sum_{j=0}^{\ell} \alpha_j = 1, n \geq 0. \tag{1.1}$$

The iteration method defined by (1.1) is due to Kirk [20].

(b) In [17], Olatinwo presented the algorithms below:

(i) for an arbitrary point $y_0 \in X$ and for $\alpha_{n,t} \geq 0, \alpha_{n,0} \neq 0, \alpha_{n,t} \in [0,1]$ *and* ℓ as a fixed integer, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$y_{n+1} = \sum_{t=0}^{\ell} \alpha_{n,t} \Gamma^t y_n, \sum_{t=0}^{\ell} \alpha_{n,t} = 1, n \geq 0 \tag{1.2}$$

(ii) for an arbitrary point $y_0 \in X$ and for $\ell \geq m, \alpha_{n,t} \beta_{n,t} \geq 0, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,t}, \beta_{n,t} \in [0,1]$ *and* ℓ, m as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \alpha_{n,0} y_n + \sum_{t=0}^{\ell} \alpha_{n,t} \Gamma^j z_n, \sum_{t=0}^{\ell} \alpha_{n,t} = 1; \\ z_n &= \sum_{t=0}^{m} \beta_{n,t} \Gamma^t y_n, \sum_{t=0}^{\ell} \beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.3}$$

and called them Kirk-Mann and Kirk-Ishikawa algorithms, respectively.

(c) Chugh and Kumar [25] presented the following iterative scheme: for an arbitrary point $y_0 \in X$ and for $\ell \geq m \geq p, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \beta_{n,0} \neq 0, \alpha_{n,s}, \gamma_{n,r}, \beta_{n,t} \in [0,1]$ *and* ℓ, m, p as fixed integers, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0} y_n + \sum_{r=1}^{\ell} \gamma_{n,r} \Gamma^r z_n, \sum_{r=0}^{\ell} \gamma_{n,r} = 1; \\ z_n &= \alpha_{n,0} y_n + \sum_{s=1}^{m} \alpha_{n,s} \Gamma^s z_n, \sum_{s=0}^{m} \alpha_{n,s} = 1; \\ z_n &= \sum_{t=0}^{p} \beta_{n,t} \Gamma^t y_n, \sum_{t=0}^{p} \beta_{n,t} = 1, n \geq 0, \end{aligned} \tag{1.4}$$

(d) Very recently, Akewe, Okeke and Olayiwola [26] presented the following general iterative scheme in the sense of Kirk [20]:

(i) for an arbitrary point $y_0 \in X$, for $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_u$, for each i, $\alpha_{n,s}^t, \gamma_{n,t} \geq 0, \gamma_{n,0}, \alpha_{n,0}, \neq 0$, for each i, $\alpha_{n,s}^i, \gamma_{n,t} \in [0,1]$ and ℓ_1, ℓ_u as fixed integers for each u, define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0} y_n + \sum_{r=1}^{\ell_1} \gamma_{n,r} \Gamma^r z_n^1, \sum_{k=0}^{\ell_1} \alpha_{n,r} = 1; \\ z_n^t &= \alpha_{n,0}^t y_n + \sum_{s=1}^{\ell_{t+1}} \alpha_{n,s}^t \Gamma^j z_n^{t+1}, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1, t = 1, 2, \cdots, u-2; \\ z_n^{u-1} &= \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} \Gamma^s y_n, \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1, u \geq 2, n \geq 0, \end{aligned} \tag{1.5}$$

(ii) for an arbitrary point $y_0 \in X$, retaining the conditions in (i), define the sequence $\{y_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} y_{n+1} &= \gamma_{n,0} z_n^1 + \sum_{r=1}^{\ell_1} \gamma_{n,k} \Gamma^r z_n^1, \sum_{r=0}^{\ell_1} \alpha_{n,r} = 1; \\ z_n^t &= \alpha_{n,0}^t z_n^{t+1} + \sum_{s=1}^{\ell_{t+1}} \alpha_{n,s}^t \Gamma^s z_n^{t+1}, \sum_{s=0}^{\ell_{t+1}} \alpha_{n,s}^t = 1, t = 1, 2, \cdots, u-2; \\ z_n^{u-1} &= \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} \Gamma^s y_n, \sum_{s=0}^{\ell_u} \alpha_{n,t}^{u-1} = 1, u \geq 2, n \geq 0, \end{aligned} \tag{1.6}$$

It is worthy to mention that in application, the stability of the iterative schemes studied above is quite invaluable. The first researcher to demonstrate this respecting the Banach contraction conditions is Ostrowski [13]. Afterwards, several authors have developed this subject basically because of its indispensable position in the current trend of computer programing. Some recent works in this direction could be seen in [1], [2], [3], [4], [12], [13], [14],
[23], [24], [26] and the references therein.

Remark 1.1. *The stability and the convergence results in the papers studied were made possible due to the sum conditions imposed on the control parameters; see, for example, [20], [17], [25], [26], etc and the references therein. But in application, especially for $\mathbb{N}$ large enough, the iterative schemes defined by* (1.1), (1.2), (1.3), (1.4) (1.5) *and* (1.6) *become practically inefficient due to the difficulties involved in generating a family of such control parameters, the windy process involved for each sum and the computational cost.*

Base on the problems mentioned in Remark 1.1, it becomes necessary to ask the following questions:

Question 1.1. *Is it possible to construct an alternative iterative scheme that would address the problems generated by the sum conditions imposed on the control parameters while maintaining, in particular, the results in [26], which in a larger sense contains the results of the other papers studied?*

Following the same argument as in [27] regarding the linear combination of the products of countably finite family of control parameters and the problems mentioned in Remark 1.1, in this paper, we provide an affirmative answer to Question 1.1.

2. Preliminary

Throughout the remaining sections, $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \mathbb{R}^+, \mathbb{N}$ *and* H will denote monotone increasing subadditive function, the set of positive integers, the set of natural numbers and a real Hilbert space, respectively. Also, the following definition, lemmas and propositions will be needed establish our results.

Definition 2.1. *([13]) Suppose* Y *is a metric space and let* $\Gamma : Y \longrightarrow Y$ *be a self-map of* Y*. Let* $\{x_n\}_{n=0}^{\infty} \subseteq Y$ *be a sequence generated by an iteration scheme*

$$x_{n+1} = g(\Gamma, x_n), \tag{2.1}$$

where $x_0 \in Y$ *is the initial approximation and* g *is some function. Suppeose* $\{x_n\}_{n=0}^{\infty}$ *converges to a fixed point* q *of* Γ*. Let* $\{t_n\}_{n=0}^{\infty} \subseteq Y$ *be an arbitrary sequence and set* $\epsilon_n = d(t_n, g(\Gamma, t_n)), n = 1, 2, \cdots$ *Then, (2.1) is said to be* Γ*-stable if and only if* $\lim_{n\to\infty} \epsilon_n = 0$ *implies* $\lim_{n\to\infty} y_n = q$*.*

Note that in practice, the sequence $\{t_n\}_{n=0}^{\infty}$ could be obtained using the following approach: let $x_0 \in Y$. Set $x_{n+1} = g(\Gamma, x_n)$ and let $t_0 = x_0$. Since, $x_1 = g(\Gamma, x_0)$ following the rounding in the function Γ, the value t_1 (which is estimated to be equal to x_1) could be calculated to give t_2, an approximate value of $g(\Gamma, t_1)$. The procedure is continued to yield the sequence $\{t_n\}_{n=0}^{\infty}$, which is approximately tha same as the sequence $\{x_n\}_{n=0}^{\infty}$.

Lemma 2.1. *(see, e.g., [26]) Let* $\{\tau_n\}_{n=0}^{\infty} \in \mathbb{R}^+ : \tau_n \to 0$ *as* $n \to \infty$*. For* $0 \le \delta < 1$*, let* $\{w_n\}_{n=0}^{\infty}$ *be a sequence of positive numbers satisfying* $w_{n+1} \le \delta w_n + \tau_n, n = 0, 1, 2, \cdots$ *Then,* $w_n \to 0$ *as* $n \to \infty$*.*

Lemma 2.2. *(see, e.g., [17]) Let* $(Y, \| .\|)$ *be a normed space, the self-map* $\Gamma : Y \longrightarrow Y$ *satisfies (1.13) and* $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *(retaining its usual meaning) be such that* $\psi(0) = 0, \phi(Mt) = M\phi(t), M \ge 0, t \in \mathbb{R}^+$*. Then,* $\forall i \in \mathbb{N}$ *and* $\forall s, t \in Y$*, we have*

$$\|\Gamma^j s - \Gamma^j t\| \le \rho^j \|s - t\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|s - \Gamma s\|). \tag{2.2}$$

Proposition 2.3. *(see,e.g., [27]) Let* $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$*, where* $k \in [0, \mathbb{R}^+]$ *is fixed and* $N \in \mathbb{N}$ *is any integer with* $k + 1 \le N$*. Then, the following holds:*

$$\alpha_k + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1} (1 - \alpha_j) + \prod_{j=k}^{N} (1 - \alpha_j) = 1. \tag{2.3}$$

Proposition 2.4. *(see,e.g., [27]) Let $t, u, v \in H$. Let $k \in [0, \mathbb{R}^+]$ be fixed and $N \in \mathbb{N}$ be such that $k+1 \leq N$. Let $\{v_i\}_{i=1}^{N-1} \subseteq H$ and $\{\alpha_i\}_{i=1}^{N} \subseteq [0,1]$. Define*

$$y = \alpha_k t + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{N}(1-\alpha_j)v.$$

Then,

$$\begin{aligned} \|y-u\|^2 \;=\; & \alpha_k\|t-u\|^2 + \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-u\|^2 + \prod_{j=k}^{N}(1-\alpha_j)\|v-u\|^2 \\ & -\alpha_k\Big[\sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v_{i-1}\|^2 + \prod_{j=k}^{i-1}(1-\alpha_j)\|t-v\|^2\Big] \\ & -(1-\alpha_k)\Big[\sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)\|v_{i-1}-(\alpha_{i+1}+w_{i+1})\|^2 \\ & +\alpha_N \prod_{j=k}^{i-1}(1-\alpha_j)\|v-v_{N-1}\|^2\Big], \end{aligned}$$

where $w_k = \sum_{i=k+1}^{N} \alpha_i \prod_{j=k}^{i-1}(1-\alpha_j)v_{i-1} + \prod_{j=k}^{i-1}(1-\alpha_j)v$, $k = 1, 2, \cdots, N$ and $w_n = (1-c_n)v$.

3. Main Results I

Let H be a Hilbert space and let $\Gamma : H \longrightarrow H$ be a self-map of X. For arbitrary $x_0 \in H$ define the sequence $\{x_{n+1}\}_{n=0}^{\infty}$ iteratively, for $s = 1, 2, \cdots, k-2$, as follows:

$$\begin{cases} x_{n+1} = \delta_{n,1}x_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}y_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}y_n^1; \\ y_n^s = \alpha_{n,1}^s x_n + \sum_{j=2}^{\ell_{s+1}} \alpha_{n,j}^s \prod_{i=1}^{j-1}(1-\alpha_{n,i}^s)\Gamma^{j-1}y_n^{s+1} + \prod_{i=1}^{\ell_{s+1}}(1-\alpha_{n,i}^s)\Gamma^{\ell_1}y_n^{s+1}; \\ y_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})\Gamma^{j-1}x_n + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})\Gamma^{\ell_k}x_n,\ k \geq 2, n \geq 1, \end{cases} \tag{3.1}$$

where $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k$, for each s, $\{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k} \in [0,1]$ for each k and $\ell_1, \ell_2, \cdots, \ell_k$ are fixed integers (for each k). We shall call the iteration scheme defined by (3.1) the multistep IH-iteration scheme.

Again, for any $x_0 \in X$, we shall call the sequence $\{x_n\}_{n=0}^{\infty}$ defined recursively, for $s = 1, 2, \cdots, k-2$, by

$$\begin{cases} x_{n+1} = \delta_{n,1}y_n^1 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}y_n^1 + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}y_n^1; \\ y_n^s = \alpha_{n,1}^s y_n^{s+1} + \sum_{j=2}^{\ell_{s+1}} \alpha_{n,j}^s \prod_{i=1}^{j-1}(1-\alpha_{n,i}^s)\Gamma^{j-1}y_n^{s+1} + \prod_{i=1}^{\ell_{s+1}}(1-\alpha_{n,i}^s)\Gamma^{\ell_1}y_n^{s+1}; \\ y_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{k-1})\Gamma^{j-1}x_n + \prod_{i=1}^{\ell_k}(1-\alpha_{n,i}^{k-1})\Gamma^{\ell_k}x_n,\ k \geq 2, n \geq 1, \end{cases} \tag{3.2}$$

where $\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_k$, for each s, $\{\{\delta_{n,i}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k}, \{\{\alpha_{n,i}\}_{n=0}^{\infty}\}_{j=1}^{\ell_k} \in [0,1]$ for each k and $\ell_1, \ell_2, \cdots, \ell_k$ are fixed integers (for each k), the multistep DI-iteration scheme.

Theorem 3.1. *Let H be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of H satisfying the contractive condition*

$$\|\Gamma^j x - \Gamma^j y\| \leq \rho^j \|x - y\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-i} \phi(\|x - \Gamma x\|), \tag{3.3}$$

where $x, y \in H, 0 \leq \rho^j < 1$, and let ϕ retain its usual meaning with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. For arbitrary $x_0 \in H$, let $\{\omega_n\}_{n=0}^{\infty}$ be the multistep H-iteration scheme defined by (3.1). Then,

(*i*) *Γ defined by (3.3) has a fixed point q;*

(*ii*) *the multistep IH-iteration scheme converges strongly to $q \in \Gamma$.*

Proof. Firstly, we show that Γ satisfying condition of (3.3) has a fixed point. Assume there exists two points $q_1, q_2 \in F(\Gamma)$ with $0 < \|q_1 - q_2\|$. Then, we have

$$\begin{aligned} 0 < \|q_1 - q_2\| = \|\Gamma^j q_1 - \Gamma^j q_2\| &\leq \rho^j \|q_1 - q_2\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-i} \phi(\|q\sqrt{1} - \Gamma q_1\|) \\ &= \rho^j \|q_1 - q_2\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-i} \phi(0) \end{aligned}$$

$\Rightarrow (1 - \rho^j)\rho^j \|q_1 - q_2\| \leq 0$. Using the fact that $\rho^j \in [[0, 1)$, we get $0 < 1 - \rho^j$ and $\|q_1 - q_2\| \leq 0$. Since the norm is a nonnegative function, we get $\|q_1 - q_2\| = 0; q_1 = q_2 = q(say)$. Therefore, Γ converges uniquely to a point of $F(\Gamma)$.

Now, we show that the sequence defined by (3.1) converges strongly to $q \in F(\Gamma)$. Using (3.3) and Proposition 2.4 with $x_{n+1} = y, u = q, x_n = t, j = i, k = 1, \Gamma^{j-1} y_n^1 = v_{j-1}$ and $\Gamma^{\ell_1} y_n^1 = v$, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \delta_{n,1}\|x_n - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1 - \delta_{n,i}) \|\Gamma^{j-1} y_n^1 - \Gamma^{j-1} q\|^2 \\ &+ \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) \|\Gamma^{\ell_1} y_n^1 - \Gamma^{\ell_1} q\|^2 \end{aligned} \tag{3.4}$$

But from (3.3), with $y = y_n^1$, we have

$$\begin{aligned} \|\Gamma^{j-1} y_n^1 - \Gamma^{j-1} q\| &\leq \rho^j \|y_n^1 - q\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|q - \Gamma q\|) \\ &= \rho^j \|y_n^1 - q\| \end{aligned} \tag{3.5}$$

Proposition 2.3, (3.4) and (3.5) imply

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \delta_{n,1}\|x_n-q\|^2+\sum_{j=2}^{\ell_1}\delta_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\delta_{n,i})\|y_n^1-q\|^2\\
&\quad+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\|y_n^1-q\|^2\\
&= \delta_{n,1}\|x_n-q\|^2+\left(1-\delta_{n,1}^1-\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\right)\|y_n^1-q\|^2\\
&\quad+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\|y_n^1-q\|^2\\
&= \delta_{n,1}\|x_n-q\|^2+\left(1-\delta_{n,1}^1\right)\|y_n^1-q\|^2 \qquad (3.6)
\end{aligned}$$

Since ℓ_1,ℓ_k are fixed integers and $\alpha_{n,i}^s\in[0,1]$ for each s, we have, using Proposition 2.3, the following estimates for $n=1,2,\cdots$ and $1\le s\le k-1$:

$$\begin{aligned}
\|y_n^1-q\|^2 &\le \alpha_{n,1}\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}\prod_{i=1}^{j-1}(1-\alpha_{n,i})\|\Gamma^{j-1}y_n^2-\Gamma^{j-1}q\|^2\\
&\quad+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i})\|\Gamma^{\ell_2}y_n^2-\Gamma^{\ell_2}q\|^2\\
&\le \alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})\|y_n^2-q\|^2\\
&\quad+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i})(\rho^j)^2\|y_n^2-q\|^2\\
&\le \alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\Big[\alpha_{n,1}^2\|x_n-q\|^2\\
&\quad+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\|y_n^3-q\|^2+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\|y_n^3-q\|^2\Big]\\
&\quad+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\Big[\alpha_{n,1}^2\|x_n-q\|^2+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\|y_n^3-q\|^2\\
&\quad+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\|y_n^3-q\|^2\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_{n,1}^{1}\|x_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\alpha_{n,1}^{2}\|x_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\|y_n^3 - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\|y_n^3 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\alpha_{n,1}^{2}\|x_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\|y_n^3 - q\|^2 \\
&\quad + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\|y_n^3 - q\|^2 \\
&\le \alpha_{n,1}^{1}\|x_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\alpha_{n,1}^{2}\|x_n - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\alpha_{n,1}^{2}\|x_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\Big[\alpha_{n,1}^{3}\|x_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_4} \alpha_{n,j}^{3}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})\|y_n^4 - q\|^2 + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{4})(\rho^j)^2\|y_n^4 - q\|^2\Big] \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\Big[\alpha_{n,1}^{3}\|x_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_4} \alpha_{n,j}^{3}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})\|y_n^4 - q\|^2 + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{4})(\rho^j)^2\|y_n^4 - q\|^2\Big] \\
&\quad + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\Big[\alpha_{n,1}^{3}\|x_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_4} \alpha_{n,j}^{3}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})\|y_n^4 - q\|^2 + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{4})(\rho^j)^2\|y_n^4 - q\|^2\Big] \\
&\quad + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\Big[\alpha_{n,1}^{3}\|x_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_4} \alpha_{n,j}^{3}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})\|y_n^4 - q\|^2 + \prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{4})(\rho^j)^2\|y_n^4 - q\|^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
= \; & \alpha_{n,1}^1\|x_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n - q\|^2 \\
& + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|x_n - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|x_n - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right) \\
& \times \left(\sum_{j=2}^{\ell_4} \alpha_{n,j}^3 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|y_n^4 - q\|^2 + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right) \\
& \times \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4 - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|x_n - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\sum_{j=2}^{\ell_4} \alpha_{n,j}^3 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|y_n^4 - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4 - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\sum_{j=2}^{\ell_4} \alpha_{n,j}^3 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|y_n^4 - q\|^2 \\
& + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4 - q\|^2 \\
& + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n - q\|^2 \\
& + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\sum_{j=2}^{\ell_4} \alpha_{n,j}^3 (\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|y_n^4 - q\|^2 \\
& + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4 - q\|^2
\end{aligned}
$$

$$
\begin{aligned}
= \;& \alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2 \\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|x_n-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|x_n-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right) \\
&\times\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2\right)\|y_n^4-q\|^2+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right) \\
&\times\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right) \\
&\times\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2)\right)\|y_n^4-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right) \\
&\times\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2)\right)\|y_n^4-q\|^2 \\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4-q\|^2 \\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2\right)\|y_n^4-q\|^2 \\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|y_n^4-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2 \\
& +\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
& +\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)(1-\alpha_{n,1}^3)(\rho^j)^2\|y_n^4-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)(1-\alpha_{n,1}^3)(\rho^j)^2)\|y_n^4-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)(1-\alpha_{n,1}^3)(\rho^j)^2)\|y_n^4-q\|^2 \\
& +\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)(1-\alpha_{n,1}^3)(\rho^j)^2\|y_n^4-q\|^2 \\
\leq \quad & \alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2 \\
& +\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|x_n-q\|^2 \\
& +\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n-q\|^2
\end{aligned}
$$

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\begin{aligned}
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2)\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2)\|x_n-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2\|x_n-q\|^2+\cdots\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\\
&\times\left(\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\times\cdots\times\left(\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\
&\times\left(\sum_{j=2}^{\ell_s}\alpha_{n,j}^{\ell_{s-1}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\alpha_{n,1}^s\|x_n-q\|^2+(\rho^j)^2\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times(\rho^j)^2\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)(\rho^j)^2\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\right)\times\cdots\times(\rho^j)^2\left(\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^j)^2\right)\\
&\times(\rho^j)^2\left(\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})(\rho^j)^2\right)\|x_n-q\|^2\\
<\ &\alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\|x_n-q\|^2\\
&+\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\|x_n-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\|x_n-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2\\
&+\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2))\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2+\cdots\\
&+\left(1-\alpha_{n,1}^1-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^2)\right)\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\\
&\times\left(1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}-\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\
&\times\left(1-\alpha_{n,1}^{\ell_{s-1}}-\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\alpha_{n,1}^s\|x_n-q\|^2+\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\\
&\times\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)\right)\times\cdots\times\left(\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\
&\times\left(\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\|x_n-q\|^2 \qquad (3.7)\\
<\ &\alpha_{n,1}^1\|x_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\|x_n-q\|^2\\
&+\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\|x_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2\\
&+\left(1-\alpha_{n,1}^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2+\cdots+\\
&+\left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right)\left(1-\alpha_{n,1}^3\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right)\\
&\times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)\alpha_{n,1}^s\|x_n-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_{n,1}^1\|x_n-q\|^2+\alpha_{n,1}^2\left(1-\alpha_{n,1}^1-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\|x_n-q\|^2\\
&\quad+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|x_n-q\|^2\\
&\quad+\left((1-\alpha_{n,1}^1-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\|x_n-q\|^2\\
&\quad+\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\|x_n-q\|^2\\
&\quad+\left((1-\alpha_{n,1}^1-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2\\
&\quad+\left(1-\alpha_{n,1}^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|x_n-q\|^2+\cdots+\\
&\quad+\left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right)\left(1-\alpha_{n,1}^3\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right)\\
&\quad\times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)\alpha_{n,1}^s\|x_n-q\|^2\\
&< [\alpha_{n,1}^1+\alpha_{n,1}^2\left(1-\alpha_{n,1}^1\right)+(1-\alpha_{n,1}^1)\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)+\left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\\
&\quad+\cdots+\left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right)\left(1-\alpha_{n,1}^3\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right)\\
&\quad\times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)]\|x_n-q\|^2 \qquad (3.8)
\end{aligned}
$$

(3.6) and (3.8) imply that

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &\leq \{\delta_{n,1}+(1-\delta_{n,1})[\alpha_{n,1}^1+\alpha_{n,1}^2\left(1-\alpha_{n,1}^1\right)+(1-\alpha_{n,1}^1)\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\\
&\quad+(1-\alpha_{n,1}^1)\left(1-\alpha_{n,1}^2\right)(1-\alpha_{n,1}^3)+\cdots+\left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right)\left(1-\alpha_{n,1}^3\right)\\
&\quad\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right)\times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)]\}\|x_n-q\|^2 \qquad (3.9)
\end{aligned}
$$

Using Lemma 2.3, we obtain (from (3.9)) that the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q\in F(\Gamma)$; and this completes the proof. □

Theorem 3.2. *Let H be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of H satisfying the contractive condition*

$$\|\Gamma^j x-\Gamma^j y\|\leq \rho^j\|x-y\|+\sum_{i=0}^{j}\binom{j}{i}\rho^{j-1}\phi(\|x-\Gamma x\|), \qquad (3.10)$$

where $x,y\in H, 0\leq\rho^j<1$, and let ϕ retain its usual meaning with $\phi(0)=0$ and $\phi(Mt)=M\phi(t), M\geq 0, t\in\mathbb{R}^+$. For arbitrary $x_0\in H$, let $\{\omega_n\}_{n=0}^{\infty}$ be the multistep DI-iteration scheme defined by (3.2). Then,

(i) Γ defined by (3.10) has a fixed point q;

(*ii*) *the multistep SH-iteration scheme converges strongly tp $q \in \Gamma$.*

Proof. We first show that Γ satisfying condition of (3.10) has a fixed point. Assume there exists two points $q_1, q_2 \in F(\Gamma)$ with $0 < \|q_1 - q_2\|$. Then, we have

$$\begin{aligned} 0 < \|q_1 - q_2\| = \|\Gamma^j q_1 - \Gamma^j q_2\| &\leq \rho^j \|q_1 - q_2\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|q\sqrt{1} - \Gamma q_1\|) \\ &= \rho^j \|q_1 - q_2\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(0) \end{aligned}$$

$\Rightarrow (1-\rho^j)\rho^j \|q_1 - q_2\| \leq 0$. Using the fact that $\rho^j \in [[0,1)$, we get $0 < 1 - \rho^j$ and $\|q_1 - q_2\| \leq 0$. Since the norm is a nonnegative function, we get $\|q_1 - q_2\| = 0; q_1 = q_2 = q(say)$. Therefore, Γ converges uniquely to a point of $F(\Gamma)$.

Now, we show that the sequence defined by (3.1) converges strongly to $q \in F(\Gamma)$. Using (3.2) and Proposition 2.4 with $x_{n+1} = y, u = q, y_n^1 = t, j = i, k = 1, \Gamma^{j-1} y_n^1 = v_{j-1}$ and $\Gamma^{\ell_1} y_n^1 = v$, we get

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \delta_{n,1} \|y_n^1 - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1 - \delta_{n,i}) \|\Gamma^{j-1} y_n^1 - \Gamma^{j-1} q\|^2 \\ &\quad + \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) \|\Gamma^{\ell_1} y_n^1 - \Gamma^{\ell_1} q\|^2 \end{aligned} \tag{3.11}$$

But from (3.10), with $y = y_n^1$, we have

$$\begin{aligned} \|\Gamma^{j-1} y_n^1 - \Gamma^{j-1} q\| &\leq \rho^j \|y_n^1 - q\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|q - \Gamma q\|) \\ &= \rho^j \|y_n^1 - q\| \end{aligned} \tag{3.12}$$

Proposition 2.3, (3.11) and (3.12) imply

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \delta_{n,1}^1 \|y_n^1 - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1 - \delta_{n,i}) \|y_n^1 - q\|^2 \\ &\quad + \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) (\rho^j)^2 \|y_n^1 - q\|^2 \\ &= \delta_{n,1}^1 \|y_n^1 - q\|^2 + \left(1 - \delta_{n,1}^1 - \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) (\rho^j)^2 \right) \|y_n^1 - q\|^2 \\ &\quad + \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) (\rho^j)^2 \|y_n^1 - q\|^2 \\ &= \|y_n^1 - q\|^2 \end{aligned} \tag{3.13}$$

Since ℓ_1, ℓ_k are fixed integers and $\alpha_{n,i}^s \in [0,1]$ for each s, we have (using Proposition 2.3, (3.2) and (3.12)) the following estimates for $n = 1, 2, \cdots$ and $1 \leq s \leq k-1$:

$$
\begin{aligned}
\|y_n^1 - q\|^2 \leq\ & \alpha_{n,1}^1 \|y_n^2 - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) \|\Gamma^{j-1} y_n^2 - \Gamma^{j-1} q\|^2 \\
& + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1) \|\Gamma^{\ell_2} y_n^2 - \Gamma^{\ell_2} q\|^2 \\
\leq\ & \left(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1)(\rho^j)^2 \right) \|y_n^2 - q\|^2 \\
\leq\ & \left(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1)(\rho^j)^2 \right) \Bigg[\alpha_{n,1}^2 \|y_n^3 - q\|^2 \\
& + \sum_{j=2}^{\ell_3} \alpha_{n,j}^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^2) \|\Gamma^{j-1} y_n^3 - \Gamma^{j-1} q\|^2 + \prod_{i=1}^{\ell_3} (1-\alpha_{n,i}^2) \|\Gamma^{\ell_3} y_n^3 - \Gamma^{\ell_3} q\|^2 \Bigg] \\
\leq\ & \left(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1)(\rho^j)^2 \right) \Bigg[\alpha_{n,1}^2 \|y_n^3 - q\|^2 \\
& + \sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) \|y_n^2 - q\|^2 + \prod_{i=1}^{\ell_3} (1-\alpha_{n,i}^2)(\rho^j)^2 \|y_n^3 - q\|^2 \Bigg] \\
=\ & \left(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1)(\rho^j)^2 \right) \\
& \times \left(\alpha_{n,1}^2 + \sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) + \prod_{i=1}^{\ell_3} (1-\alpha_{n,i}^2)(\rho^j)^2 \right) \|y_n^3 - q\|^2 \qquad (3.14) \\
\leq\ & \left(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1)(\rho^j)^2 \right) \\
& \times \left(\alpha_{n,1}^2 + \sum_{j=2}^{\ell_3} \alpha_{n,j}^2 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) + \prod_{i=1}^{\ell_3} (1-\alpha_{n,i}^2)(\rho^j)^2 \right) \Bigg[\alpha_{n,1}^3 \|y_n^4 - q\|^2 \\
& + \sum_{j=2}^{\ell_4} \alpha_{n,j}^3 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^3) \|\Gamma^{j-1} y_n^4 - \Gamma^{j-1} q\|^2 + \prod_{i=1}^{\ell_4} (1-\alpha_{n,i}^3) \|\Gamma^{\ell_4} y_n^4 - \Gamma^{\ell_4} q\|^2 \Bigg] \\
\leq\ & \left(\alpha_{n,1}^1 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1 (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}^1) + \prod_{i=1}^{\ell_2} (1-\alpha_{n,i}^1)(\rho^j)^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\alpha_{n,1}^2+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\Bigg[\alpha_{n,1}^3\|y_n^4-q\|^2\\
&+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|y_n^4-q\|^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\|y_n^4-q\|^2\Bigg]\\
= &\left(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^2+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\right)\|y_n^4-q\|^2\\
\leq &\left(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^2+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\right)\Bigg[\alpha_{n,1}^4\|y_n^5-q\|^2\\
&+\sum_{j=2}^{\ell_5}\alpha_{n,j}^4\prod_{i=1}^{j-1}(1-\alpha_{n,i}^4)\|\Gamma^{j-1}y_n^5-\Gamma^{j-1}q\|^2+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^4)\|\Gamma^{\ell_5}y_n^5-\Gamma^{\ell_5}q\|^2\Bigg]\\
\leq &\left(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^2+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\right)\Bigg[\alpha_{n,1}^4\|y_n^5-q\|^2\\
&+\sum_{j=2}^{\ell_5}\alpha_{n,j}^4(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^4)\|y_n^5-q\|^2+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^4)(\rho^j)^2\|y_n^5-q\|^2\Bigg]\\
= &\left(\alpha_{n,1}^1+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^2+\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^3+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\alpha_{n,1}^{4}+\sum_{j=2}^{\ell_5}\alpha_{n,j}^{4}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{4})+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^{4})(\rho^{j})^{2}\right)\|y_n^5-q\|^2\\
\leq\ &\left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{3}+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{4}+\sum_{j=2}^{\ell_5}\alpha_{n,j}^{4}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{4})+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^{4})(\rho^{j})^{2}\right)\\
&\times\cdots\times\left(\alpha_{n,1}^{\ell_{s-2}}+\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{\ell_{s-1}}+\sum_{j=2}^{\ell_{s}}\alpha_{n,j}^{\ell_{s-1}}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})+\prod_{i=1}^{\ell_{s}}(1-\alpha_{n,i}^{\ell_{s-1}})(\rho^{j})^{2}\right)\\
&\times\|x_n-q\|^2 \qquad (3.15)
\end{aligned}
$$

(3.13) and (3.15) imply that

$$
\begin{aligned}
\|x_{n+1}-q\|^2\ \leq\ &\left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{3}+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{4}+\sum_{j=2}^{\ell_5}\alpha_{n,j}^{4}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{4})+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^{4})(\rho^{j})^{2}\right)\\
&\times\cdots\times\left(\alpha_{n,1}^{\ell_{s-2}}+\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{\ell_{s-1}}+\sum_{j=2}^{\ell_{s}}\alpha_{n,j}^{\ell_{s-1}}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})+\prod_{i=1}^{\ell_{s}}(1-\alpha_{n,i}^{\ell_{s-1}})(\rho^{j})^{2}\right)\\
&\times\|x_n-q\|^2 \qquad (3.16)
\end{aligned}
$$

Since $\rho^j \in [0,1]$, we obtain using Proposition 2.3, for $j = 1, 2, 3, \cdots, s-1$, that

$$Q \leq P = 1, \tag{3.17}$$

where

$$\begin{aligned} Q &= \left(\alpha^3_{n,1} + \sum_{j=2}^{\ell_4} \alpha^3_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^3_{n,i}) + \prod_{i=1}^{\ell_4} (1-\alpha^3_{n,i})(\rho^j)^2\right) \\ &\times \left(\alpha^4_{n,1} + \sum_{j=2}^{\ell_5} \alpha^4_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^4_{n,i}) + \prod_{i=1}^{\ell_5} (1-\alpha^4_{n,i})(\rho^j)^2\right) \\ &\times \cdots \times \left(\alpha^{\ell_{s-2}}_{n,1} + \sum_{j=2}^{\ell_{s-1}} \alpha^{\ell_{s-2}}_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^{\ell_{s-2}}_{n,i}) + \prod_{i=1}^{\ell_{s-1}} (1-\alpha^{\ell_{s-2}}_{n,i})(\rho^j)^2\right) \\ &\times \left(\alpha^{\ell_{s-1}}_{n,1} + \sum_{j=2}^{\ell_s} \alpha^{\ell_{s-1}}_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^{\ell_{s-1}}_{n,i}) + \prod_{i=1}^{\ell_s} (1-\alpha^{\ell_{s-1}}_{n,i})(\rho^j)^2\right) \end{aligned}$$

and

$$\begin{aligned} P &= \left(\alpha^3_{n,1} + \sum_{j=2}^{\ell_4} \alpha^3_{n,j} \prod_{i=1}^{j-1} (1-\alpha^3_{n,i}) + \prod_{i=1}^{\ell_4} (1-\alpha^3_{n,i})\right) \\ &\times \left(\alpha^4_{n,1} + \sum_{j=2}^{\ell_5} \alpha^4_{n,j} \prod_{i=1}^{j-1} (1-\alpha^4_{n,i}) + \prod_{i=1}^{\ell_5} (1-\alpha^4_{n,i})\right) \\ &\times \cdots \times \left(\alpha^{\ell_{s-2}}_{n,1} + \sum_{j=2}^{\ell_{s-1}} \alpha^{\ell_{s-2}}_{n,j} \prod_{i=1}^{j-1} (1-\alpha^{\ell_{s-2}}_{n,i}) + \prod_{i=1}^{\ell_{s-1}} (1-\alpha^{\ell_{s-2}}_{n,i})\right) \\ &\times \left(\alpha^{\ell_{s-1}}_{n,1} + \sum_{j=2}^{\ell_s} \alpha^{\ell_{s-1}}_{n,j} \prod_{i=1}^{j-1} (1-\alpha^{\ell_{s-1}}_{n,i}) + \prod_{i=1}^{\ell_s} (1-\alpha^{\ell_{s-1}}_{n,i})\right) \end{aligned}$$

Applying (3.17) in (3.16), we obtain, using Lemma 2.3 that the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.2) converges strongly to the fixed point q in $F(\Gamma)$. Thus, the proof is completed. □

Example 3.1. *Let the operator* $\Gamma : [0,1] \longrightarrow [0,1]$ *be defined as*

$$\Gamma z = \frac{z}{3}, \forall z \in [0,1].$$

Clearly, Γ *is quasi-contractive satisfying* (2.2) *with a unique fixed point* 0*; see, for example, [26] for details. Set*

$$\alpha^1_{n,1} = \delta^1_{n,1} = \frac{1}{\sqrt{n+1}}, n = 1, 2, \cdots, n_0, \ for\ n_0 \in \mathbb{N};$$

$$\delta_{n,i} = 1 - \delta^1_{n,1}, \ for\ i = 1, 2, \cdots, \ell_1 \ and$$

$$\alpha^s_{n,i} = 1 - 2\alpha^1_{n,1}, \ for\ i = 1, 2, \cdots, \ell_{s+1}, s = 1, 2, \cdots, n_0.$$

It is not hard to see that all the conditions of Theorem 3.1 and Theorem 3.2 has been satisfied by Example 3.1.

4. Main Results II

Here, we consider stablity results for the multistep IH-iteration scheme and the multistep DI-iteration scheme defined by (3.1) and (3.2) for operators satisfying (2.2), respectively.

Theorem 4.1. *Let H be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of H satisfying the contractive condition*

$$\|\Gamma^j x - \Gamma^j y\| \leq \rho^j \|x - y\| + \sum_{i=0}^{j} \binom{j}{i} \rho^{j-1} \phi(\|x - \Gamma x\|), \tag{4.1}$$

where $x, y \in H, 0 \leq \rho^j < 1$, and let ϕ retains its usual meaning with $\phi(0) = 0$ and $\phi(Mt) = M\phi(t), M \geq 0, t \in \mathbb{R}^+$. For arbitrary $x_0 \in H$, let $\{x_n\}_{n=0}^{\infty}$ be the multistep DI-iteration scheme defined by (3.2). Assume $F(\Gamma) \neq \emptyset, q \in F(\Gamma)$. Then, the multisetp DI-iterative scheme is Γ-stable.

Proof. Let $\{v_n\}_{n=0}^{\infty}$, be a real sequences in H. Suppose $\{t_n\}_{n=0}^{\infty} \subset X$ is an arbitrary sequence, set

$$\epsilon_n = \|t_{n+1} - \delta_{n,1} v_{n,1}^1 - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1 - \delta_{n,i}) \Gamma^{j-1} v_n^1 - \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) \Gamma^{\ell_1} v_n^1\|^2 \tag{4.2}$$

where, for $s = 1, 2, \cdots, k-2$,

$$v_n^s = \alpha_{n,1}^s v_n^{s+1} + \sum_{j=2}^{\ell_{s+1}} \alpha_{n,j}^s \prod_{i=1}^{j-1} (1 - \alpha_{n,i}^s) \Gamma^{j-1} v_n^{s+1} + \prod_{i=1}^{\ell_{s+1}} (1 - \alpha_{n,i}^s) \Gamma^{\ell_1} v_n^{s+1} \tag{4.3}$$

and, for $k \geq 2$,

$$v_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1} (1 - \alpha_{n,i}^{k-1}) \Gamma^{j-1} t_n + \prod_{i=1}^{\ell_k} (1 - \alpha_{n,i}^{k-1}) \Gamma^{\ell_k} t_n, n \geq 1, \tag{4.4}$$

Now, suppose $\epsilon_n \to 0$ *as* $n \to \infty$. Then, we show that $t_n \to q$ *as* $n \to \infty$ using contractive mapping defined by (4.1).

Indeed, using Proposition 2.4 with $u = q, v_n^1 = t, j = i, k = 1, \Gamma^{j-1} v_n^1 = v_{j-1}$ and $\Gamma^{\ell_1} v_n^1 = v_,$, we obtain

$$\begin{aligned} \|t_{n+1} - q\|^2 &= \|\delta_{n,1} v_{n,1}^1 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1 - \delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q \\ &\quad - [\delta_{n,1} v_{n,1}^1 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1 - \delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1 - \delta_{n,i}) \Gamma^{\ell_1} v_n^1 - t_{n+1}]\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \| -[t_{n+1}-\delta_{n,1}v^1_{n,1}-\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v^1_n-\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v^1_n]\|^2\\
&\quad +\|\delta_{n,1}v^1_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v^1_n+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v^1_n-q\|^2\\
&= \|t_{n+1}-\delta_{n,1}v^1_{n,1}-\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v^1_n-\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v^1_n\|^2\\
&\quad +\|\delta_{n,1}v^1_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v^1_n+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v^1_n-q\|^2\\
&= \epsilon_n+\|\delta_{n,1}v^1_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v^1_n+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v^1_n-q\|^2\\
&\leq \epsilon_n+\delta_{n,1}\|v^1_{n,1}-q\|^2+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\|\Gamma^{j-1}v^1_n-q\|^2\\
&\quad +\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\|\Gamma^{\ell_1}v^1_n-q\|^2\\
&\leq \epsilon_n+\delta_{n,1}\|v^1_{n,1}-q\|^2+\sum_{j=2}^{\ell_1}\delta_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\delta_{n,i})\|v^1_n-q\|^2\\
&\quad +\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\|v^1_n-q\|^2\\
&\leq \epsilon_n+\left(\delta_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\delta_{n,i})+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\right)\\
&\quad \times\|v^1_n-q\|^2 \qquad (4.5)
\end{aligned}$$

Since ℓ_1,ℓ_k are fixed integers and $\alpha^s_{n,i}\in[0,1]$ for each s, using (3.2) and (3.12), the estimations below are obtained, for $n=1,2,\cdots$ and $1\leq s\leq k-1$:,

$$\begin{aligned}
\|v^1_n-q\|^2 &\leq \alpha^1_{n,1}\|v^2_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha^1_{n,j}\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})\|\Gamma^{j-1}v^2_n-\Gamma^{j-1}q\|^2\\
&\quad +\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})\|\Gamma^{\ell_2}v^2_n-\Gamma^{\ell_2}q\|^2\\
&\leq \left(\alpha^1_{n,1}+\sum_{j=2}^{\ell_2}\alpha^1_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\rho^j)^2\right)\|v^2_n-q\|^2
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\left[\alpha_{n,1}^{2}\|v_n^{3}-q\|^{2}\right.\\
&\left.+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\|\Gamma^{j-1}v_n^{3}-\Gamma^{j-1}q\|^{2}+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})\|\Gamma^{\ell_3}v_n^{3}-\Gamma^{\ell_3}q\|^{2}\right]\\
\leq\ & \left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\left[\alpha_{n,1}^{2}\|v_n^{3}-q\|^{2}\right.\\
&\left.+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})\|v_n^{3}-q\|^{2}+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\|v_n^{3}-q\|^{2}\right]\\
=\ & \left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\right)\|v_n^{3}-q\|^{2}\\
\leq\ & \left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\right)\left[\alpha_{n,1}^{3}\|v_n^{4}-q\|^{2}\right.\\
&\left.+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})\|\Gamma^{j-1}v_n^{4}-\Gamma^{j-1}q\|^{2}+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})\|\Gamma^{\ell_4}v_n^{4}-\Gamma^{\ell_4}q\|^{2}\right]\\
\leq\ & \left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\right)\left[\alpha_{n,1}^{3}\|v_n^{4}-q\|^{2}\right.\\
&\left.+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})\|v_n^{4}-q\|^{2}+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})(\rho^{j})^{2}\|v_n^{4}-q\|^{2}\right]\\
=\ & \left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^{j})^{2}\right)\\
&\times\left(\alpha_{n,1}^{3}+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}(\rho^{j})^{2}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})(\rho^{j})^{2}\right)\|v_n^{4}-q\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\alpha^1_{n,1} + \sum_{j=2}^{\ell_2} \alpha^1_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^1_{n,i}) + \prod_{i=1}^{\ell_2} (1-\alpha^1_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^2_{n,1} + \sum_{j=2}^{\ell_3} \alpha^2_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) + \prod_{i=1}^{\ell_3} (1-\alpha^2_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^3_{n,1} + \sum_{j=2}^{\ell_4} \alpha^3_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^3_{n,i}) + \prod_{i=1}^{\ell_4} (1-\alpha^3_{n,i})(\rho^j)^2\right) \Bigg[\alpha^4_{n,1}\|v^5_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_5} \alpha^4_{n,j} \prod_{i=1}^{j-1} (1-\alpha^4_{n,i}) \|\Gamma^{j-1} v^5_n - \Gamma^{j-1} q\|^2 + \prod_{i=1}^{\ell_5} (1-\alpha^4_{n,i}) \|\Gamma^{\ell_5} v^5_n - \Gamma^{\ell_5} q\|^2\Bigg] \\
&\leq \left(\alpha^1_{n,1} + \sum_{j=2}^{\ell_2} \alpha^1_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^1_{n,i}) + \prod_{i=1}^{\ell_2} (1-\alpha^1_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^2_{n,1} + \sum_{j=2}^{\ell_3} \alpha^2_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) + \prod_{i=1}^{\ell_3} (1-\alpha^2_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^3_{n,1} + \sum_{j=2}^{\ell_4} \alpha^3_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^3_{n,i}) + \prod_{i=1}^{\ell_4} (1-\alpha^3_{n,i})(\rho^j)^2\right) \Bigg[\alpha^4_{n,1}\|v^5_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_5} \alpha^4_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^4_{n,i}) \|v^5_n - q\|^2 + \prod_{i=1}^{\ell_5} (1-\alpha^4_{n,i})(\rho^j)^2 \|v^5_n - q\|^2\Bigg] \\
&= \left(\alpha^1_{n,1} + \sum_{j=2}^{\ell_2} \alpha^1_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^1_{n,i}) + \prod_{i=1}^{\ell_2} (1-\alpha^1_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^2_{n,1} + \sum_{j=2}^{\ell_3} \alpha^2_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) + \prod_{i=1}^{\ell_3} (1-\alpha^2_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^3_{n,1} + \sum_{j=2}^{\ell_4} \alpha^3_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^3_{n,i}) + \prod_{i=1}^{\ell_4} (1-\alpha^3_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^4_{n,1} + \sum_{j=2}^{\ell_5} \alpha^4_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^4_{n,i}) + \prod_{i=1}^{\ell_5} (1-\alpha^4_{n,i})(\rho^j)^2\right) \|v^5_n - q\|^2 \\
&\leq \left(\alpha^1_{n,1} + \sum_{j=2}^{\ell_2} \alpha^1_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^1_{n,i}) + \prod_{i=1}^{\ell_2} (1-\alpha^1_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^2_{n,1} + \sum_{j=2}^{\ell_3} \alpha^2_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha_{n,i}) + \prod_{i=1}^{\ell_3} (1-\alpha^2_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^3_{n,1} + \sum_{j=2}^{\ell_4} \alpha^3_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^3_{n,i}) + \prod_{i=1}^{\ell_4} (1-\alpha^3_{n,i})(\rho^j)^2\right) \\
&\quad \times \left(\alpha^4_{n,1} + \sum_{j=2}^{\ell_5} \alpha^4_{n,j} (\rho^j)^2 \prod_{i=1}^{j-1} (1-\alpha^4_{n,i}) + \prod_{i=1}^{\ell_5} (1-\alpha^4_{n,i})(\rho^j)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\cdots\times\left(\alpha_{n,1}^{\ell_{s-2}}+\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{\ell_{s-1}}+\sum_{j=2}^{\ell_{s}}\alpha_{n,j}^{\ell_{s-1}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})+\prod_{i=1}^{\ell_{s}}(1-\alpha_{n,i}^{\ell_{s-1}})(\rho^j)^2\right)\\
&\times\|t_n-q\|^2 \qquad (4.6)
\end{aligned}
$$

(4.5) and (4.6)imply that

$$
\begin{aligned}
\|t_{n+1}-q\|^2 \quad\leq\quad &\left(\delta_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\delta_{n,i})+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{1}+\sum_{j=2}^{\ell_2}\alpha_{n,j}^{1}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{2}+\sum_{j=2}^{\ell_3}\alpha_{n,j}^{2}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{3}+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{4}+\sum_{j=2}^{\ell_5}\alpha_{n,j}^{4}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{4})+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^{4})(\rho^j)^2\right)\\
&\times\cdots\times\left(\alpha_{n,1}^{\ell_{s-2}}+\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{\ell_{s-1}}+\sum_{j=2}^{\ell_{s}}\alpha_{n,j}^{\ell_{s-1}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})+\prod_{i=1}^{\ell_{s}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^j)^2\right)\\
&\times\|t_n-q\|^2+\epsilon_n \qquad (4.7)
\end{aligned}
$$

Note that (4.7) is valid since $\Gamma q=q$ *and* $\phi(0)=0$.

Now, since $\rho^j\in[0,1]$, we obtain using Proposition 2.3, for $j=1,2,3,\cdots,s-1$, that

$$\tau_n<\eta_n=1, \qquad (4.8)$$

where

$$
\begin{aligned}
\tau_n \quad=\quad &\left(\alpha_{n,1}^{3}+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})(\rho^j)^2\right)\\
&\times\left(\alpha_{n,1}^{4}+\sum_{j=2}^{\ell_5}\alpha_{n,j}^{4}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{4})+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^{4})(\rho^j)^2\right)\\
&\times\cdots\times\left(\alpha_{n,1}^{\ell_{s-2}}+\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^j)^2\right)
\end{aligned}
$$

$$\times\left(\alpha_{n,1}^{\ell_{s-1}}+\sum_{j=2}^{\ell_s}\alpha_{n,j}^{\ell_{s-1}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})+\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})(\rho^j)^2\right)$$

and

$$\begin{aligned}\eta_n &= \left(\alpha_{n,1}^{3}+\sum_{j=2}^{\ell_4}\alpha_{n,j}^{3}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{3})+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^{3})\right)\\ &\quad\times\left(\alpha_{n,1}^{4}+\sum_{j=2}^{\ell_5}\alpha_{n,j}^{4}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{4})+\prod_{i=1}^{\ell_5}(1-\alpha_{n,i}^{4})\right)\\ &\quad\times\cdots\times\left(\alpha_{n,1}^{\ell_{s-2}}+\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\ &\quad\times\left(\alpha_{n,1}^{\ell_{s-1}}+\sum_{j=2}^{\ell_s}\alpha_{n,j}^{\ell_{s-1}}\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})+\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\end{aligned}$$

Putting (4.8) in (4.7), we obtain, using Lemma 2.3 that the sequence $\{t_n\}_{n=0}^{\infty}$ converges strongly to the point q in $F(\Gamma)$.

On the other hand, suppose $t_n \to q$ *as* $n \to \infty$. Then, we show that $\epsilon \to 0$ *as* $n \to \infty$. Indeed, from (3.5) with $v_n^1 = y_n^1$, (4.2) and Proposition 2.4 with $u = q, v_n^1 = t, j = i, k = 1, \Gamma^{j-1}v_n^1 = v_{j-1}$ and $\Gamma^{\ell_1}v_n^1 = v_{,,}$ we have

$$\begin{aligned}\epsilon_n &= \|t_{n+1}-\delta_{n,1}v_{n,1}^1-\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v_n^1-\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v_n^1\|^2\\ &= \|t_{n+1}-q-\left(\delta_{n,1}v_{n,1}^1+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v_n^1+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v_n^1-q\right)\|^2\\ &\le \|t_{n+1}-q\|^2+\|\delta_{n,1}v_{n,1}^1+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\Gamma^{j-1}v_n^1+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\Gamma^{\ell_1}v_n^1-q\|^2\\ &\le \|t_{n+1}-q\|^2+\delta_{n,1}\|v_{n,1}^1-q\|^2+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})\|\Gamma^{j-1}v_n^1-\Gamma^{j-1}q\|^2\\ &\quad+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})\|\Gamma^{\ell_1}v_n^1-\Gamma^{\ell_1}q\|^2\\ &\le \|t_{n+1}-q\|^2+\delta_{n,1}\|v_{n,1}^1-q\|^2+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})(\rho^j)^2\|v_n^1-q\|^2\\ &\quad+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\|v_n^1-q\|^2\end{aligned}$$

$$
\begin{aligned}
&= \|t_{n+1}-q\|^2+\left(\delta_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})(\rho^j)^2+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\right)\\
&\quad\times\|v_n^1-q\|^2 \qquad (4.9)
\end{aligned}
$$

Putting (4.6) into (4.9), and using (4.8), we get

$$
\begin{aligned}
\epsilon_n &\le \|t_{n+1}-q\|^2+\left(\delta_{n,1}+\sum_{j=2}^{\ell_1}\delta_{n,j}\prod_{i=1}^{j-1}(1-\delta_{n,i})(\rho^j)^2+\prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\right)\\
&\quad\times\left(\alpha^1_{n,1}+\sum_{j=2}^{\ell_2}\alpha^1_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha^1_{n,i})+\prod_{i=1}^{\ell_2}(1-\alpha^1_{n,i})(\rho^j)^2\right)\\
&\quad\times\left(\alpha^2_{n,1}+\sum_{j=2}^{\ell_3}\alpha^2_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i})+\prod_{i=1}^{\ell_3}(1-\alpha^2_{n,i})(\rho^j)^2\right)\\
&\quad\times\left(\alpha^3_{n,1}+\sum_{j=2}^{\ell_4}\alpha^3_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha^3_{n,i})+\prod_{i=1}^{\ell_4}(1-\alpha^3_{n,i})(\rho^j)^2\right)\\
&\quad\times\left(\alpha^4_{n,1}+\sum_{j=2}^{\ell_5}\alpha^4_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha^4_{n,i})+\prod_{i=1}^{\ell_5}(1-\alpha^4_{n,i})(\rho^j)^2\right)\\
&\quad\times\cdots\times\left(\alpha^{\ell_{s-2}}_{n,1}+\sum_{j=2}^{\ell_{s-1}}\alpha^{\ell_{s-2}}_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha^{\ell_{s-2}}_{n,i})+\prod_{i=1}^{\ell_{s-1}}(1-\alpha^{\ell_{s-2}}_{n,i})(\rho^j)^2\right)\\
&\quad\times\left(\alpha^{\ell_{s-1}}_{n,1}+\sum_{j=2}^{\ell_s}\alpha^{\ell_{s-1}}_{n,j}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha^{\ell_{s-1}}_{n,i})+\prod_{i=1}^{\ell_s}(1-\alpha^{\ell_{s-1}}_{n,i})(\rho^j)^2\right)\\
&\quad\times\|t_n-q\|^2\\
&\le \|t_{n+1}-q\|^2+\tau_n\|t_n-q\|^2 \qquad (4.10)
\end{aligned}
$$

Thus, from our assumption, we obtain from (4.10) that $\epsilon_n \to 0$ *as* $n \to \infty$. Hence, the multistep DI-iteration scheme (3.2) is Γ-stable. Thus, tje proof is completed. □

Theorem 4.2. *Let H be a Hilbert space, $\Gamma : H \longrightarrow H$ be a self-map of H satisfying the contractive condition*

$$\|\Gamma^j x-\Gamma^j y\|\le\rho^j\|x-y\|+\sum_{i=0}^{j}\binom{j}{i}\rho^{j-1}\phi(\|x-\Gamma x\|), \qquad (4.11)$$

where $x,y\in H, 0\le\rho^j<1$, and let ϕ retains its usual meaning with $\phi(0)=0$ and $\phi(Mt)=M\phi(t), M\ge 0, t\in\mathbb{R}^+$. For arbitrary $x_0\in H$, let $\{\omega_n\}_{n=0}^{\infty}$ be the multistep IH-iteration scheme defined by (3.1). Assume $F(\Gamma)\neq\emptyset, q\in F(\Gamma)$. Then, the multisetp IH-iteration scheme is Γ-stable.

Proof. Let $\{t_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, for $i = 1, 2, \cdots, s-1$, be two real sequences in H. Set

$$\epsilon_n = \|t_{n+1} - \delta_{n,1} t_n - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 - \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1\|^2 \tag{4.12}$$

where, for $s = 1, 2, \cdots, k-2$,

$$v_n^s = \alpha_{n,1}^s t_n + \sum_{j=2}^{\ell_{s+1}} \alpha_{n,j}^s \prod_{i=1}^{j-1} (1-\alpha_{n,i}^s) \Gamma^{j-1} v_n^{s+1} + \prod_{i=1}^{\ell_{s+1}} (1-\alpha_{n,i}^s) \Gamma^{\ell_1} v_n^{s+1} \tag{4.13}$$

and, for $k \geq 2$,

$$v_n^{k-1} = \sum_{j=1}^{\ell_k} \alpha_{n,j}^{k-1} \prod_{i=1}^{j-1} (1-\alpha_{n,i}^{k-1}) \Gamma^{j-1} t_n + \prod_{i=1}^{\ell_k} (1-\alpha_{n,i}^{k-1}) \Gamma^{\ell_k} t_n, \, n \geq 1, \tag{4.14}$$

Now, suppose $\epsilon_n \to 0$ *as* $n \to \infty$. Then, we show that $t_n \to q$ *as* $n \to \infty$ using contractive mapping defined by (4.1).

Indeed, Using Proposition 2.4 with $u = q, t_n = t, j = i, k = 1, \Gamma^{j-1} v_n^1 = v_{j-1}$ and $\Gamma^{\ell_1} v_n^1 = v$„ we obtain

$$\begin{aligned}
\|t_{n+1} - q\|^2 &= \|\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q \\
&\quad - [\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - t_{n+1}]\|^2 \\
&\leq \| - [t_{n+1} - \delta_{n,1} t_n - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 - \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1]\|^2 \\
&\quad + \|\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q\|^2 \\
&= \|t_{n+1} - \delta_{n,1} t_n - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 - \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1\|^2 \\
&\quad + \|\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q\|^2 \\
&= \epsilon_n + \|\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q\|^2 \\
&\leq \epsilon_n + \delta_{n,1} \|t_n - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \|\Gamma^{j-1} v_n^1 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \|\Gamma^{\ell_1} v_n^1 - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon_n + \delta_{n,1}\|t_n - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\delta_{n,i})\|v_n^1 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\|v_n^1 - q\|^2 \\
&= \epsilon_n + \delta_{n,1}\|t_n - q\|^2 + \left(1 - \delta_{n,1} - \prod_{i=1}^{\ell_1}(1-\delta_{n,i})\right)(\rho^j)^2\|v_n^1 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_1}(1-\delta_{n,i})(\rho^j)^2\|v_n^1 - q\|^2 \\
&< \epsilon_n + \delta_{n,1}\|t_n - q\|^2 + (1-\delta_{n,1})\|v_n^1 - q\|^2 \qquad (4.15)
\end{aligned}
$$

Since ℓ_1, ℓ_k are fixed integers and $\alpha_{n,i}^s \in [0,1]$ for each s, the estimations below are obtained for $n = 1, 2, \cdots$ and $1 \leq s \leq k-1$:

$$
\begin{aligned}
\|v_n^1 - q\|^2 &\leq \alpha_{n,1}\|t_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j} \prod_{i=1}^{j-1}(1-\alpha_{n,i})\|\Gamma^{j-1}v_n^2 - \Gamma^{j-1}q\|^2 \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i})\|\Gamma^{\ell_2}v_n^2 - \Gamma^{\ell_2}q\|^2 \\
&\leq \alpha_{n,1}^1\|t_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i})\|v_n^2 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i})(\rho^j)^2\|v_n^2 - q\|^2 \\
&\leq \alpha_{n,1}^1\|t_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\Big[\alpha_{n,1}^2\|t_n - q\|^2 \\
&\quad + \sum_{j=2}^{\ell_3} \alpha_{n,j}^2(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\|v_n^3 - q\|^2 + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\|v_n^3 - q\|^2\Big] \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\Big[\alpha_{n,1}^2\|t_n - q\|^2 + \sum_{j=2}^{\ell_3} \alpha_{n,j}^2(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\|v_n^3 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\|v_n^3 - q\|^2\Big] \\
&= \alpha_{n,1}^1\|t_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^1(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^1(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^2(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\|v_n^3 - q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|v_n^3-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\|v_n^3-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\|v_n^3-q\|^2\\
\leq\ & \alpha_{n,1}^1\|t_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\Bigg[\alpha_{n,1}^3\|t_n-q\|^2\\
&+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|v_n^4-q\|^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\|v_n^4-q\|^2\Bigg]\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\Bigg[\alpha_{n,1}^3\|t_n-q\|^2\\
&+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|v_n^4-q\|^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\|v_n^4-q\|^2\Bigg]\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\Bigg[\alpha_{n,1}^3\|t_n-q\|^2\\
&+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|v_n^4-q\|^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\|v_n^4-q\|^2\Bigg]\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\Bigg[\alpha_{n,1}^3\|t_n-q\|^2\\
&+\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\|v_n^4-q\|^2+\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\|v_n^4-q\|^2\Bigg]\\
=\ & \alpha_{n,1}^1\|t_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|t_n-q\|^2
\end{aligned}
$$

$$+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|t_n-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)$$

$$\times\left(\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|v_n^4-q\|^2+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)$$

$$\times\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|t_n-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|v_n^4-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|t_n-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|v_n^4-q\|^2$$

$$+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2$$

$$+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|t_n-q\|^2$$

$$+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^3)\right)\|v_n^4-q\|^2$$

$$+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2$$

$$=\alpha_{n,1}^1\|t_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2$$

$$+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\alpha_{n,1}^2\|t_n-q\|^2$$

$$
\begin{aligned}
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|t_n-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\\
&\times\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2\right)\|v_n^4-q\|^2+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\\
&\times\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\\
&\times\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2)\right)\|v_n^4-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2)\right)\|v_n^4-q\|^2\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left((1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3))(\rho^j)^2\right)\|v_n^4-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^4)(\rho^j)^2\right)\|v_n^4-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_{n,1}^{1}\|t_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\alpha_{n,1}^{2}\|t_n - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\alpha_{n,1}^{2}\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\alpha_{n,1}^{3}\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\alpha_{n,1}^{3}\|t_n - q\|^2 \\
&\quad + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\alpha_{n,1}^{3}\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)(1-\alpha_{n,1}^{3})(\rho^j)^2\|v_n^4 - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)(1-\alpha_{n,1}^{3})(\rho^j)^2)\|v_n^4 - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)(1-\alpha_{n,1}^{3})(\rho^j)^2)\|v_n^4 - q\|^2 \\
&\quad + \left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)(1-\alpha_{n,1}^{3})(\rho^j)^2\|v_n^4 - q\|^2 \\
&\leq \alpha_{n,1}^{1}\|t_n - q\|^2 + \sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\alpha_{n,1}^{2}\|t_n - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\alpha_{n,1}^{2}\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\alpha_{n,1}^{3}\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_2} \alpha_{n,j}^{1}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{1})\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^{2})(\rho^j)^2\right)\|t_n - q\|^2 \\
&\quad + \left(\sum_{j=2}^{\ell_3} \alpha_{n,j}^{2}(\rho^j)^2 \prod_{i=1}^{j-1}(1-\alpha_{n,i}^{2})\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})(\rho^j)^2\right)\alpha_{n,1}^{3}\|t_n - q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2)\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2)\|t_n-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)(\rho^j)^2\|t_n-q\|^2+\cdots\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\sum_{j=2}^{\ell_3}\alpha_{n,j}^2(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\\
&\times\left(\sum_{j=2}^{\ell_4}\alpha_{n,j}^3(\rho^j)^3\prod_{i=1}^{j-1}(1-\alpha_{n,i}^2)\right)\times\cdots\times\left(\sum_{j=2}^{\ell_{s-1}}\alpha_{n,j}^{\ell_{s-2}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\
&\times\left(\sum_{j=2}^{\ell_s}\alpha_{n,j}^{\ell_{s-1}}(\rho^j)^2\prod_{i=1}^{j-1}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\alpha_{n,1}^s\|t_n-q\|^2+(\rho^j)^2\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)(\rho^j)^2\right)\\
&\times(\rho^j)^2\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)(\rho^j)^2\right)(\rho^j)^2\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)(\rho^j)^2\right)\times\cdots\times(\rho^j)^2\left(\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})(\rho^j)^2\right)\\
&\times(\rho^j)^2\left(\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})(\rho^j)^2\right)\|t_n-q\|^2\\
<\ &\alpha_{n,1}^1\|t_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\|t_n-q\|^2\\
&+\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\|t_n-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|t_n-q\|^2\\
&+\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2))\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|t_n-q\|^2\\
&+\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|t_n-q\|^2+\cdots\\
&+\left(1-\alpha_{n,1}^1-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^2)\right)\left(1-\alpha_{n,1}^2-\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\\
&\times\left(1-\alpha_{n,1}^3-\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}-\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\
&\times\left(1-\alpha_{n,1}^{\ell_{s-1}}-\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\alpha_{n,1}^s\|t_n-q\|^2+\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\\
&\times\left(\prod_{i=1}^{\ell_3}(1-\alpha_{n,i}^2)\right)\left(\prod_{i=1}^{\ell_4}(1-\alpha_{n,i}^3)\right)\times\cdots\times\left(\prod_{i=1}^{\ell_{s-1}}(1-\alpha_{n,i}^{\ell_{s-2}})\right)\\
&\times\left(\prod_{i=1}^{\ell_s}(1-\alpha_{n,i}^{\ell_{s-1}})\right)\|t_n-q\|^2 \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
<\;&\alpha_{n,1}^1\|t_n-q\|^2+\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2\\
&+\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\alpha_{n,1}^2\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\|t_n-q\|^2\\
&+\alpha_{n,1}^3\left(1-\alpha_{n,1}^2\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^1)\right)\|t_n-q\|^2\\
&+\left(\sum_{j=2}^{\ell_2}\alpha_{n,j}^1\prod_{i=1}^{j-1}(1-\alpha_{n,i}^1)\right)\left(1-\alpha_{n,1}^2\right)\alpha_{n,1}^3(1-\alpha_{n,1}^3)\|t_n-q\|^2
\end{aligned}
$$

$$
\begin{aligned}
& +\left(1-\alpha_{n,1}^{2}\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\right)\alpha_{n,1}^{3}(1-\alpha_{n,1}^{3})\|t_n-q\|^2+\cdots+ \\
& +\left(1-\alpha_{n,1}^{1}\right)\left(1-\alpha_{n,1}^{2}\right)\left(1-\alpha_{n,1}^{3}\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right) \\
& \times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)\alpha_{n,1}^{s}\|t_n-q\|^2 \\
= \; & \alpha_{n,1}^{1}\|t_n-q\|^2+\alpha_{n,1}^{2}\left(1-\alpha_{n,1}^{1}-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\right)\|t_n-q\|^2 \\
& +\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\alpha_{n,1}^{2}\|t_n-q\|^2 \\
& +\left((1-\alpha_{n,1}^{1}-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\right)\alpha_{n,1}^{3}\left(1-\alpha_{n,1}^{2}\right)\|t_n-q\|^2 \\
& +\alpha_{n,1}^{3}\left(1-\alpha_{n,1}^{2}\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\right)\|t_n-q\|^2 \\
& +\left((1-\alpha_{n,1}^{1}-\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\right)\left(1-\alpha_{n,1}^{2}\right)\alpha_{n,1}^{3}(1-\alpha_{n,1}^{3})\|t_n-q\|^2 \\
& +\left(1-\alpha_{n,1}^{2}\right)\left(\prod_{i=1}^{\ell_2}(1-\alpha_{n,i}^{1})\right)\alpha_{n,1}^{3}(1-\alpha_{n,1}^{3})\|t_n-q\|^2+\cdots+ \\
& +\left(1-\alpha_{n,1}^{1}\right)\left(1-\alpha_{n,1}^{2}\right)\left(1-\alpha_{n,1}^{3}\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right) \\
& \times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)\alpha_{n,1}^{s}\|t_n-q\|^2 \\
< \; & \Big[\alpha_{n,1}^{1}+\alpha_{n,1}^{2}\left(1-\alpha_{n,1}^{1}\right)+(1-\alpha_{n,1}^{1})\alpha_{n,1}^{3}\left(1-\alpha_{n,1}^{2}\right)+\left(1-\alpha_{n,1}^{1}\right)\left(1-\alpha_{n,1}^{2}\right)\alpha_{n,1}^{3}(1-\alpha_{n,1}^{3}) \\
& +\cdots+\left(1-\alpha_{n,1}^{1}\right)\left(1-\alpha_{n,1}^{2}\right)\left(1-\alpha_{n,1}^{3}\right)\times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right) \\
& \times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)\Big]\|t_n-q\|^2 \qquad (4.17)
\end{aligned}
$$

(4.15) and (4.17) imply that

$$
\begin{aligned}
\|t_{n+1}-q\|^2 \;\le\; & \{\delta_{n,1}+(1-\delta_{n,1})\,[\alpha_{n,1}^{1}+\alpha_{n,1}^{2}\left(1-\alpha_{n,1}^{1}\right)+(1-\alpha_{n,1}^{1})\alpha_{n,1}^{3}\left(1-\alpha_{n,1}^{2}\right) \\
& +(1-\alpha_{n,1}^{1})\left(1-\alpha_{n,1}^{2}\right)\left(1-\alpha_{n,1}^{3}\right)+\cdots+\left(1-\alpha_{n,1}^{1}\right)\left(1-\alpha_{n,1}^{2}\right)\left(1-\alpha_{n,1}^{3}\right) \\
& \times\cdots\times\left(1-\alpha_{n,1}^{\ell_{s-2}}\right)\times\left(1-\alpha_{n,1}^{\ell_{s-1}}\right)]\}\|t_n-q\|^2 \qquad (4.18)
\end{aligned}
$$

Using Lemma 2.3, we obtain (from (4.18)) that the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(\Gamma)$.

Conversely, suppose $t_n \to q$ *as* $n \to \infty$. Then, we show that $\epsilon \to 0$ *as* $n \to \infty$. Indeed, from (3.5) with $v_n^1 = y_n^1$, (4.12) and Proposition 2.4 with $u = q, t_n = t, j = i, k = 1, \Gamma^{j-1}v_n^1 = v_{j-1}$ and

$\Gamma^{\ell_1} v_n^1 = v$„ we have

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - \delta_{n,1} t_n - \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 - \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 \|^2 \\
&= \|t_{n+1} - q - \left(\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q \right) \|^2 \\
&\leq \|t_{n+1} - q\|^2 + \|\delta_{n,1} t_n + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \Gamma^{j-1} v_n^1 + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \Gamma^{\ell_1} v_n^1 - q\|^2 \\
&\leq \|t_{n+1} - q\|^2 + \delta_{n,1} \|t_n - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) \|\Gamma^{j-1} v_n^1 - \Gamma^{j-1} q\|^2 \\
&\quad + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \|\Gamma^{\ell_1} v_n^1 - \Gamma^{\ell_1} q\|^2 \\
&\leq \|t_{n+1} - q\|^2 + \delta_{n,1} \|t_n - q\|^2 + \sum_{j=2}^{\ell_1} \delta_{n,j} \prod_{i=1}^{j-1} (1-\delta_{n,i}) (\rho^j)^2 \|v_n^1 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) (\rho^j)^2 \|v_n^1 - q\|^2 \\
&= \|t_{n+1} - q\|^2 + \delta_{n,1} \|t_n - q\|^2 + \left(1 - \delta_{n,1} - \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) \right) (\rho^j)^2 \|v_n^1 - q\|^2 \\
&\quad + \prod_{i=1}^{\ell_1} (1-\delta_{n,i}) (\rho^j)^2 \|v_n^1 - q\|^2 \\
&= \|t_{n+1} - q\|^2 + \delta_{n,1} \|t_n - q\|^2 + (1-\delta_{n,1}) \|v_n^1 - q\|^2 \qquad (4.19)
\end{aligned}$$

(4.17) and (4.19) imply

$$\begin{aligned}
\epsilon_n \leq\ & \|t_{n+1} - q\|^2 + \Big\{ \delta_{n,1} + (1-\delta_{n,1}) \Big[\alpha_{n,1}^1 + \alpha_{n,1}^2 \left(1 - \alpha_{n,1}^1\right) \\
& + (1-\alpha_{n,1}^1) \alpha_{n,1}^3 \left(1-\alpha_{n,1}^2\right) + \left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right) \alpha_{n,1}^3 (1-\alpha_{n,1}^3) \\
& + \cdots + \left(1-\alpha_{n,1}^1\right)\left(1-\alpha_{n,1}^2\right)\left(1-\alpha_{n,1}^3\right) \times \cdots \times \left(1-\alpha_{n,1}^{\ell_{s-2}}\right) \\
& \times \left(1-\alpha_{n,1}^{\ell_{s-1}}\right) \Big] \Big\} \|t_n - q\|^2 \qquad (4.20)
\end{aligned}$$

Again, from our assumption, we obtain from (4.20) that $\epsilon_n \to 0$ *as* $n \to \infty$. Hence, the multistep IH-iteration scheme (3.1) is Γ-stable, and this completes the proof.

□

Remark 4.1. *The following areas are still open:*

(i) *to reconstruct, approximate the fixed points and the stability results of some existing iterative schemes in the current literature, other than the ones under study, for finite family of certain class of contractive-type map;*

(ii) *to compare convergent rates of the iterative schemes defined by* (3.1) *and* (3.2) *with those of* (1.5) *and* (1.6).

Competing interest

The authors declare that there is no conflict of interest.

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