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A Population Model with Smooth Functions and Application

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Abstract. The purpose of this paper is to give sufficient conditions for the existence and uniqueness of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain Ω in R^n. Also considered are the effects of perturbations on the coexistence state and uniqueness. The techniques used in this paper are super-sub solutions method, eigenvalues of operators, maximum principles, spectrum estimates, inverse function theory, and general elliptic theory. The arguments also rely on some detailed properties for the solution of logistic equations. These results yield an algebraically computable criterion for the positive coexistence of species of animals with predator-prey relation in many biological models.

1. Introduction

One of the prominent subjects of study and analysis in mathematical biology concerns the survival of two or more species of animals in the same environment. Especially, pertinent areas of investigation include the conditions under which the species can coexist, as well as the conditions under which any one of the species becomes extinct, that is, one of the species is excluded by the others. In this paper, we focus on the general predator-prey model to better understand the competitive interactions between two species. Specifically, we investigate the conditions needed for the coexistence of two species when the factors affecting them are fixed or perturbed.

2. Literature Review

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple predator-prey model, commonly known as the Lotka-Volterra predator-prey model. This system describes the predator-prey interaction of two species residing in the same environment in the following manner:

Suppose two species of animals with predator-prey interaction, rabbits and tigers for instance, are residing in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of preys and predators

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in the place x of Ω at time t, respectively. Then we have the dynamic predator-prey model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \alpha u(x,t) - au^2(x,t) - bu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + \beta v(x,t) - dv^2(x,t) + cu(x,t)v(x,t) \end{cases} \text{ in } \Omega \times [0,\infty),$$
$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0,$$

where $\alpha, \beta > 0$ are growth rates, $a, d > 0$ are self-limitation rates, and $b, c > 0$ are competition rates, and the homogeneous boundary conditions imply that the two species are not staying on the boundary of Ω. Here, we are interested in the time independent, positive solutions, i.e., the positive solution $u(x), v(x)$ of

$$\begin{cases} \Delta u(x) + u(x)(\alpha - au(x) - bv(x)) = 0 \\ \Delta v(x) + v(x)(\beta - dv(x) + cu(x)) = 0 \end{cases} \text{ in } \Omega, \qquad (1)$$
$$u|_{\partial\Omega} = v|_{\partial\Omega} = 0,$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so, its existence means that the two species of animals can live peacefully and forever.

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (1)([8], [9], [19]).

One of the initial important results for the time-independent Lotka-Volterra model was obtained by Korman and Leung. In 1986, they published the following sufficient conditions for the existence of a positive steady state solution to (1):

Theorem 2.1. *(in [8])*
If $ad > bc, \alpha > \frac{ad(\lambda_1 + \frac{b}{d}\beta)}{ad - bc}$ and $\beta > \lambda_1$, where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$ with homogeneous boundary condition as stated in the Lemma 3.2, then (1) has a positive solution.

Biologically, the conditions in Theorem 2.1 implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, in other words, if members of each species interact strongly among themselves and weakly with members of the other species, then there is a positive steady state solution to (1), that is, the two species within the same domain will coexist indefinitely at population densities.

Another important result was obtained by Zhengyuan and Mottoni. In 1992, they published the following characterization of non-negative solutions to (1) in terms of growth rates (α, β):

Theorem 2.2. *(in [19])*
There exist two functions $\gamma_0(\alpha), \mu_0(\beta)$ such that the set S of non-negative solutions to (1) is characterized as follows:

(1) If $\alpha \le \lambda_1, \beta \le \lambda_1$, then $S = \{(0,0)\}$.
(2) If $\alpha \le \lambda_1, \beta > \lambda_1$, then $S = \{(0,0), (0, \theta_{\frac{\beta}{d}})\}$.

(3) If $\alpha > \lambda_1, \beta < \gamma_0(\alpha)$, *then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0)\}$.
(4) If $\lambda_1 < \alpha < \mu_0(\beta), \beta > \lambda_1$, *then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (0, \theta_{\frac{\beta}{d}})\}$.
(5) If $\alpha > \lambda_1, \gamma_0(\alpha) < \beta \leq \lambda_1$, *then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (u^+, v^+)\}$, *where* (u^+, v^+) *is a positive solution to (1).*
(6) If $\beta > \lambda_1, \alpha > \mu_0(\beta)$, *then* $S = \{(0,0), (\theta_{\frac{\alpha}{a}}, 0), (0, \theta_{\frac{\beta}{d}}), (u^+, v^+)\}$.

These results provide insight into the predator-prey interactions of two species operating under the conditions described in the Lotka-Volterra model. However, their results are somewhat limited by a few key assumptions. In the Lotka-Volterra model that they studied, the rates of change of densities largely depend on constant rates of reproduction, self-limitation, and competition. The model also assumes a linear relationship of the terms affecting the rates of change for both population densities.

However, in reality, the rates of change of population densities may vary in a more complicated and irregular manner than can be described by the simple predator-prey model. Therefore, in the last decades, significant research has been focused on the existence and uniqueness of the positive steady state solution of the general predator-prey model for multiple species,

$$\begin{cases} (u_i)_t(x,t) = \Delta u_i(x,t) + g_i(u_1(x,t), ..., u_i(x,t), ..., , u_N(x,t)) & \text{in } \Omega \times R^+, \\ (u_i)(x,t)|_{\partial\Omega} = 0, i = 1, ..., N. \end{cases}$$

or the positive solution to

$$\begin{cases} \Delta u_i(x) + g_i(u_1(x), ..., u_i(x), ..., u_N(x)) = 0 & \text{in } \Omega, \\ (u_i)(x)|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{2}$$

where $g_i \in C^2, i = 1, ..., N$ designate reproduction, self-limitation and competition rates that satisfy certain growth conditions.

Because of its broader applicability, the general predator-prey model has become a more popular subject of research within the mathematical community over the past few years.

The functions g_i's describe how species interact among themselves and with each other.

The followings are questions raised in the general model with nonlinear growth rates.

Problem 1 : What are the sufficient conditions for existence of positive solutions?
Problem 2 : What are the sufficient conditions for uniqueness of positive solutions?
Problem 3 : What is the effect of perturbation for existence and uniqueness? In our analysis, we focus on the conditions required for the maintenance of the coexistence state of the model when interaction rates $(g_1, ..., g_N)$ are slightly perturbed. Biologically, our conclusion implies that the species may slightly relax ecologically and yet continue to coexist at unique densities.

In sections 4 and 5, we establish sufficient conditions for the existence and non-existence of positive solution of the system that generalizes the Theorems 2.1 and 2.2. We also achieve solution estimates in the section 4 to prove the uniqueness and the invertibility of linearization in sections 6, 7 and 8, where we investigate the effect of perturbation for existence and uniqueness.

An especially significant aspect of the global uniqueness result is the stability of the positive steady state solution, which has become an important subject of mathematical study. Indeed, researchers have obtained several stability results for the Lotka-Volterra model with constant rates.([2], [3]) The research presented in this paper therefore begins the mathematical community's discussion on the stability of the steady state solution for the general predator-prey model.

3. Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition **3.1.** *(Super and Sub solutions)*
The vector functions $(\bar{u}^1, ..., \bar{u}^N), (\underline{u}^1, ..., \underline{u}^N)$ form a super/sub solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1, ..., u^N) = 0 \ \ in \ \ \Omega, \\ \qquad\qquad u^i = 0 \ \ on \ \ \partial\Omega, \end{cases}$$

if for $i = 1, ..., N$,

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1, ..., u^{i-1}, \bar{u}^i, u^{i+1}, ..., u^N) \leq 0 \\ \Delta \underline{u}^i + g^i(u^1, ..., u^{i-1}, \underline{u}^i, u^{i+1}, ..., u^N) \geq 0 \end{cases} \ in \ \ \Omega \ \ for \ \ \underline{u}^j \leq u^j \leq \bar{u}^j, j \neq i,$$

and

$$\begin{cases} \underline{u}^i \leq \bar{u}^i \ \ on \ \ \Omega, \\ \underline{u}^i \leq 0 \leq \bar{u}^i \ \ on \ \ \partial\Omega. \end{cases}$$

Lemma 3.1. *If g^i in the Definition 3.1 are in C^1 and the system admits a super/sub solution pair $(\underline{u}^1, ..., \underline{u}^N), (\bar{u}^1, ..., \bar{u}^N)$, respectively, then there is a solution $(u_1, ..., u_N)$ to the system in Definition 3.1 with $\underline{u}^i \leq u^i \leq \bar{u}^i$ in $\bar{\Omega}$. If*

$$\begin{cases} \Delta \bar{u}^i + g^i(\bar{u}^1, ..., \bar{u}^N) \neq 0, \\ \Delta \underline{u}^i + g^i(\underline{u}^1, ..., \underline{u}^N) \neq 0 \end{cases}$$

in Ω for $i = 1, ..., N$, then $\underline{u}^i < u^i < \bar{u}^i$ in Ω.

Lemma 3.2. *(The first eigenvalue) Consider*

$$\begin{cases} -\Delta u + q(x)u = \lambda u \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

where $q(x)$ is a smooth function from Ω to R and Ω is a bounded domain.
(A) The first eigenvalue $\lambda_1(q)$ of (3), denoted by simply λ_1 when $q \equiv 0$, is simple with a positive

eigenfunction ϕ_q.
(B) If $q_1(x) < q_2(x)$ *for all* $x \in \Omega$, *then* $\lambda_1(q_1) < \lambda_1(q_2)$.
(C)(Variational Characterization of the first eigenvalue)

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (|\nabla \phi|^2 + q\phi^2)dx}{\int_\Omega \phi^2 dx}$$

.

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

Lemma 3.3. *Consider*

$$\begin{cases} \Delta u + uf(u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f *is a decreasing* C^1 *function such that there exists* $c_0 > 0$ *such that* $f(u) \leq 0$ *for* $u \geq c_0$ *and* Ω *is a bounded domain.*

(1) *If* $f(0) > \lambda_1$, *then the above equation has a unique positive solution. We denote this unique positive solution as* θ_f.
(2) *If* $f(0) \leq \lambda_1$, *then* $u \equiv 0$ *is the only nonnegative solution to the above equation.*

The main property about this positive solution is that θ_f is increasing as f is increasing.

Especially, for $\alpha > \lambda_1$, we denote θ_α as the unique positive solution of

$$\begin{cases} \Delta u + u(\alpha - u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0. \end{cases}$$

Hence, θ_α is increasing as $\alpha > 0$ is increasing.

Having established these preliminaries, we now commence our investigation of the general predator-prey model.

4. Existence and solution estimate

A general type of elliptic interacting system of two functions with homogeneous boundary condition is

$$\begin{cases} \Delta u_i + g_i(u_1, ..., u_N) \ = \ 0 \quad \text{in } \ \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{4}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $g_i \in C^2, i = 1, ..., N$ are relative growth rates.

The following inequalities will be useful for later use which are true for any C^2 functions such that $(g_i)_{u_i u_i} < 0$ and $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ for all $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N$ and $i = 1, ..., N$.

Lemma 4.1. *For all u_i with $u_i \geq 0, i = 1, ..., N$,*

$$g_i(u_1, ..., u_i, ..., u_N) \geq u_i(g_i)_{u_i}(u_1, ..., u_i, ..., u_N), i = 1, ..., N.$$

Proof. By the condition and the Mean Value Theorem, there is $\bar{u_i}$ such that $0 \leq \bar{u_i} \leq u_i$ and

$$\begin{aligned} g_i(u_1, ..., u_i, ..., u_N) &= g_i(u_1, ..., u_i, ..., u_N) - g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ &= u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N). \end{aligned}$$

But, by the monotonicity of $(g_i)_{u_i}$, we have

$$\begin{aligned} g_i(u_1, ..., u_i, ..., u_N) &= g_i(u_1, ..., u_i, ..., u_N) - g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ &= u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) \\ &\geq u_i(g_i)_{u_i}(u_1, ..., u_i, ..., u_N). \end{aligned}$$

We establish the following existence result:

Theorem 4.2. *If*

(1)

$$(g_i)_{u_iu_j} \begin{cases} < 0, i = 1, j = 1, ..., N \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N, \end{cases}$$

(2) $g_i(u_1, ..., u_i, ..., u_N) \leq u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N)$ *for all* $u_1, ..., u_N$ *and* $i = 1, ..., N$,

(3) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ *for all* $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N$,

(4) $\sup(g_1)_{u_1u_1} \sup(g_i)_{u_iu_i} + 4(N-1)\inf(g_1)_{u_1u_i} \sup(g_i)_{u_1u_i} > 0$, *for* $i = 2, ..., N$

(5) $(g_1)_{u_1}(0, ..., 0) > \frac{4\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}(\lambda_1 + \frac{(N-1)\inf(g_1)_{u_1u_i}}{\sup(g_i)_{u_iu_i}}(g_i)_{u_i}(0,...,0))}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i} + 4(N-1)\inf(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}}$

and $(g_i)_{u_i}(0, ..., 0) > \lambda_1 - (N-2)\inf_{j=2,...,N, j\neq i}\{\bar{u_j}\inf(g_i)_{u_iu_j}\}$, *where* $\bar{u_j}$ *is defined below, then (4) has a positive solution.*

Proof. Let $\underline{u_i} = \gamma_i\omega$, where $\gamma_i > 0, i = 1, ..., N$ are constants and ω is the eigenfunction of (3) with $q(x) \equiv 0$ corresponding to the first eigenvalue λ_1, and $\bar{u_1} = -\frac{2(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}, \bar{u_i} = -\frac{4}{\sup(g_i)_{u_iu_i}}[(g_i)_{u_i}(0, ..., 0) - \frac{(g_1)_{u_1}(0,...,0)\sup(g_i)_{u_1u_i}}{\sup(g_1)_{u_1u_1}}], i = 2, ..., N.$
Then by the Mean Value Theorem, for all u_i such that $\underline{u_i} \leq u_i \leq \bar{u_i}, i = 2, ..., N$,

$$\begin{aligned} &\Delta\bar{u_1} + g_1(\bar{u_1}, u_2, ..., u_N) \\ \leq &\bar{u_1}(g_1)_{u_1}(\frac{\bar{u_1}}{2}, u_2, ..., u_N) \\ = &\bar{u_1}[(g_1)_{u_1}(0, ..., 0) \\ &+ (g_1)_{u_1}(\frac{\bar{u_1}}{2}, u_2, ..., u_N) - (g_1)_{u_1}(0, u_2, ..., u_N) \\ &+ (g_1)_{u_1}(0, u_2, ..., u_N) - (g_1)_{u_1}(0, 0, u_3, ..., u_N)] \\ &+ \\ &. \end{aligned}$$

$$
\begin{aligned}
&.\\
&.\\
&+(g_1)_{u_1}(0,...,0,u_N)-(g_1)_{u_1}(0,...,0)]\\
\le&\bar{u_1}[(g_1)_{u_1}(0,...,0)+\frac{\bar{u_1}}{2}\sup(g_1)_{u_1u_1}+\sum_{j=2}^{N}u_j\sup(g_1)_{u_1u_j}]\\
=&\sum_{j=2}^{N}\bar{u_1}u_j\sup(g_1)_{u_1u_j}\\
<&0,
\end{aligned}
$$

and for all u_1 such that $\underline{u_1}\le u_1\le\bar{u_1}$ and $i=2,...,N$,

$$
\begin{aligned}
&\Delta\bar{u_i}+g_i(u_1,...,u_{i-1},\bar{u_i},u_{i+1},...,u_N)\\
\le\;&\bar{u_i}(g_i)_{u_i}(u_1,...,u_{i-1},\tfrac{\bar{u_i}}{2},u_{i+1},...,u_N)\\
\le\;&\bar{u_i}[(g_i)_{u_i}(0,...,0)+\tfrac{\bar{u_i}}{2}\sup(g_i)_{u_iu_i}+\textstyle\sum_{j=1,j\neq i}^{N}u_j\sup(g_i)_{u_iu_j}]\\
\le\;&\bar{u_i}[(g_i)_{u_i}(0,...,0)+\tfrac{\bar{u_i}}{2}\sup(g_i)_{u_iu_i}+u_1\sup(g_i)_{u_iu_1}]\\
\le\;&\bar{u_i}[(g_i)_{u_i}(0,...,0)+\tfrac{\bar{u_i}}{2}\sup(g_i)_{u_iu_i}+\bar{u_1}\sup(g_i)_{u_iu_1}]\\
=\;&-\bar{u_i}(g_i)_{u_i}(0,...,0)\\
<\;&0.
\end{aligned}
$$

By the conditions, the Lemma 4.1 and the Mean Value Theorem again, for all u_i such that $\underline{u_i}\le u_i\le\bar{u_i}$, $i=2,...,N$,

$$
\begin{aligned}
&\Delta\underline{u_1}+g_1(\underline{u_1},u_2,...,u_N)\\
\ge\;&\Delta\underline{u_1}+\underline{u_1}(g_1)_{u_1}(\underline{u_1},u_2,...,u_N)\\
=\;&\Delta(\gamma_1\omega)+\gamma_1\omega(g_1)_{u_1}(\gamma_1\omega,u_2,...,u_N)\\
\ge\;&-\gamma_1\lambda_1\omega+\gamma_1\omega[(g_1)_{u_1}(0,...,0)+\gamma_1\omega\inf(g_1)_{u_1u_1}+\textstyle\sum_{i=2}^{N}u_i\inf(g_1)_{u_1u_i}]\\
\ge\;&-\gamma_1\lambda_1\omega+\gamma_1\omega[(g_1)_{u_1}(0,...,0)+\gamma_1\omega\inf(g_1)_{u_1u_1}+\textstyle\sum_{i=2}^{N}\bar{u_i}\inf(g_1)_{u_1u_i}]\\
\ge\;&-\gamma_1\lambda_1\omega+\gamma_1\omega[(g_1)_{u_1}(0,...,0)+(N-1)\inf_{i=2,...,N}\{\bar{u_i}\inf(g_1)_{u_1u_i}\}\\
&+\gamma_1\omega\inf(g_1)_{u_1u_1}]\\
=\;&\gamma_1\omega[-\lambda_1+\inf_{i=2,...,N}\{(g_1)_{u_1}(0,...,0)(1+\tfrac{4(N-1)\sup(g_i)_{u_1u_i}\inf(g_1)_{u_1u_i}}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}})\\
&-\tfrac{4(N-1)\inf(g_1)_{u_1u_i}(g_i)_{u_i}(0,...,0)}{\sup(g_i)_{u_iu_i}}\}+\gamma_1\omega\inf(g_1)_{u_1u_1}]\\
>\;&\gamma_1\omega[-\lambda_1+\inf_{i=2,...,N}\{\tfrac{4\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}[\lambda_1+\frac{(N-1)\inf(g_1)_{u_1u_i}(g_i)_{u_i}(0,...,0)}{\sup(g_i)_{u_iu_i}}]}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}+4(N-1)\inf(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}}\\
&\tfrac{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}+4(N-1)\inf(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}}-\tfrac{4(N-1)\inf(g_1)_{u_1u_i}(g_i)_{u_i}(0,...,0)}{\sup(g_i)_{u_iu_i}}\}\\
&+\gamma_1\omega\inf(g_1)_{u_1u_1}]\\
=\;&\gamma_1\omega[-\lambda_1+4\lambda_1+\gamma_1\omega\inf(g_1)_{u_1u_1}]\\
=\;&\gamma_1\omega[3\lambda_1+\gamma_1\omega\inf(g_1)_{u_1u_1}]\\
>\;&0
\end{aligned}
$$

with small enough $\gamma_1 > 0$, and for all u_1 such that $\underline{u_1} \leq u_1 \leq \bar{u_1}$ and $i = 2, ..., N$,

$$\begin{aligned}
& \Delta\underline{u_i} + g_i(u_1, ..., u_{i-1}, \underline{u_i}, u_{i+1}, ..., u_N) \\
\geq \quad & -\gamma_i\lambda_1\omega + \gamma_i\omega(g_i)_{u_i}(u_1, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
= \quad & -\gamma_i\lambda_1\omega + \gamma_i\omega[(g_i)_{u_i}(0, ..., 0) + (g_i)_{u_i}(u_1, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
& -(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
& +(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N) \\
& -(g_i)_{u_i}(0, 0, u_3, ..., u_{i-1}, \gamma_i\omega, u_{i+1}, ..., u_N)] \\
& +.... \\
& +(g_i)_{u_i}(0, ..., 0, \gamma_i\omega, u_{i+1}, ..., u_N) - (g_i)_{u_i}(0, ..., 0, u_{i+1}, ..., u_N) \\
& +(g_i)_{u_i}(0, ..., 0, u_{i+1}, ..., u_N) - (g_i)_{u_i}(0, ..., 0, u_{i+2}, ..., u_N) \\
& +... \\
& +(g_i)_{u_i}(0, ..., 0, u_N) - (g_i)_{u_i}(0, ..., 0)] \\
\geq \quad & -\gamma_i\lambda_1\omega + \gamma_i\omega[(g_i)_{u_i}(0, ..., 0) + \textstyle\sum_{j=1, j\neq i}^{N} u_j \inf(g_i)_{u_iu_j} + \gamma_i\omega \inf(g_i)_{u_iu_i}] \\
\geq \quad & \gamma_i\omega[(g_i)_{u_i}(0, ..., 0) - \lambda_1 + \gamma_i\omega \inf(g_i)_{u_iu_i} + \textstyle\sum_{j=2, j\neq i}^{N} \bar{u_j} \inf(g_i)_{u_iu_j}] \\
> 0 &
\end{aligned}$$

with small enough $\gamma_i > 0$. Furthermore,

$$\underline{u_i} = \bar{u_i} = 0 \ \text{ on } \ \partial\Omega$$

and

$$\underline{u_i} \leq \bar{u_i}$$

with small enough $\gamma_i > 0$. Hence, by the Lemma 3.1, there is a solution $(u_1, ..., u_N)$ to (4) with

$$\underline{u_i} \leq u_i \leq \bar{u_i}.$$

In order to prove further results, we will need the following solution estimate.

Lemma 4.3. *Let $(u_1, ..., u_N)$, where $u_i \geq 0, i = 1, ..., N$ be a solution of the problem*

$$\begin{cases} \Delta u_i + g_i(u_1, ..., u_N) = 0 \quad in \ \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{5}$$

where

(A)

$$(g_i)_{u_iu_j} \quad \begin{cases} < 0, i = 1, j = 1, ..., N \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N, \end{cases}$$

(B) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ *for all* $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N$,

(C) $g_i(u_1, ..., u_i, ..., u_N) \leq u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N)$ *for all* $u_1, ..., u_N$ *and* $i = 1, ..., N$,

(D) $\sup(g_1)_{u_1u_1} \sup(g_i)_{u_iu_i} + 4(N-1) \inf(g_1)_{u_1u_i} \sup(g_i)_{u_1u_i} > 0$, *for* $i = 2, ..., N$

(E) $(g_1)_{u_1}(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N)$

$+\inf(g_1)_{u_1u_i}[\frac{4(g_1)_{u_1}(0,...,0)\sup(g_i)_{u_1u_i}-2(g_i)_{u_i}(0,...,0)\sup(g_1)_{u_1u_1}}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}}]>\lambda_1$, *and*
$(g_i)_{u_i}(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N)>\lambda_1, i=2,...,N$, *where* $M_i, i=2,...,N$ *is defined below,*
(F) $\lim_{x_i\to\infty}(g_i)_{x_i}(x_1,...,x_i,...,x_N)=-\infty, i=1,...,N$,
then
(1)

$$\begin{aligned} u_1 &\le -\tfrac{2(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}, \\ u_i \le M_i &= \tfrac{4\sup(g_i)_{u_1u_i}(g_1)_{u_1}(0,...,0)-2(g_i)_{u_i}(0,...,0)\sup(g_1)_{u_1u_1}}{\sup(g_i)_{u_iu_i}\sup(g_1)_{u_1u_1}}, i=2,...,N. \end{aligned}$$

(2)

$$\begin{array}{rcccl} \frac{1}{2}\theta_{M(g_1,g_i)} & \le & u_1 & \le & 2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, \\ \theta_{(g_i)_{u_i}(0,M_2,...,M_{i-1},\cdot,M_{i+1},...,M_N)} & \le & u_i & \le & 2\theta_{K(g_1,g_i)}, i=2,...,N, \end{array}$$

where

$$\begin{aligned} M(g_1,g_i) &= (g_1)_{u_1}(\cdot,M_2,...,M_{i-1},0,M_{i+1},...,M_N) \\ &\quad +\inf(g_1)_{u_1u_i}[\tfrac{4(g_1)_{u_1}(0,...,0)\sup(g_i)_{u_1u_i}-2(g_i)_{u_i}(0,...,0)\sup(g_1)_{u_1u_1}}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}}], i=2,...,N, \\ K(g_1,g_i) &= (g_i)_{u_i}(0,...,0,\cdot,0,...,0)-\tfrac{2(g_1)_{u_1}(0,...,0)\sup(g_i)_{u_1u_i}}{\sup(g_1)_{u_1u_1}}, i=2,...,N. \end{aligned}$$

Proof.
(1) Since $(u_1,...,u_N)$ is a solution to (5), by the condition and the Mean Value Theorem,

$$\begin{aligned} & \Delta u_1+u_1[(g_1)_{u_1}(0,...,0)+\tfrac{u_1}{2}\sup(g_1)_{u_1u_1})+\textstyle\sum_{i=2}^N\sup(g_1)_{u_1u_i}u_i] \\ \ge\ & \Delta u_1+u_1[(g_1)_{u_1}(0,...,0)+(g_1)_{u_1}(\tfrac{u_1}{2},u_2,...,u_N)-(g_1)_{u_1}(0,u_2,...,u_N) \\ & +(g_1)_{u_1}(0,u_2,...,u_N)-(g_1)_{u_1}(0,0,u_3,...,u_N) \\ & +(g_1)_{u_1}(0,0,u_3,...,u_N)-(g_1)_{u_1}(0,0,0,u_4,...,u_N)] \\ & + \\ & . \\ & . \\ & . \\ & + \\ & +(g_1)_{u_1}(0,0,...0,u_N)-(g_1)_{u_1}(0,0,...,0)]] \\ =\ & \Delta u_1+u_1(g_1)_{u_1}(\tfrac{u_1}{2},u_2,...,u_N) \\ \ge\ & \Delta u_1+g_1(u_1,...,u_N) \\ =\ & 0, \end{aligned}$$

and so,

$$\Delta u_1+u_1[(g_1)_{u_1}(0,...,0)+\sup(g_1)_{u_1u_1}\frac{u_1}{2}]\ge -u_1\sup(g_1)_{u_1u_i})u_i>0.$$

Hence, by the Maximum Principles,

$$(g_1)_{u_1}(0,...,0)+\sup(g_1)_{u_1u_1})\frac{u_1}{2}\ge 0,$$

equivalently,

$$u_1\le -\frac{2(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}.$$

Since $(u_1, ..., u_N)$ is a solution to (5), by the condition and the Mean Value Theorem, for $i = 2, ..., N$,

$$\begin{array}{rl} & \Delta u_i + u_i[(g_i)_{u_i}(0, ..., 0) + \sup(g_i)_{u_iu_i}\frac{u_i}{2} + \sup(g_i)_{u_iu_1}u_1] \\ \geq & \Delta u_i + u_i[(g_i)_{u_i}(0, ..., 0) + \sup(g_i)_{u_iu_i}\frac{u_i}{2} + \sum_{j=1,j\neq i}^{N}\sup(g_i)_{u_iu_j}u_j] \\ \geq & \Delta u_i + u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N) \\ \geq & \Delta u_i + g_i(u_1, ..., u_i, ..., u_N) \\ = & 0, \end{array}$$

and so,

$$\begin{array}{rl} & \Delta u_i + u_i[(g_i)_{u_i}(0, ..., 0) + \sup(g_i)_{u_iu_i}\frac{u_i}{2} + \sup(g_i)_{u_1u_i}(-2\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}})] \\ \geq & -\sup(g_i)_{u_1u_i}u_1u_i + u_i + \sup(g_i)_{u_1u_i}(-2\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}} \\ = & \sup(g_i)_{u_iu_1}u_i[-u_1 - \frac{2(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}] \\ \geq & 0. \end{array}$$

Hence, by the Maximum Principles,

$$(g_i)_{u_i}(0, ..., 0) + \sup(g_i)_{u_iu_i}\frac{u_i}{2} + \sup(g_i)_{u_1u_i}(-2\frac{(g_1)_{u_1}(0, ..., 0)}{\sup(g_1)_{u_1u_1}}) \geq 0, i = 2, ..., N,$$

equivalently,

$$u_i \leq \frac{4\sup(g_i)_{u_1u_i}(g_1)_{u_1}(0, ..., 0) - 2(g_i)_{u_i}(0, ..., 0)\sup(g_1)_{u_1u_1}}{\sup(g_i)_{u_iu_i}\sup(g_1)_{u_1u_1}}, i = 2, ..., N.$$

(2) By the conditions and monotonicity of g_1,

$$\begin{array}{rcl} \Delta\frac{u_1}{2} + \frac{u_1}{2}(g_1)_{u_1}(\frac{u_1}{2}, 0, ..., 0) & \geq & \frac{1}{2}[\Delta u_1 + g_1(u_1, 0, ..., 0)] \\ & \geq & \frac{1}{2}[\Delta u_1 + g_1(u_1, ..., u_N)] \\ & = & 0, \end{array}$$

and so, $\frac{u_1}{2}$ is a subsolution to

$$\begin{array}{l} \Delta Z + Z(g_1)_{u_1}(Z, 0, ..., 0) = 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} = 0. \end{array}$$

But, since any sufficiently large positive constant is a supersolution to

$$\begin{array}{l} \Delta Z + Z(g_1)_{u_1}(Z, 0, ..., 0) = 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} = 0, \end{array}$$

by the Lemmas 3.1 and 3.3, we conclude that

$$u_1 \leq 2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}. \tag{6}$$

Since for $i = 2, ..., N$,

$$\begin{aligned}
& \Delta 2u_1 + 2u_1[(g_1)_{u_1}(2u_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& + \inf(g_1)_{u_1u_i}[\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}}] \\
\leq\ & \Delta 2u_1 + 2u_1(g_1)_{u_1}(u_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& +2u_1\inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}}) \\
=\ & 2u_1(g_1)_{u_1}(u_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) - 2g_1(u_1, ..., u_N) \\
& +2u_1\inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}}) \\
\leq\ & 2u_1(g_1)_{u_1}(u_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& -2u_1(g_1)_{u_1}(u_1, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) \\
& +2u_1\inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}}) \\
\leq\ & -2u_1\inf(g_1)_{u_1u_i}u_i + +2u_1\inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}}) \\
=\ & -2u_1\inf(g_1)_{u_1u_i}[u_i - \tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}}] \\
\leq\ & 0
\end{aligned}$$

by the Mean Value Theorem, the Lemma 4.1, the condition and (1), $2u_1$ is a supersolution to

$$\begin{aligned}
& \Delta Z + Z[(g_1)_{u_1}(Z, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& \quad + \inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}})] = 0 \text{ in } \Omega, \\
& Z|_{\partial\Omega} = 0.
\end{aligned}$$

But, by the continuity of $(g_1)_{u_1}$ and the condition, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}
& \Delta\epsilon\phi_1 + \epsilon\phi_1[(g_1)_{u_1}(\epsilon\phi_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& + \inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}})] \\
=\ & \epsilon\phi_1[-\lambda_1 + (g_1)_{u_1}(\epsilon\phi_1, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& + \inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}})] \\
>\ & 0,
\end{aligned}$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\begin{aligned}
& \Delta Z + Z[(g_1)_{u_1}(Z, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) \\
& \quad + \inf(g_1)_{u_1u_i}(\tfrac{4\sup(g_i)_{u_1u_i}[-\frac{(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}]+2(g_i)_{u_i}(0,...,0)}{-\sup(g_i)_{u_iu_i}})] = 0 \text{ in } \Omega, \\
& Z|_{\partial\Omega} = 0.
\end{aligned}$$

Therefore, by the Lemmas 3.1 and 3.3, we have

$$\tfrac{1}{2}\theta_{M(g_1,g_i)} \leq u_1, i = 2, ..., N. \tag{7}$$

By the monotonicity of $(g_i)_{u_i}, i = 2, ..., N$ and the Lemma 4.1,

$$\begin{aligned} & \Delta u_i + u_i (g_i)_{u_i}(0, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) \\ = \ & u_i (g_i)_{u_i}(0, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) - g_i(u_1, ..., u_N) \\ \leq \ & u_i (g_i)_{u_i}(0, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) \\ & -u_i (g_i)_{u_i}(u_1, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) \\ < \ & 0, i = 2, ..., N, \end{aligned}$$

and so, u_i is a supersolution to

$$\begin{aligned} & \Delta Z + Z(g_i)_{u_i}(0, M_2, ..., M_{i-1}, Z, M_{i+1}, ..., M_N) = 0 \ \text{ in } \ \Omega, \\ & Z|_{\partial\Omega} = 0, i = 2, ..., N. \end{aligned}$$

But, by the continuity of $(g_i)_{u_i}$ and the condition, for sufficiently small $\epsilon > 0$,

$$\begin{aligned} & \Delta \epsilon\phi_1 + \epsilon\phi_1 (g_i)_{u_i}(0, M_2, ..., M_{i-1}, \epsilon\phi_1, M_{i+1}, ..., M_N) \\ = \ & \epsilon\phi_1[-\lambda_1 + (g_i)_{u_i}(0, M_2, ..., M_{i-1}, \epsilon\phi_1, M_{i+1}, ..., M_N)] \\ > \ & 0, i = 2, ..., N, \end{aligned}$$

and so, $\epsilon\phi_1$ is a subsolution to

$$\begin{aligned} & \Delta Z + Z(g_i)_{u_i}(0, M_2, ..., M_{i-1}, Z, M_{i+1}, ..., M_N) = 0 \ \text{ in } \ \Omega, \\ & Z|_{\partial\Omega} = 0, i = 2, ..., N. \end{aligned}$$

Hence, by the Lemmas 3.1 and 3.3, we have

$$\theta_{(g_i)_{u_i}(0, M_2, ..., M_{i-1}, \cdot, M_{i+1}, ..., M_N)} \leq u_i, i = 2, ..., N. \tag{8}$$

Since $(u_1, ..., u_N)$ is a solution to (5), by the Mean Value Theorem, the condition and (1),

$$\begin{aligned} & \Delta \tfrac{u_i}{2} + \tfrac{u_i}{2}[(g_i)_{u_i}(0, ..., 0, \tfrac{u_i}{2}, 0, ..., 0) - \tfrac{2 \sup (g_i)_{u_1 u_i} (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] \\ \geq \ & \tfrac{u_i}{2}[-(g_i)_{u_i}(u_1, ..., u_{i-1}, \tfrac{u_i}{2}, u_{i+1}, ..., u_N) + (g_i)_{u_i}(0, ..., 0, \tfrac{u_i}{2}, 0, ..., 0) \\ & -\tfrac{2 \sup (g_i)_{u_1 u_i} (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] \\ \geq \ & \tfrac{u_i}{2}[-(g_i)_{u_i}(u_1, ..., u_{i-1}, \tfrac{u_i}{2}, u_{i+1}, ..., u_N) + (g_i)_{u_i}(0, u_2, ..., u_{i-1}, \tfrac{u_i}{2}, u_{i+1}, ..., u_N) \\ & -\tfrac{2 \sup (g_i)_{u_1 u_i} (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] \\ \geq \ & \tfrac{u_i}{2}[-\sup (g_i)_{u_1 u_i}) u_1 - \tfrac{2 \sup (g_i)_{u_1 u_i} (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] \\ = \ & \sup (g_i)_{u_1 u_i} \tfrac{u_i}{2}[-u_1 - \tfrac{2 (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] \\ \geq \ & 0, i = 2, ..., N. \end{aligned}$$

Therefore, $\frac{u_i}{2}$ is a subsolution to

$$\begin{aligned} & \Delta Z + Z[(g_i)_{u_i}(0, ..., 0, Z, 0, ..., 0) - \tfrac{2 \sup (g_i)_{u_1 u_i} (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] = 0 \ \text{ in } \ \Omega, \\ & Z|_{\partial\Omega} = 0, i = 2, ..., N. \end{aligned}$$

But, since any sufficiently large constant is a supersolution to

$$\begin{aligned} & \Delta Z + Z[(g_i)_{u_i}(0, ..., 0, Z, 0, ..., 0) - \tfrac{2 \sup (g_i)_{u_1 u_i} (g_1)_{u_1}(0,...,0)}{\sup (g_1)_{u_1 u_1}}] = 0 \ \text{ in } \ \Omega, \\ & Z|_{\partial\Omega} = 0, i = 2, ..., N. \end{aligned}$$

by the Lemmas 3.1 and 3.3, we have

$$u_i \leq 2\theta_{(g_i)_{u_i}(0,...,0,\cdot,0,...,0)-\frac{2\sup(g_i)_{u_1u_i}(g_1)_{u_1}(0,...,0)}{\sup(g_1)_{u_1u_1}}}, i = 2, ..., N. \tag{9}$$

By (6), (7), (8) and (9), we establish the desired inequalities.

5. Nonexistence

We consider

$$\begin{cases} \Delta u_i + g_i(u_1, ..., u_N) &= 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{10}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $g_i \in C^2, i = 1, ..., N$ are such that

(A)

$$(g_i)_{u_iu_j} \begin{cases} < 0, i = 1, j = 1, ..., N \\ > 0, i = 2, ..., N, j = 1, \\ < 0, i = 2, ..., N, j = 2, ..., N, \end{cases}$$

(B) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ for all $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N$ and $i = 1, ..., N$ and

(C) $(g_i)_{u_j}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0, i, j = 1, ..., N, i \neq j$.

First, we see that the two species can not coexist when the reproduction capacities are not robust enough.

Theorem 5.1. *Suppose $(g_i)_{u_i}(0, ..., 0) \leq \lambda_1$ for all $i = 1, ..., N$. Then $(0, ..., 0)$ is the only nonnegative solution to (10).*

Proof. Let $(u_1, ..., u_N)$ be a nonnegative solution to (10). By the Mean Value Theorem, for each $i = 2, ..., N$, there are $\tilde{u_1}, \bar{u_1}, \tilde{u_i}, \bar{u_i}, w, z$ such that $0 \leq \tilde{u_1} \leq u_1, 0 \leq \bar{u_1} \leq u_1, 0 \leq \tilde{u_i} \leq u_i, 0 \leq \bar{u_i} \leq u_i, 0 \leq w \leq u_1, 0 \leq z \leq u_i$ and

$$\begin{aligned} & \Delta u_1 + u_1[(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ & +u_i(g_1)_{u_1u_i}(w, u_2, ..., u_{i-1}, \tilde{u_i}, u_{i+1}, ..., u_N)] \\ = \; & \Delta u_1 + u_1(g_1)_{u_1}(\tilde{u_1}, , u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ & +u_1u_i(g_1)_{u_1u_i}(w, u_2, ..., u_{i-1}, \tilde{u_i}, u_{i+1}, ..., u_N) \\ = \; & \Delta u_1 + u_1(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ & +u_i(g_1)_{u_i}(u_1, ..., u_{i-1}, \tilde{u_i}, u_{i+1}, ..., u_N) - u_i(g_1)_{u_i}(0, u_2, ..., u_{i-1}, \tilde{u_i}, u_{i+1}, ..., u_N) \\ = \; & \Delta u_1 + g_1(u_1, ..., u_i, ..., u_N) - g_1(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ & +g_1(u_1, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) - g_1(0, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ = \; & \Delta u_1 + g_1(u_1, ..., u_N) \\ = \; & 0, \end{aligned}$$

and

$$\begin{aligned}
& \Delta u_i + u_i[(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) \\
& +u_1(g_i)_{u_1u_i}(\bar{u_1}, u_2, ..., u_{i-1}, z, u_{i+1}, ..., u_N)] \\
= \; & \Delta u_i + u_i(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) \\
& +u_1u_i(g_i)_{u_1u_i}(\bar{u_1}, u_2, ..., u_{i-1}, z, u_{i+1}, ..., u_N) \\
= \; & \Delta u_i + u_i(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) + u_1(g_i)_{u_1}(\bar{u_1}, u_2, ..., u_i, ..., u_N) \\
& -u_1(g_i)_{u_1}(\bar{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\
= \; & \Delta u_i + g_i(u_1, ..., u_N) - g_i(0, u_2, ..., u_N) + g_i(0, u_2, ..., u_i, ..., u_N) \\
& -g_i(0, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\
= \; & \Delta u_i + g_i(u_1, ..., u_N) \\
= \; & 0.
\end{aligned}$$

Hence, for $i = 2, ..., N$,

$$\begin{aligned}
& \Delta u_1 + u_1[(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\
& + \sup(g_1)_{u_1u_i}u_i] \geq 0 \;\text{ in }\; \Omega, \\
& \Delta u_i + u_i[(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) \\
& + \sup(g_i)_{u_1u_i}u_1] \geq 0 \;\text{ in }\; \Omega.
\end{aligned}$$

Therefore, for $i = 2, ..., N$,

$$\begin{aligned}
& \sup(g_i)_{u_1u_i}\phi_1\Delta u_1 + \sup(g_i)_{u_1u_i}\phi_1u_1[(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\
& + \sup(g_1)_{u_1u_i}u_i] \geq 0 \;\text{ in }\; \Omega, \\
& -\sup(g_1)_{u_1u_i}\phi_1\Delta u_i - \sup(g_1)_{u_1u_i}\phi_1u_i[(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) \\
& + \sup(g_i)_{u_1u_i}u_1] \geq 0 \;\text{ in }\; \Omega.
\end{aligned}$$

So, for $i = 2, ..., N$,

$$\begin{aligned}
& \int_\Omega -\sup(g_i)_{u_1u_i}\phi_1\Delta u_1 dx \\
\leq \; & \int_\Omega[(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)\sup(g_i)_{u_1u_i}u_1 \\
& + \sup(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}u_1u_i]\phi_1 dx, \\
& \int_\Omega \sup(g_1)_{u_1u_i}\phi_1\Delta u_i dx \\
\leq \; & \int_\Omega[-(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N)\sup(g_1)_{u_1u_i}u_i \\
& - \sup(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}u_1u_i]\phi_1 dx.
\end{aligned}$$

Hence, by Green's Identity, for $i = 2, ..., N$, we have

$$\begin{aligned}
& \int_\Omega \sup(g_i)_{u_1u_i}\lambda_1\phi_1u_1 dx \\
\leq \; & \int_\Omega[(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)\sup(g_i)_{u_1u_i}u_1 \\
& + \sup(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}u_1u_i]\phi_1 dx, \\
& \int_\Omega -\sup(g_{uv})\lambda_1\phi_1 v dx \\
\leq \; & \int_\Omega[-(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N)\sup(g_1)_{u_1u_i}u_i \\
& - \sup(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}u_1u_i]\phi_1 dx.
\end{aligned}$$

Therefore, for $i = 2, ..., N$,

$$\begin{aligned}&\textstyle\int_\Omega \sup(g_i)_{u_1u_i}[\lambda_1 - (g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)]u_1\phi_1 \\ &-\sup(g_1)_{u_1u_i}[\lambda_1 - (g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N)]u_i\phi_1 dx \leq 0.\end{aligned}$$

Since for $i = 2, ..., N$, the left hand side is nonnegative from

$$\begin{aligned}&(g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \leq (g_1)_{u_1}(0, ..., 0) \leq \lambda_1,\\ &(g_i)_{u_i}(0, u_2, ..., u_{i-1}, \bar{u_i}, u_{i+1}, ..., u_N) \leq (g_i)_{u_i}(0, ..., 0) \leq \lambda_1,\end{aligned}$$

we conclude that $u_1 = u_i \equiv 0, i = 2, ..., N$.

Theorem 5.2. *Let $(u_1, ..., u_N)$, where $u_i \geq 0, i = 1, ..., N$ be a solution to (10). If $(g_1)_{u_1}(0, ..., 0) \leq \lambda_1$, then $u_1 \equiv 0$.*

Proof. Proceeding as in the proof of theorem 5.1, we obtain

$$\begin{aligned}0 &\leq \textstyle\int_\Omega[\lambda_1 - (g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)]u_1\phi_1 dx\\ &\leq \textstyle\int_\Omega \sup(g_1)_{u_1u_i}u_1u_i\phi_1 dx\\ &\leq 0.\end{aligned}$$

But, since $\lambda_1 - (g_1)_{u_1}(\tilde{u_1}, u_2, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) > \lambda_1 - (g_1)_{u_1}(0, ..., 0) \geq 0$, $u_1 \equiv 0$.

We consider

$$\begin{cases} \Delta u_i + g_i(u_1, ..., u_N) = 0 \quad \text{in } \Omega,\\ u_i|_{\partial\Omega} = 0, i = 1, ..., N, \end{cases} \tag{11}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $g_i \in C^2, i = 1, ..., N$ are such that

(A)

$$(g_i)_{u_iu_j} \begin{cases} < 0, i = 1, j = 1, ..., N\\ > 0, i = 2, ..., N, j = 1,\\ < 0, i = 2, ..., N, j = 2, ..., N, \end{cases}$$

(B) $g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0$ for all $u_1, ..., u_{i-1}, u_{i+1}, ..., u_N$ and for all $i = 1, ..., N$,

(C) $g_i(u_1, ..., u_i, ..., u_N) \leq u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N)$ for all $u_1, ..., u_N$ and $i = 1, ..., N$, and

(D) there is $c > 0$ such that $(g_1)_{u_1}(u_1, 0, ..., 0) < 0$ and $(g_i)_{u_i}(0, M_2, ..., M_{i-1}, u_i, M_{i+1}, ..., M_N) < 0$ for all $u_i > c, i = 1, ..., N$.

Theorem 5.3. *(1) If $(g_1)_{u_1}(0, ..., 0) > \lambda_1$ and $\lambda_1[-(g_i)_{u_i}(2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 0, ..., 0)] \geq 0$ for all $i = 2, ..., N$, then there is no positive solution to (11).*
(2) If $(g_i)_{u_i}(0, ..., 0) > \lambda_1$ and $\lambda_1[-(g_1)_{u_1}(0, \theta_{(g_2)_{u_2}(0,\cdot,M_3,...,M_N)}, ..., \theta_{(g_N)_{u_N}(0,M_2,...,M_{i-1},\cdot)})] \geq 0$ for all $i = 2, ..., N$, then there is no positive solution to (11).

Proof. (1) Let $(\tilde{u_1}, ..., \tilde{u_N})$, where $\tilde{u_i} \geq 0, i = 1, ..., N$ be a solution to (11). Suppose $\tilde{u_i} > 0$ for some $i = 2, ..., N$ in Ω. Then since

$$\begin{array}{rl} & \Delta\tilde{u_i} + \tilde{u_i}(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_i}, ..., \tilde{u_N}) \\ \leq & \Delta\tilde{u_i} + g_i(\tilde{u_1}, ..., \tilde{u_i}, ..., \tilde{u_N}) \\ = & 0, \end{array}$$

by the Lemma 4.1, $\tilde{u_i}$ is a supersolution to

$$\begin{array}{rcl} \Delta Z + Z(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_{i-1}}, Z, \tilde{u_{i+1}}, ..., \tilde{u_N}) & = & 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} & = & 0. \end{array}$$

Furthermore,

$$\begin{array}{rl} & \Delta\frac{\tilde{u_i}}{2} + \frac{\tilde{u_i}}{2}(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_{i-1}}, \frac{\tilde{u_i}}{2}, \tilde{u_{i+1}}, ..., \tilde{u_N}) \\ = & \frac{1}{2}[\Delta\tilde{u_i} + \tilde{u_i}(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_{i-1}}, \frac{\tilde{u_i}}{2}, \tilde{u_{i+1}}, ..., \tilde{u_N})] \\ \geq & \frac{1}{2}[\Delta\tilde{u_i} + (g_i)(\tilde{u_1}, ..., \tilde{u_i}, ..., \tilde{u_N})] \\ = & 0, \end{array}$$

and so, $\frac{\tilde{u_i}}{2}$ is a subsolution to

$$\begin{array}{rcl} \Delta Z + Z(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_{i-1}}, Z, \tilde{u_{i+1}}, ..., \tilde{u_N}) & = & 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} & = & 0. \end{array}$$

Therefore, by the Lemma 3.1, there is a solution u_i to

$$\begin{array}{rcl} \Delta Z + Z(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_{i-1}}, Z, \tilde{u_{i+1}}, ..., \tilde{u_N}) & = & 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} & = & 0 \end{array}$$

with $\frac{\tilde{u_i}}{2} \leq u_i \leq \tilde{u_i}$.

Hence, by the monotonicity and the Lemma 4.3, we have

$$\begin{array}{rl} & (g_i)_{u_i}(0, ..., 0) \\ = & \lambda_1[-(g_i)_{u_i}(\tilde{u_1}, ..., \tilde{u_{i-1}}, u_i, \tilde{u_{i+1}}, ..., \tilde{u_N}) + (g_i)_{u_i}(0, ..., 0)] \\ > & \lambda_1[-(g_i)_{u_i}(2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 0, ..., 0) + (g_i)_{u_i}(0, ..., 0)] \\ = & \lambda_1[-(g_i)_{u_i}(2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 0, ..., 0)] + (g_i)_{u_i}(0, ..., 0), \end{array}$$

and so,

$$\lambda_1[-(g_i)_{u_i}(2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}, 0, ..., 0)] < 0,$$

which is a contradiction to the assumption.

(2) Let $(\tilde{u_1}, ..., \tilde{u_N})$, where $\tilde{u_i} \geq 0, i = 1, ..., N$ be a solution to (11).

If $\tilde{u_1} > 0$, then by the same way in (1), there is a solution u_1 to

$$\begin{array}{rcl} \Delta Z + Z(g_1)_{u_1}(Z, \tilde{u_2}, ..., \tilde{u_N}) & = & 0 \text{ in } \Omega, \\ Z|_{\partial\Omega} & = & 0 \end{array}$$

with $\frac{\tilde{u_1}}{2} \le u_1 \le \tilde{u_1}$.

Hence, by the monotonicity and the Lemma 4.3, we have

$$\begin{aligned}
& (g_1)_{u_1}(0,...,0) \\
= \;& \lambda_1[-(g_1)_{u_1}(u_1, \tilde{u_2}, ..., \tilde{u_N}) + (g_1)_{u_1}(0,...,0)] \\
> \;& \lambda_1[-(g_1)_{u_1}(0, \theta_{(g_2)_{u_2}(0,\cdot,M_3,...,M_N)}, ..., \theta_{(g_N)_{u_N}(0,M_2,...,M_{N-1},\cdot)}) + (g_1)_{u_1}(0,...,0)] \\
= \;& \lambda_1[-(g_1)_{u_1}(0, \theta_{(g_2)_{u_2}(0,\cdot,M_3,...,M_N)}, ..., \theta_{(g_N)_{u_N}(0,M_2,...,M_{N-1},\cdot)})] + (g_1)_{u_1}(0,...,0),
\end{aligned}$$

and so,

$$\lambda_1[-(g_1)_{u_1}(0, \theta_{(g_2)_{u_2}(0,\cdot,M_3,...,M_N)}, ..., \theta_{(g_N)_{u_N}(0,M_2,...,M_{N-1},\cdot)})] < 0,$$

which is a contradiction to the assumption.

6. Uniqueness

In this section, we prove the uniqueness of positive solution.

For our convenience, we denote $A(a,b)$ for the average of real numbers a and b.

Definition **6.1.** *We say that a real valued function f has the Mean Value Property in the sense of average if $A(x_1,x_2) \ge A(w_1,w_2)$, then there are $\bar{x}$ between x_1 and x_2, and $\bar{w}$ between w_1 and w_2 such that*

$$\begin{aligned}
&\bar{x} - \bar{w} \ge A(x_1,x_2) - A(w_1,w_2), \\
&f(x_1) - f(x_2) = (x_1 - x_2) f'(\bar{x}), \\
&f(w_1) - f(w_2) = (w_1 - w_2) f'(\bar{w}).
\end{aligned}$$

We consider

$$\begin{cases} \Delta u_i + g_i(u_1,...,u_N) \;=\; 0 \quad \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1,...,N, \end{cases} \tag{12}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $g_i \in C^2, i = 1,...,N$ are such that $g_i(x_1,...,x_{i-1},\cdot,x_{i+1},...,x_N), i = 1,...,N$ has the Mean Value Property in the sense of average for all $x_1,...,x_{i-1},x_{i+1},...,x_N \in R$, and

(U1)

$$(g_i)_{u_iu_j} \begin{cases} < 0, i = 1, j = 1,...,N \\ > 0, i = 2,...,N, j = 1, \\ < 0, i = 2,...,N, j = 2,...,N, \end{cases}$$

(U2) $g_i(u_1,...,u_i,...,u_N) \le u_i(g_i)_{u_i}(u_1,...,u_{i-1},\frac{u_i}{2},u_{i+1},...,u_N)$ for all $u_1,...,u_N$ and $i = 1,...,N$,

(U3) $g_i(u_1,...,u_{i-1},0,u_{i+1},...,u_N) = 0$ for all $u_1,...,u_{i-1},u_{i+1},...,u_N$ and $i = 1,...,N$ and $(g_i)_{u_j}(u_1,...,u_{i-1},0,u_{i+1},...,u_N) = 0, i,j = 1,...,N, i \ne j$.

(U4) $\lim_{x_i\to\infty}(g_i)_{x_i}(x_1,...,x_i,...,x_N) = -\infty, i = 1,...,N$.

We have the following uniqueness result.

Theorem 6.1. *If*

(A) $\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}+4(N-1)\inf(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}>0$ *for* $i=2,...,N$,

(B) $(g_1)_{u_1}(0,...,0)>\frac{4\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}(\lambda_1+\frac{(N-1)\inf(g_1)_{u_1u_i}}{\sup(g_i)_{u_iu_i}}(g_i)_{u_i}(0,...,0))}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}+4(N-1)\inf(g_1)_{u_1u_i}\sup(g_i)_{u_1u_i}}$,

(C) $(g_1)_{u_1}(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N)$
$+\inf(g_1)_{u_1u_i}[\frac{4(g_1)_{u_1}(0,...,0)\sup(g_i)_{u_1u_i}-2(g_i)_{u_i}(0,...,0)\sup(g_1)_{u_1u_1}}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}}]>\lambda_1$,

(D) $(g_i)_{u_i}(0,M_2,...,M_{i-1},0,M_{i+1},...,M_N)>\lambda_1, i=2,...,N$,

(E) $(g_i)_{u_i}(0,...,0)>\lambda_1-(N-2)\inf_{j=2,...,N,j\neq i}\{\bar{u_j}\inf(g_i)_{u_iu_j}\}$, *and*

(F) *for* $i=2,...,N$,

$$\begin{aligned}&\sup(g_1)_{u_1u_1}+\textstyle\sum_{i=2}^N\sup(g_i)_{u_iu_1}\\ <\;&\textstyle\sum_{j=2}^N[\inf(g_1)_{u_1u_j}+\inf(g_j)_{u_1u_j}\frac{\theta_{(g_j)_{u_j}(0,M_2,...,M_{j-1},\cdot,M_{j+1},...,M_N)}}{2\theta_{(g_1)_{u_1}(\cdot,0,...,0)}}],\\ &\sup(g_i)_{u_iu_i}+\sup(g_i)_{u_iu_1}\\ <\;&\textstyle\sum_{j=2,j\neq i}^N[\inf(g_i)_{u_iu_j}+\inf(g_j)_{u_iu_j}\frac{2\theta_{(g_j)_{u_j}(0,...,0,\cdot,0,...,0)-\frac{2(g_1)_{u_1}(0,...,0)\sup(g_j)_{u_1u_j}}{\sup(g_1)_{u_1u_1}}}}{\theta_{(g_i)_{u_i}(0,M_2,...,M_{i-1},\cdot,M_{i+1},...,M_N)}},\end{aligned}$$

then (12) has a unique posotive solution.

The conditions imply that species 1 interacts strongly among themselves and weakly with species 2. Similarly for species 2, they interact more strongly among themselves than they do with species 1.

Proof. The existence was already proved in the last section. We prove the uniqueness.
Let $(u_1,...,u_N),(w_1,...,w_N)$ be positive solutions to (4), and let $p_i=u_i-w_i, i=1,...,N$. We want to show that $p_i\equiv 0, i=1,...,N$.
Since $(u_1,...,u_N),(w_1,...,w_N)$ are solutions to (12), for $i=1,...,N$,

$$\Delta(p_i)+g_i(u_1,...,u_N)-g_i(w_1,...,w_N)=0.$$

Hence, by the Mean Value Property in the sense of average and the Mean Value Theorem, there are $\bar{u}_i,\bar{u}_{ij}, j=1,...,N, j\neq i,\tilde{u}_i$ such that $\bar{u}_i$ is between u_i and w_i, $\bar{u}_{ij}$ is between u_j and w_j, $j=1,...,N, j\neq i$, $0\leq\tilde{u}_i\leq u_i$, $\bar{u}_i-\tilde{u}_i\geq\frac{u_i+w_i}{2}-\frac{u_i}{2}=\frac{w_i}{2}$ and

$$\begin{aligned}&\Delta(p_i)+p_i(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\\ &+p_i[(g_i)_{u_i}(u_1,...,u_{i-1},\bar{u}_i,u_{i+1},...,u_N)-(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\\ &+\textstyle\sum_{j=1,j\neq i}^N p_j(g_i)_{u_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_j,...,u_{i-1},w_i,u_{i+1},...,u_N)=0.\end{aligned}$$

Since

$$\Delta(u_i)+(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)u_i=\Delta(u_i)+g_i(u_1,...,u_N)=0,$$

by the Lemma 3.2, we have

$$\textstyle\int_\Omega -p_i\Delta(p_i)-(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)p_i^2dx\geq 0,$$

and so

$$\begin{aligned}&\int_\Omega -[(g_i)_{u_i}(u_1,...,u_{i-1},\bar{u}_i,u_{i+1},...,u_N)-(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)]p_i^2\\&-\sum_{j=1,j\neq i}^N p_ip_j(g_i)_{u_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},w_i,u_{i+1},...,u_N)dx\leq 0.\end{aligned}$$

Hence, taking the sum $i=1,...,N$, we have

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^N p_i^2[(g_i)_{u_i}(u_1,...,u_{i-1},\bar{u}_i,u_{i+1},...,u_N)\\&-(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)]\\&-\sum_{i,j=1,j\neq i}^N p_ip_j[(g_i)_{u_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},w_i,u_{i+1},...,u_N)\\&-(g_i)_{u_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},0,u_{i+1},...,u_N)]dx\\&\leq 0\end{aligned}$$

by (U_3).

By the Mean Value Theorem again, there are $\hat{u}_i,\bar{w}_i, i=1,...,N$, such that $\tilde{u}_i\leq\hat{u}_i\leq\bar{u}_i, 0\leq\bar{w}_i\leq w_i$ and

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^N p_i^2(\bar{u}_i-\tilde{u}_i)(g_i)_{u_iu_i}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\\&-\sum_{i,j=1,j\neq i}^N p_ip_jw_i(g_i)_{u_iu_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},\bar{w}_i,u_{i+1},...,u_N)dx\\&\leq 0.\end{aligned}$$

But, since

$$\bar{u}_i-\tilde{u}_i\geq\frac{w_i}{2}, i=1,...,N,$$

we have

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^N p_i^2\tfrac{w_i}{2}(g_i)_{u_iu_i}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\\&-\sum_{i,j=1,j\neq i}^N p_ip_jw_i(g_i)_{u_iu_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},\bar{w}_i,u_{i+1},...,u_N)dx\\&\leq 0.\end{aligned}$$

So,

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^N p_i^2\tfrac{w_i}{2}(g_i)_{u_iu_i}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\\&-\sum_{i=2}^N p_ip_1w_i(g_i)_{u_iu_1}(\bar{u_{i1}},u_2,...,u_{i-1},\bar{w}_i,u_{i+1},...,u_N)\\&-\sum_{i=1}^N\sum_{j=2,j\neq i}^N p_ip_jw_i(g_i)_{u_iu_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},\bar{w}_i,u_{i+1},...,u_N)dx\\&\leq 0.\end{aligned}$$

But, since $p_ip_1\leq\frac{p_i^2}{2}+\frac{p_1^2}{2}$ for $i=1,...,N$ and $(g_i)_{u_iu_1}>0$ for $i=2,...,N$, we have

$$\begin{aligned}&\int_\Omega -\sum_{i=1}^N p_i^2\tfrac{w_i}{2}(g_i)_{u_iu_i}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\\&-\sum_{i=2}^N(\tfrac{p_i^2}{2}+\tfrac{p_1^2}{2})w_i(g_i)_{u_iu_1}(\bar{u_{i1}},u_2,...,u_{i-1},\bar{w}_i,u_{i+1},...,u_N)\\&-\sum_{i=1}^N\sum_{j=2,j\neq i}^N p_ip_jw_i(g_i)_{u_iu_j}(w_1,...,w_{j-1},\bar{u}_{ij},u_{j+1},...,u_{i-1},\bar{w}_i,u_{i+1},...,u_N)dx\\&\leq 0.\end{aligned}$$

If the integrand on the left side is positive definite, then $p_i \equiv 0, i = 1, ..., N$, which means the uniqueness.

But,

$$
\begin{aligned}
& -(g_i)_{u_iu_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, \bar{w}_i, u_{i+1}, ..., u_N)p_ip_jw_i \\
\leq \quad & -(g_i)_{u_iu_j}(w_1, ..., w_{j-1}, \bar{u}_{ij}, u_{j+1}, ..., u_{i-1}, \bar{w}_i, u_{i+1}, ..., u_N)(\frac{p_i^2}{2} + \frac{p_j^2}{2})w_i
\end{aligned}
$$

for $i = 1, ..., N, j = 2, ..., N, j \neq i$, and so, the integrand is positive definite if the condition is satisfied by the solution estimates in the Lemma 4.3.

7. Uniqueness with perturbation

We consider the model

$$
\begin{cases}
\Delta u_i + g_i(u_1, ..., u_i, ..., u_N) = 0 \quad \text{in } \Omega, \\
u_i|_{\partial\Omega} = 0, i = 1, ..., N,
\end{cases} \tag{13}
$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $g_i \in C^2$ are relative growth rates such that

$$
(g_i)_{u_iu_j} \begin{cases}
< 0, i = 1, j = 1, ..., N \\
> 0, i = 2, ..., N, j = 1, \\
< 0, i = 2, ..., N, j = 2, ..., N,
\end{cases}
$$

Define $B = \{(g_1, ..., g_N) \in [C^2]^N | g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) = 0,$

$g_i(u_1, ..., u_i, ..., u_N) \leq u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \frac{u_i}{2}, u_{i+1}, ..., u_N)$

for all $u_1, ..., u_N, \lim_{x_i \to \infty}(g_i)_{x_i}(x_1, ..., x_i, ..., x_N) = -\infty, i = 1, ..., N\}$ with

$$
\| (g_1, ..., g_N) \|_B = \sum_{i=1}^{N} |(g_i)_{u_i}(0, ..., 0)| + \sum_{i,j,k=1}^{N} \sup |(g_i)_{u_ju_k}|
$$

for all $(g_1, ..., g_N) \in B$.

Then by the functional analysis theory, $(B, \| \cdot \|_B)$ is a Banach space.

If $(g_1, ..., g_N) \in B$, then by the definition of B, for each $i = 1, ..., N$, there is $c_i > 0$ such that $g_i(0, ..., 0, x_i, 0, ..., 0) < 0$ for all $x_i > c_i$. Then by the Maximum Principles, we have $\theta_{g_i(0,...,0,\cdot,0,...,0)}(x) \leq c_i$ for all $x \in \Omega$.

The following theorem is our main result about the perturbation of uniqueness.

Theorem 7.1. *Suppose* $(g_1, ..., g_N) \in B$ *is such that*

(A) $\lambda_1[-(g_1)_{u_1}(0, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}, ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)})] < 0,$

$\lambda_1[-(g_i)_{u_i}(\frac{1}{2}\theta_{M(g_1,g_i)}, 2\theta_{K(g_1,g_2)}, ..., 2\theta_{K(g_1,g_{i-1})}, 0, 2\theta_{K(g_1,g_{i+1})}, ..., 2\theta_{K(g_1,g_N)})] < 0,$

(B) $\sup(g_1)_{u_1u_1} \sup(g_i)_{u_iu_i} + 4(N-1)\inf(g_1)_{u_1u_i} \sup(g_i)_{u_1u_i} > 0$, *for* $i = 2, ..., N$

(C) $(g_1)_{u_1}(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N)$

$+ \inf(g_1)_{u_1u_i}[\frac{4(g_1)_{u_1}(0,...,0)\sup(g_i)_{u_1u_i} - 2(g_i)_{u_i}(0,...,0)\sup(g_1)_{u_1u_1}}{\sup(g_1)_{u_1u_1}\sup(g_i)_{u_iu_i}}] > \lambda_1$, *and*

$(g_i)_{u_i}(0, M_2, ..., M_{i-1}, 0, M_{i+1}, ..., M_N) > \lambda_1, i = 2, ..., N,$
(D) for every $T \in R$, there is $c > 0$ such that $(g_1)_{u_1}(T, y) < 0$ for all $y > c$, (E) (13) has a unique coexistence state $(u_1, ..., u_N)$,
(F) the Frechet derivative of (13) at $(u_1, ..., u_N)$ is invertible.
Then there is a neighborhood V of $(g_1, ..., g_N)$ in B such that if $(\bar{g_1}, ..., \bar{g_N}) \in V$, then (13) with $(\bar{g_1}, ..., \bar{g_N})$ has a unique positive solution.

Biologically, the conditions in Theorem 7.1 indicates that the rates of reproduction are relatively large, and the birth rate of the prey must be larger than that of predator. Similarly, the condition, which requires the invertibility of the Frechet derivative, signifies that the rates of self-limitation are relatively larger than the rates of competition, a relationship that will be established in Lemma 7.2. When these conditions are fulfilled, the conclusion of our theorem asserts that small perturbations of the rates do not affect the existence and uniqueness of the positive steady state. That is, the two species implied can continue to coexist even if the factors determining the population densities vary slightly.

Now, at first glance, Theorem 7.1 may appear to be a consequence of the Implicit Function Theorem. However, the Implicit Function Theorem only guarantees local uniqueness. In contrast, our result in Theorem 7.1 guarantees global uniqueness. The techniques we will use in the proof of Theorem 7.1 include the Implicit Function Theorem and a priori estimates on solutions of (13).

Proof. Since the Frechet derivative of (13) at $(u_1, ..., u_N)$ is invertible, by the Implicit Function Theorem, there is a neighborhood V of $(g_1, ..., g_N)$ in B and a neighborhood W of $(u_1, ..., u_N)$ in $[C_0^{2,\alpha}(\bar{\Omega})]^N$ such that for all $(\bar{g_1}, ..., \bar{g_N}) \in V$, there is a unique positive solution $(\bar{u_1}, ..., \bar{u_N}) \in W$ of (13) with $(\bar{g_1}, ..., \bar{g_N})$. Thus, the local uniqueness of the solution is guaranteed.
To prove global uniqueness, suppose that the conclusion of Theorem 7.1 is false. Then, there are sequences $(g_{1n}, g_{2n}, ..., g_{Nn}, u_{1n}, u_{2n}, ..., u_{Nn})$ and
$(g_{1n}, g_{2n}, ..., g_{Nn}, u^*_{1n}, u^*_{2n}, ..., u^*_{Nn})$ in $V \times [C_0^{2,\alpha}(\bar{\Omega})]^N$ such that $(u_{1n}, ..., u_{Nn})$ and $(u^*_{1n}, ..., u^*_{Nn})$ are positive solutions of (13) with $(g_{1n}, ..., g_{Nn})$,
$(u_{1n}, ..., u_{Nn}) \neq (u^*_{1n}, ..., u^*_{Nn})$ and $(g_{1n}, ..., g_{Nn}) \to (g_1, ..., g_N)$. By Schauder's estimate in elliptic theory and the solution estimate in the Lemma 4.3, there are uniformly convergent subsequences of $\{u_{1n}\}, ..., \{u_{Nn}\}$, which again will be denoted by $\{u_{1n}\}, ..., \{u_{Nn}\}$.
Thus, let

$$(u_{1n}, ..., u_{Nn}) \to (\hat{u_1}, ..., \hat{u_N}),$$
$$(u^*_{1n}, ..., u^*_{Nn}) \to (u^*_1, ..., u^*_N).$$

Then $(\hat{u_1}, ..., \hat{u_N}), (u^*_1, ..., u^*_N) \in (C^{2,\alpha})^N$ are also solutions to (13) with $(g_1, ..., g_N)$. We claim that $\hat{u_1} > 0, ..., \hat{u_N} > 0, u^*_1 > 0, ..., u^*_N > 0$.
Let $T_i = \sup_{x \in \bar{\Omega}} \hat{u}_i, i = 1, ..., N$.

Then by the conditions for g_i's, there is $M > 0$ such that $M > \sup_{2\le i\le N}\sup_{x\in\bar\Omega}\hat{u_i}$ and

$$(g_1)_{u_1}(T_1, M, M, ..., M) \le 0,$$
$$(g_i)_{u_i}(0, T_2, ..., T_{i-1}, M, T_{i+1}, ..., T_N) \le 0, i = 2, ..., N.$$

Furthermore, by the monotonicity,

$$\begin{aligned} & \Delta\hat{u_1} + \hat{u_1}(g_1)_{u_1}(T_1, M, ..., M) \\ \le\ & \Delta\hat{u_1} + \hat{u_1}(g_1)_{u_1}(T_1, \hat{u_2}, ..., \hat{u_N}) \\ \le\ & \Delta\hat{u_1} + \hat{u_1}(g_1)_{u_1}(\hat{u_1}, ..., \hat{u_N}) \\ \le\ & \Delta\hat{u_1} + g_1(\hat{u_1}, ..., \hat{u_N}) \\ =\ & 0, \end{aligned}$$

and

$$\begin{aligned} & \Delta\hat{u_i} + \hat{u_i}(g_i)_{u_i}(0, T_2, ..., T_{i-1}, M, T_{i+1}, ..., T_N) \\ \le\ & \Delta\hat{u_i} + \hat{u_i}(g_i)_{u_i}(0, T_2, ..., T_{i-1}, \hat{u_i}, T_{i+1}, ..., T_N) \\ \le\ & \Delta\hat{u_i} + \hat{u_i}(g_i)_{u_i}(\hat{u_1}, ..., \hat{u_N}) \\ \le\ & \Delta\hat{u_i} + g_i(\hat{u_1}, ..., \hat{u_N}) \\ =\ & 0, i = 2, ..., N. \end{aligned}$$

Hence, by the Maximum Principles, it suffices to claim that $\hat{u_1}, ..., \hat{u_N}, u_1^*, ..., u_N^*$ are not identically zero.
Suppose that it is not true. First, assume $\hat{u_1} \equiv 0$.
Then let $\tilde{u}_{1n} = \frac{u_{1n}}{\|u_{1n}\|_\infty}$ and $\tilde{u}_{in} = u_{in}, i = 2, ..., N$. By the elliptic theory again, there is $\tilde{u_1}$ such that $\tilde{u}_{1n} \to \tilde{u_1}$.

$$\begin{aligned} & \Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(u_{1n}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \\ =\ & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + u_{1n}(g_{1n})_{u_1}(u_{1n}, ..., u_{Nn})] \\ \le\ & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + g_{1n}(u_{1n}, ..., u_{Nn})] \\ =\ & 0, \end{aligned}$$

and

$$\begin{aligned} & \Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(\tfrac{u_{1n}}{2}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \\ =\ & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + u_{1n}(g_{1n})_{u_1}(\tfrac{u_{1n}}{2}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn})] \\ \ge\ & \frac{1}{\|u_{1n}\|_\infty}[\Delta u_{1n} + g_{1n}(u_{1n}, ..., u_{Nn})] \\ =\ & 0. \end{aligned}$$

But, since

$$\Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(u_{1n}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \to \Delta\tilde{u_1} + \tilde{u_1}(g_1)_{u_1}(0, \hat{u_2}, ..., \hat{u_N}),$$
$$\Delta\tilde{u}_{1n} + \tilde{u}_{1n}(g_{1n})_{u_1}(\tfrac{u_{1n}}{2}, \tilde{u}_{2n}, ..., \tilde{u}_{Nn}) \to \Delta\tilde{u_1} + \tilde{u_1}(g_1)_{u_1}(0, \hat{u_2}, ..., \hat{u_N}),$$

we conclude that

$$\Delta\tilde{u_1} + \tilde{u_1}(g_1)_{u_1}(0, \hat{u_2}, ..., \hat{u_N}) = 0.$$

Since

$$\begin{aligned}
& \Delta\tilde{u}_{in}+\tilde{u}_{in}(g_{in})_{u_i}(u_{1n},0,...,0,\tfrac{\tilde{u}_{in}}{2},0,...,0) \\
= \; & \Delta(u_{in})+u_{in}(g_{in})_{u_i}(u_{1n},0,...,0,\tfrac{u_{in}}{2},0,...,0) \\
\geq \; & \Delta(u_{in})+u_{in}(g_{in})_{u_i}(u_{1n},...,u_{(i-1)n},\tfrac{u_{in}}{2},u_{(i+1)n},...,u_{Nn}) \\
\geq \; & \Delta(u_{in})+g_{in}(u_{1n},...,u_{Nn}) \\
= \; & 0, i=2,...,N,
\end{aligned}$$

and

$$\Delta\tilde{u}_{in}+\tilde{u}_{in}(g_{in})_{u_i}(u_{1n},0,...,0,\frac{\tilde{u}_{in}}{2},0,...,0)\to\Delta\hat{u}_i+\hat{u}_i(g_i)_{u_i}(0,...,0,\frac{\hat{u}_i}{2},0,...,0),$$

$$\Delta\hat{u}_i+\hat{u}_i(g_i)_{u_i}(0,...,0,\frac{\hat{u}_i}{2},0,...,0)\geq 0, i=2,...,N,$$

and so, $\frac{\hat{u}_i}{2}$ is a subsolution to

$$\begin{aligned}
\Delta(Z)+Z(g_i)_{u_i}(0,...,0,Z,0,...,0) &= 0, \\
Z|_{\partial\Omega} &= 0, i=2,...,N.
\end{aligned}$$

But, since any large enough constant is a supersolution to

$$\begin{aligned}
\Delta(Z)+Z(g_i)_{u_i}(0,...,0,Z,0,...,0) &= 0, \\
Z|_{\partial\Omega} &= 0, i=2,...,N,
\end{aligned}$$

by the Lemma 3.1, we have

$$\frac{\hat{u}_i}{2}\leq\theta_{(g_i)_{u_I}(0,...,0,\cdot,0,...,0)},$$

and so, by the monotonicity,

$$\begin{aligned}
& \lambda_1[-(g_1)_{u_1}(0,2\theta_{(g_2)_{u_2}(0,...,0,\cdot,0,...,0,...)},...,2\theta_{(g_N)_{u_N}(0,...,0,\cdot)}] \\
\geq \; & \lambda_1[-(g_1)_{u_1}(0,\hat{u_2},...,\hat{u_N})] \\
= \; & 0,
\end{aligned}$$

which is a contradiction.

Suppose $\hat{u}_i\equiv 0$ for some $i=2,...,N$.

Let $\tilde{u}_{1n}=u_{1n},\tilde{u}_{2n}=u_{2n},...,\tilde{u}_{(i-1)n}=u_{(i-1)n},\tilde{u}_{in}=\frac{u_{in}}{\|u_{in}\|_\infty}$,

$\tilde{u}_{(i+1)n}=u_{(i+1)n},...,\tilde{u}_{Nn}=u_{Nn}$.

Then by the elliptic theory again, there is $\tilde{u}_i$ such that $\tilde{u}_{in}\to\tilde{u}_i$.

$$\begin{aligned}
& \Delta\tilde{u}_{in}+\tilde{u}_{in}(g_{in})_{u_i}(\tilde{u}_{1n},...,\tilde{u}_{(i-1)n},u_{in},\tilde{u}_{(i+1)n},...,\tilde{u}_{Nn}) \\
= \; & \tfrac{1}{\|u_{in}\|_\infty}[\Delta u_{in}+u_{in}(g_{in})_{u_i}(u_{1n},...,u_{Nn})] \\
\leq \; & \tfrac{1}{\|u_{in}\|_\infty}[\Delta u_{in}+g_{in}(u_{1n},...,u_{Nn})] \\
= \; & 0,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta\tilde{u}_{in}+\tilde{u}_{in}(g_{in})_{u_i}(\tilde{u}_{1n},...,\tilde{u}_{(i-1)n},\tfrac{u_{in}}{2},\tilde{u}_{(i+1)n},...,\tilde{u}_{Nn}) \\
= \; & \tfrac{1}{\|u_{in}\|_\infty}[\Delta u_{in}+u_{in}(g_{in})_{u_i}(u_{1n},...,u_{(i-1)n},\tfrac{u_{in}}{2},u_{(i+1)n},...,u_{Nn})] \\
\geq \; & \tfrac{1}{\|u_{in}\|_\infty}[\Delta u_{in}+g_{in}(u_{1n},...,u_{Nn})] \\
= \; & 0,
\end{aligned}$$

But, since

$$\begin{aligned}&\Delta\tilde{u}_{in}+\tilde{u}_{in}(g_{in})_{u_i}(\tilde{u}_{1n},...,\tilde{u}_{(i-1)n},u_{in},u_{(i+1)n},...,u_{Nn})\\&\to\Delta\tilde{u_i}+\tilde{u_i}(g_i)_{u_i}(\hat{u_1},...,\hat{u_{i-1}},0,\hat{u_{i+1}},...,\hat{u_N}),\\&\Delta\tilde{u}_{in}+\tilde{u}_{in}(g_{in})_{u_i}(\tilde{u}_{1n},...,\tilde{u}_{(i-1)n},\tfrac{u_{in}}{2},u_{(i+1)n},...,u_{Nn})\\&\to\Delta\tilde{u_i}+\tilde{u_i}(g_i)_{u_i}(\hat{u_1},...,\hat{u_{i-1}},0,\hat{u_{i+1}},...,\hat{u_N}),\end{aligned}$$

we conclude that $\Delta\tilde{u_i}+\tilde{u_i}(g_i)_{u_i}(\hat{u_1},...,\hat{u_{i-1}},0,\hat{u_{i+1}},...,\hat{u_N})=0$.
Hence,

$$\begin{aligned}&\lambda_1[-(g_i)_{u_i}(\tfrac{1}{2}\theta_{M(g_1,g_i)},2\theta_{K(g_1,g_2)},...,2\theta_{K(g_1,g_{i-1})},0,2\theta_{K(g_1,g_{i+1})},...,2\theta_{K(g_1,g_N)})]\\\geq\;&\lambda_1[-(g_i)_{u_i}(\hat{u_1},...,\hat{u_{i-1}},0,\hat{u_{i+1}},...,\hat{u_N})]\\=\;&0,\end{aligned}$$

which is also a contradictionn.
Therefore, we conclude that $\hat{u_1}>0,...,\hat{u_N}>0$.
By the same procedure with the sequence $(u^*_{1n},...,u^*_{Nn})$, we also have
$u^*_1>0,...,uN^*>0$.
Consequently, $(\hat{u_1},...,\hat{u_N})$ and $(u^*_1,...,u^*_N)$ are positive solutions to (13) with $(g_1,...,g_N)$, and so $(\hat{u_1},...,\hat{u_N})=(u^*_1,...,u^*_N)=(u_1,...,u_N)$ by the uniqueness condition. But, this is a contradiction to the Implicit Function Theorem, since $(u_{1n},...,u_{Nn})\neq(u^*_{1n},...,u^*_{Nn})$.

In biological terms, the proof of our theorem indicates that if one of two species living in the same domain becomes extinct, that is, if one species is excluded by the other, then the reproduction rates of both must be small. In other words, the region condition of reproduction rates (A) is reasonable.

Now, the condition (C) in Theorem 7.1 requiring the invertibility of the Frechet derivative is too artificial to have any direct biological implications. We therefore turn our attention to more applicable conditions that will guarantee the invertibility of the Frechet derivative. We then obtain the following relationship:

$$\begin{cases}\Delta u_i+g_i(u_1,...,u_i,...,u_N)\;=\;0\quad\text{in }\Omega,\\u_i|_{\partial\Omega}=0,i=1,...,N,\end{cases}\tag{14}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $g_i\in C^2$ are relative growth rates such that

$$(g_i)_{u_iu_j}\begin{cases}<0,i=1,j=1,...,N\\>0,i=2,...,N,j=1,\\<0,i=2,...,N,j=2,...,N,\end{cases}$$

$g_i(u_1,...,u_{i-1},0,u_{i+1},...,u_N)=(g_i)_{u_j}(u_1,...,u_{i-1},0,u_{i+1},...,u_N)=0$ for all $i,j=1,...,N,i\neq j$ and $u_1,...,u_{i-1},u_{i+1},...,u_N\in R$, and
$g_i(u_1,...,u_{i-1},\cdot,u_{i+1},...,u_N)$ has the Mean value property in the sense of average for all $i=1,...,N$ and $u_1,...,u_{i-1},u_{i+1},...,u_N\in R$.

Lemma 7.2. *Suppose* $(u_1, ..., u_N)$ *is a positive solution to (14). If*

$$\sup(g_1)_{u_1u_1} + \textstyle\sum_{i=2}^N \sup(g_i)_{u_iu_1}\frac{u_i}{u_1} < \sum_{j=2}^N[\inf(g_1)_{u_1u_j} + \inf(g_j)_{u_1u_j}\frac{u_j}{u_1}],$$
$$\sup(g_i)_{u_iu_i} + \sup(g_i)_{u_iu_1} < \textstyle\sum_{j=2,j\neq i}^N[\inf(g_i)_{u_iu_j} + \inf(g_j)_{u_iu_j}\frac{u_j}{u_i}, i = 2, ..., N,$$

then the Frechet derivative of (14) at $(u_1, ..., u_N)$ *is invertible.*

Proof. The Frechet derivative of (14) at $(u_1, ..., u_N)$ is $A = (A_{ij})$, where $A = (A_{ij})$, where

$$A_{ij} = \begin{cases} \Delta + (g_i)_{u_i}(u_1, ..., u_N), & i = j \\ (g_i)_{u_j}(u_1, ..., u_N), & i \neq j \end{cases}$$

for $i, j = 1, ..., N$. We need to show that $N(A) = \{0\}$ by the Fredholm Alternative, where $N(A)$ is the null space of A. By the Mean Value Property in the sense of average, there are $\bar{u}_i$ such that $u_i - \bar{u}_i \geq \frac{u_i}{2}$ and

$$\begin{aligned} g_i(u_1, ..., u_N) &= g_i(u_1, ..., u_{i-1}, u_i, u_{i+1}, ..., u_N) - g_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N) \\ &= u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N), i = 1, ..., N. \end{aligned}$$

If

$$\Delta\phi_i + (g_i)_{u_i}(u_1, ..., u_N)\phi_i + \textstyle\sum_{j=1,j\neq i}^N (g_i)_{u_j}(u_1, ..., u_N)\phi_j = 0, i = 1, ..., N,$$

then

$$\begin{aligned} &\Delta\phi_i + (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\phi_i \\ &+[(g_i)_{u_i}(u_1, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)]\phi_i \\ &+ \textstyle\sum_{j=1,j\neq i}^N (g_i)_{u_j}(u_1, ..., u_N)\phi_j \\ &= 0, i = 1, ..., N. \end{aligned}$$

Hence,

$$\begin{aligned} &\textstyle\int_\Omega |\nabla\phi_i|^2 - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\phi_i^2 \\ &-[(g_i)_{u_i}(u_1, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)]\phi_i^2 \\ &- \textstyle\sum_{j=1,j\neq i}^N (g_i)_{u_j}(u_1, ..., u_N)\phi_i\phi_j dx \\ &= 0, i = 1, ..., N. \end{aligned}$$

But, since

$$\Delta(u_i) + u_i(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., , u_N) = \Delta(u_i) + g_i(u_1, ..., u_N) = 0,$$
$$i = 1, ..., N,$$
$$\lambda_1[-(g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)] = 0, i = 1, ..., N.$$

Hence, by the Lemma 3.2, we have

$$\textstyle\int_\Omega |\nabla\phi_i|^2 - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)\phi_i^2 dx \geq 0, i = 1, ..., N.$$

Therefore,

$$\textstyle\int_\Omega - \sum_{i=1}^N[(g_i)_{u_i}(u_1, ..., u_N) - (g_i)_{u_i}(u_1, ..., u_{i-1}, \bar{u}_i, u_{i+1}, ..., u_N)]\phi_i^2$$
$$- \textstyle\sum_{i=1}^N \sum_{j=1,j\neq i}^N[(g_i)_{u_j}(u_1, ..., u_N) - (g_i)_{u_j}(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_N)]\phi_i\phi_j dx$$

$\leq 0.$

But, by the Mean Value Theorem, there are $\tilde{u}_i, \hat{u}_{ij}$ such that

$$\bar{u}_i \leq \tilde{u}_i \leq u_i, 0 \leq \hat{u}_{ij} \leq u_i$$

and

$$\begin{aligned}
& (g_i)_{u_i}(u_1,...,u_{i-1},u_i,u_{i+1},...,u_N) - (g_i)_{u_i}(u_1,...,u_{i-1},\bar{u}_i,u_{i+1},...,u_N) \\
= \quad & (u_i - \bar{u}_i)(g_i)_{u_iu_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N), \\
& (g_i)_{u_j}(u_1,...,u_{i-1},u_i,u_{i+1},...,u_N) - (g_i)_{u_j}(u_1,...,u_{i-1},0,u_{i+1},...,u_N) \\
= \quad & u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_{ij},u_{i+1},...,u_N), i,j = 1,...,N, j \neq i.
\end{aligned}$$

Therefore,

$\int_\Omega -\sum_{i=1}^N (u_i - \bar{u}_i)(g_i)_{u_iu_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\phi_i^2$

$-\sum_{i=1}^N \sum_{j=1,j\neq i}^N u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_{ij},u_{i+1},...,u_N)\phi_i\phi_j dx \leq 0.$

But, since $u_i - \bar{u}_i \geq \frac{u_i}{2}, i = 1,...,N$, we have

$\int_\Omega -\sum_{i=1}^N \frac{u_i}{2}(g_i)_{u_iu_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\phi_i^2$

$-\sum_{i=1}^N \sum_{j=1,j\neq i}^N u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_i\phi_j dx \leq 0.$

Hence,

$$\begin{aligned}
& \int_\Omega -\sum_{i=1}^N \frac{u_i}{2}(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\phi_i^2 \\
& -\sum_{j=2}^N u_1(g_1)_{u_1u_j}(\hat{u}_1,u_2,...,u_N)\phi_1\phi_j \\
& -\sum_{i=2}^N u_i(g_i)_{u_iu_1}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_i\phi_1 \\
& -\sum_{i=2}^N \sum_{j=2,j\neq i}^N u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_i\phi_j dx \\
& \leq 0.
\end{aligned}$$

Since $\phi_i\phi_1 \leq \frac{\phi_i^2}{2} + \frac{\phi_1^2}{2}$ and $(g_i)_{u_iu_1} > 0$ for $i = 2,...,N$,

$$\begin{aligned}
& \int_\Omega -\sum_{i=1}^N \frac{u_i}{2}(g_i)_{u_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N)\phi_i^2 \\
& -\sum_{i=2}^N u_i(g_i)_{u_iu_1}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\frac{\phi_i^2}{2} + \frac{\phi_1^2}{2} \\
& -\sum_{j=2}^N u_1(g_1)_{u_1u_j}(\hat{u}_1,u_2,...,u_N)\phi_1\phi_j \\
& -\sum_{i=2}^N \sum_{j=2,j\neq i}^N u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_i\phi_j dx \\
& \leq 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_\Omega -\frac{u_1}{2}(g_1)_{u_1u_1}(\tilde{u_1},u_2,...,u_N)\phi_1^2 \\
& -\frac{1}{2}[\sum_{i=2}^N u_i(g_i)_{u_iu_1}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_1^2 \\
& -\sum_{i=2}^N [\frac{u_i}{2}(g_i)_{u_iu_i}(u_1,...,u_{i-1},\tilde{u}_i,u_{i+1},...,u_N) \\
& +\frac{u_i}{2}(g_i)_{u_iu_1}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)]\phi_i^2 \\
& -\sum_{j=2}^N u_1(g_1)_{u_1u_j}(\hat{u}_1,u_2,...,u_N)\phi_1\phi_j \\
& -\sum_{i=2}^N \sum_{j=2,j\neq i}^N u_i(g_i)_{u_iu_j}(u_1,...,u_{i-1},\hat{u}_i,u_{i+1},...,u_N)\phi_i\phi_j dx \\
& \leq 0.
\end{aligned}$$

Hence, $\phi_i \equiv 0, i = 1,...,N$ if the integrand is positive definite, which is true by the condition.

8. Uniqueness with perturbation of region

The following Theorem is the main result.

Theorem 8.1. *Suppose that $\Gamma \subseteq B$ is a closed, bounded, convex region in B such that*
(A) for all $(g_1, ..., g_N,$ $\lambda_1[-(g_1)_{u_1}(0, 2\theta_{(g_2)_{u_2}(0,\cdot,0,...,0)}, ..., 2\theta_{(g_N)_{u_N}(0,...,0,\cdot)})] < 0,$
$\lambda_1[-(g_i)_{u_i}(\frac{1}{2}\theta_{M(g_1,g_i)}, 2\theta_{K(g_1,g_2)}, ..., 2\theta_{K(g_1,g_{i-1})}, 0, 2\theta_{K(g_1,g_{i+1})}, ..., 2\theta_{K(g_1,g_N)})] < 0,$
(B) for all $(g_1, ..., g_N) \in \partial_L\Gamma$, (13) has a unique positive solution, where $\partial_L\Gamma =$ $\{(\lambda_{g_2,...,g_N}, g_2, ..., g_N) \in \Gamma |$ for any fixed $g_2, ..., g_N,$
$\| \lambda_{g_2,...,g_N} \| = \inf\{\| g_1 \| \,|(g_1, g_2, ..., g_N) \in \Gamma\}\},$
(C) for all $(g_1, ..., g_N) \in \Gamma$, the Frechet derivative of (13) at every positive solution $(u_1, ..., u_N)$ is invertible.
Then for all $(g_1, ..., g_N) \in \Gamma$, (13) has a unique positive solution. Furthermore, there is an open set W in B such that $\Gamma \subseteq W$ and for every $(g_1, ..., g_N) \in W$, (13) has a unique positive solution.

Theorem 8.1 goes even further than Theorem 7.1 which states uniqueness in the whole region of (g, h) whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

Proof. For each fixed $g_2, ..., g_N$, consider $(g_1, ..., g_N) \in \partial_L\Gamma$ and $(\bar{g}_1, g_2, ..., g_N) \in \Gamma$. We need to show that for all $0 \le t \le 1$, (13) with $(1 - t)(g_1, ..., g_N) + t(\bar{g}_1, g_2, ..., g_N)$ has a unique positive solution. Since (13) with $(g_1, ..., g_N)$ has a unique positive solution $(u_1, ..., u_N)$ and the Frechet derivative of (13) at $(u_1, ..., u_N)$ is invertible, Theorem 7.1 implies that there is an open neighborhood V of $(g_1, ..., g_N)$ in B such that if $(g_{10}, ..., g_{N0}) \in V$, then (13) with $(g_{10}, ..., g_{N0})$ has a unique positive solution.
Let $\lambda_s = \sup\{0 \le \lambda \le 1 |$ (13) with $(1 - t)(g_1, ..., g_N) + t(\bar{g}_1, g_2, ..., g_N)$ has a unique coexistence state for $0 \le t \le \lambda.\}$.
We need to show that $\lambda_s = 1$. Suppose $\lambda_s < 1$. From the definition of λ_s, there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_s^-$ and there is a sequence $(u_{1n}, ..., u_{Nn})$ of the unique positive solutions of (13) with $(1-\lambda_n)(g_1, ..., g_N)+\lambda_n(\bar{g}_1, g_2, ..., g_N)$. Then by elliptic theory, there is $(u_{10}, ..., u_{N0})$ such that $(u_{1n}, ..., u_{Nn})$ converges to $(u_{10}, ..., u_{N0})$ uniformly and $(u_{10}, ..., u_{N0})$ is a solution of (13) with $(1 - \lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g}_1, g_2, ..., g_N)$.
But, by the same proof as in the section 7, $u_{10} > 0, ..., u_{N0} > 0$.
We claim that (13) has a unique coexistence state with $(1 - \lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g}_1, g_2, ..., g_N)$. In fact, if not, assume that $(\bar{u}_{10}, ..., \bar{u}_{N0}) \neq (u_{10}, ..., u_{N0})$ is another coexistence state. By the Implicit Function Theorem, there exists $0 < \tilde{a} < \lambda_s$ and very close to λ_s such that (13) with $(1-\tilde{a})(g_1, ..., g_N) + \tilde{a}(\bar{g}_1, g_2, ..., g_N)$ has a coexistence state very close to $(\bar{u}_{10}, ..., \bar{u}_{N0})$, which means that (13) with $(1-\tilde{a})(g_1, ..., g_N)+\tilde{a}(\bar{g}_1, g_2, ..., g_N)$ has more than one coexistence state. This is a contradiction to the definition of λ_s. But, since (13) with $(1 - \lambda_s)(g_1, ..., g_N) + \lambda_s(\bar{g}_1, g_2, ..., g_N)$ has a unique coexistence state and the Frechet derivative is invertible, Theorem 7.1 implies that λ_s can not be as defined. Therefore, for each $(g_1, ..., g_N) \in \Gamma$, (13) with $(g_1, ..., g_N)$ has a

unique coexistence state $(u_1, ..., u_N)$. Furthermore, by the assumption, for each $(g_1, ..., g_N) \in \Gamma$, the Frechet derivative of (13) with $(g_1, ..., g_N)$ at the unique solution $(u_1, ..., u_N)$ is invertible. Hence, Theorem 7.1 concluded that for each $(g_1, ..., g_N) \in \Gamma$, there is an open neighborhood $V_{(g_1,...,g_N)}$ of $(g_1, ..., g_N)$ in B such that if $(\bar{g_1}, ..., \bar{g_N}) \in V_{(g_1,...,g_N)}$, then (13) with $(\bar{g_1}, ..., \bar{g_N})$ has a unique coexistence state. Let $W = \bigcup_{(g_1,...,g_N)\in\Gamma} V_{(g_1,...,g_N)}$. Then W is an open set in B such that $\Gamma \subseteq W$ and for each $(\bar{g_1}, ..., \bar{g_N}) \in W$, (13) with $(\bar{g_1}, ..., \bar{g_N})$ has a unique coexistence state.

Apparently, Theorem 8.1 generalizes Theorem 7.1.

9. Conclusions

In this paper, our investigation of the effects of perturbations on the general predator-prey model resulted in the development and proof of Theorem 7.1, Lemma 7.2 as detailed above. The three together assert that given the existence of a unique solution $(u_1, ..., u_N)$ to the system (13), perturbations of the birth rates $(g_1, ..., g_N)$, within a specified neighborhood, will maintain the existence and uniqueness of the positive steady state. Indeed, our results specifically outline conditions sufficient to maintain the positive, steady state solution when the general predator-prey model is perturbed within some region.

Applying this mathematical result to real world situations, our results establish that the species residing in the same environment can vary their interactions, within certain bounds, and continue to survive together indefinitely at unique densities. The conditions necessary for coexistence, as described in the theorem, simply require that members of each species interact strongly with themselves and weakly with members of the other species.

The research presented in this paper has a number of strengths, which confirm both the validity and the applicability of the project. Perturbations of the general model render its implications more applicable both mathematically and biologically. Because our theorem extends the steady state to any value within some neighborhood of $(g_1, ..., g_N)$, results for the general model pertain to a far wider variety of values. Biologically, perturbations extend the model's description to species affected by factors that vary slightly yet erratically. Thus, the description of competitive interactions given by the model becomes a closer approximation of real-world population dynamics.

While our research therefore represents a progression in the field, the results obtained have an important limitation. Theorem 7.1 and Lemma 7.2 establish that a region of perturbation exists within which the coexistence state is maintained for the general predator-prey model. However, the exact extent of that region remains unknown. Therefore, the results presented in this paper may serve as a platform for research of the question given above. Mathematicians should now attempt to establish the exact extent of the perturbation region in which coexistence is maintained for the general model. Such information would prove very useful not only mathematically but also biologically. Specifically, knowledge of the extent of the region would imply exactly how far the species can relax and yet continue to coexist. Thus, the results achieved through our research will enable the field to continue the development of theory on predator-prey interaction of populations.

In the future, we also want to develop population models to generalize mathematical results about other models related to COVID-19.(See [14], [15], [16], [17].)

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