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Generalized Beta Distribution: Theoretical Properties and Applications in Health Data Modeling

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Abstract. In this publication, a generalization of the classical Beta distribution by five parameters, called the Generalized Beta (GB) distribution, is introduced and well examined, particularly on the usage of the health data. The GB distribution provides the highest flexibility ever in the modeling of complex data patterns in med and public health analyses including the incidence of illnesses, response to treatments, and the values of biomarkers. Systematically basic statistical properties, including correct formulations of moments, moment-generating functions, characteristic functions, hazard functions, and entropy measures, are obtained in this paper. The article gives novel recurrence relations of moments and incomplete moments, which are computationally useful in their application. The paper delves deep into correlation with other important statistical distributions like Kumaraswamy, generalized Gamma, Power Function and Mc-Donalds generalized distributions, by in-exhaustive mathematical analysis. This paper constructs robust parameter estimations practices on both frequentist (maximum likelihood), and in the Bayesian paradigm which have been examined in long simulating studies of parameter regimes, and sample sizes. The practical usefulness of the distribution could be demonstrated by the extensive simulation experiments re-creating the real world conditions of health data, including the disease prevalence modeling, the efficacy of disease treatments analysis, biomarker analysis. GB distribution has excellent flexibility to observe distributional properties of complexity like multimodality, heavy tails and fine skewness patterns which are often nonparametrically challenging in current statistical models applied in health research. Extensive goodness-of-fit statistics, model comparison conditions and diagnostic tests are used to establish the improved performance of the distribution compared to the available options.

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1. Introduction and Motivation

1.1. **Historical Context and Theoretical Background in Health Applications.** Beta distribution is among the most basic and most used continuous distributions of probability in the field of statistics, having its roots in the original work on the foundations of Bayesian statistics and inference. The classical Beta distribution is defined on the finite interval [0,1] and is a useful mathematical model of proportions, probabilities and other constrained random variables that are often used in health studies, such as disease prevalence rates, treatment efficacy proportions and normalized biomarker levels [7]. The mathematical beauty of the distribution, its interpretable parameters, and computational convenience, have made it a mainstay of statistical methodology in medical applications.

The standard Beta distribution with parameters $\alpha > 0$ and $\beta > 0$ possesses the probability density function:

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, \tag{1}$$

where $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ denotes the Beta function. This expression has a high degree of flexibility in describing different shapes of distributions such as symmetric, skewed, U-shaped and J-shaped and is therefore especially useful in Bayesian analysis to provide conjugate prior distributions of binomial and geometric distributions [8] which are essential in clinical trial analysis and epidemiological trials.

1.2. **Limitations of Classical Beta Distribution in Health Research.** Although the Beta distribution is widely used, the growing complexity of modern health datasets has increasingly revealed the limitations of the traditional Beta distribution in capturing complex distributional properties. The data in most of the modern health applications may have characteristics like:

- Multi-modal behavior: Standard Beta distributions are unimodal and hence will not be sufficient to capture data with multiple maxima, like bimodal disease prevalence by age group.
- **Heavy-tailed phenomena:** The current health data is often characterized by the infrequence of the occurrence of a disease or extreme values of a biomarker, and the usual tail behavior control of the Beta distribution does not fit these phenomena.
- **Complex patterns of asymmetry:** There are numerous treatment response patterns that are asymmetries that the conventional skewness parameters of the classic Beta model cannot account for.
- Heterogeneous variance structures: In multi-center clinical trials, the data may have differing patterns of dispersion and thus may require more shape parameters to be modeled.
- Boundary accumulation: The health data would accumulate around boundaries (i.e. near-zero or near-one efficacy values) which are not available in the classical Beta form.

These limitations have led to extensive effort on generalized versions of Beta models which enhance the flexibility without decreasing the mathematical tractability or interpretability of

the classical Beta model. These generalized forms have been used in lower-limited modeling of health measures and the probability distributions of discrete health observations as well have been computed using them as well.

1.3. **Extensions Implemented and Their Limitations in the Medical Field.** The classical Beta distribution has been generalized many times, and these generalizations have helped significantly in expanding the statistical literature. Some of the most important contributions include:

McDonald's Generalized Beta Distribution: It was generalized with the introduction of an additional power parameter, which renders it flexible in modeling income distributions and other economic phenomena [12]. In spite of its usefulness, this extension facilitates scaling and power transformations, unlike the simple shape properties needed in complex health data patterns.

Kumaraswamy Distribution: This distribution is due to the work of [11], and gives the same flexibility as Beta, only with a simpler set of quantile functions, giving it particular popularity in hydrology. However, it lacks the rich theoretical connections to other distributions that characterize the Beta family and is little used in health sciences. **Beta-Generated Distributions:** The authors of the article by [4], and later [20], used the Beta distribution as a generator of new families like the Beta-Normal or Beta-Gamma distributions. These distributions are theoretically beautiful, but they tend to reduce the support of the Beta distribution, which is directly proportional to health data.

Generalized Beta of the First and Second Kind: It is an extension of the Beta distribution to unbounded supports but loses the intuitive interpretation of proportional data, which is significant in health analysis where measurements are naturally bounded.

Even though such extensions do offer useful improvements, they are frequently constrained by a similar aspect: they can be used to improve certain properties of flexibility but often at the cost of mathematical tractability or other useful properties [3]. This natural trade-off motivates our current research, whereby we develop a more generalized form of the distribution that respects mathematical beauty and offers greater modeling power for complex health data structures.

1.4. **Our Contribution to Health Statistics.** This paper has some important extensions of the theory and practice of the generalized Beta distribution in the health research:

(1) A five-parameter Generalized Beta distribution, with additional shape parameters, but fixed at the proportional extremes of the health data.
(2) We also extend to the derivations of basic statistical properties, such as moments, generating functions, hazard functions and entropy measures, and build a strong theoretical framework on the distribution in health practice.

(3) Construction of incomplete moments and creative recurrence relation of moments are developed, which are very useful in the practical implementation of health research in large scale.
(4) In the GB distribution, we see what other significant statistical distributions are like by putting them into perspective in the context of the statistical theory of medicine [14,19].
(5) We apply robust parameter estimation tools of the frequentist or the Bayesian type [6,13], which survive the test of very large simulation processes in scenarios involving health data [18].
(6) This indicates usefulness in empirical application with the application of the model to simulated health data, which is that our model is better than existing competitors in modeling disease prevalence, treatment response, and biomarker distributions [15, 16].

The remaining part of this paper is organized in the following way: Section 2 provides the formal definition and basic properties of the Generalized Beta distribution, such as probability density function, cumulative distribution function, survival functions, hazard functions, and special cases of interest in health applications. Section 3 has detailed simulation studies using health data applications, simulation design methodology, parameter estimation method performance evaluation and comparing it with other distributions. Section 4 illustrates usefulness to simulated health data, such as modeling disease prevalence, modeling treatment efficacy, and modeling biomarker levels with in-depth goodness-of-fit tests. Section 5 ends by discussing the theoretical and practical implications, limitations and future research directions in health statistics.

2. Definition and Basic Properties

2.1. Formal Definition and Parameter Interpretation in Health Context.

Definition 2.1. *A random variable X follows a Generalized Beta distribution with parameters $a > 0$, $b > 0$, $c > 0$, $p > 0$, and $q > 0$, denoted $X \sim GB(a, b, c, p, q)$, if its probability density function (PDF) is given by:*

$$f(x; a, b, c, p, q) = Kx^{p-1}(1 - x^c)^{q-1} \left[-\ln(1 - x^c)\right]^{a-1}, \quad 0 < x < 1, \tag{2}$$

where the normalization constant K is:

$$K = \frac{c}{\Gamma(a)B\left(\frac{p}{c}, q\right)}, \tag{3}$$

with $B(\cdot, \cdot)$ representing the complete Beta function and $\Gamma(\cdot)$ denoting the Gamma function [1].

The parameters in the Generalized Beta distribution possess the following interpretations in health contexts:

- **Parameter a:** Governs the behavior of the distribution near the upper boundary (x $\to$ 1). In health applications, larger values of a increase the density concentration near

complete efficacy (100%) or maximum biomarker levels, while smaller values promote more uniform spread across the range.

- **Parameter b:** Controls the distribution's scale characteristics and interacts with other parameters to fine-tune distributional shape, useful for modeling different population characteristics in multi-center studies.
- **Parameter c:** This parameter mostly has an impact on distributional asymmetry and tail behavior. The positive values of c tend to cause right-skewness (common in the age distribution of disease onset), whereas the negative ones cause left-skewness (characteristic of a recovery time distribution).
- **Parameter p:** This determines the behaviour along the lower limit ($x \to 0$). Higher values bring the concentration of density towards zero, which is a scenario of large proportions of non-responders or undetectable concentrations of biomarkers.
- **Parameter q:** Tail and modality parameters. Smaller values tend to induce heavier tails and possibly multimodality, and this is useful in the modeling of heterogeneous treatment responses or disease subtypes [17].

2.2. **Existence and Normalization.**

Theorem 2.2 (Existence and Normalization)**.** *The function defined in Equation (2) constitutes a valid probability density function over the interval (0,1).*

Proof. To verify that $f(x)$ represents a proper density function, we must establish that $\int_0^1 f(x)dx = 1$. Employing the transformation $u = x^c$, which yields $du = cx^{c-1}dx$, we obtain:

$$\begin{aligned}\int_0^1 f(x)dx &= K\int_0^1 x^{p-1}(1-x^c)^{q-1}\left[-\ln(1-x^c)\right]^{a-1}dx \\ &= \frac{K}{c}\int_0^1 u^{\frac{p}{c}-1}(1-u)^{q-1}\left[-\ln(1-u)\right]^{a-1}du.\end{aligned}$$

Now applying the substitution $t = -\ln(1-u)$, we have $u = 1-e^{-t}$, $du = e^{-t}dt$, and when $u = 0$, $t = 0$; when $u = 1$, $t \to \infty$. Thus:

$$\begin{aligned}\int_0^1 f(x)dx &= \frac{K}{c}\int_0^\infty (1-e^{-t})^{\frac{p}{c}-1}(e^{-t})^{q-1}t^{a-1}e^{-t}dt \\ &= \frac{K}{c}\int_0^\infty t^{a-1}e^{-qt}(1-e^{-t})^{\frac{p}{c}-1}dt.\end{aligned}$$

Recognizing this as an integral representation involving the Beta function and using the identity [1]:

$$\int_0^\infty t^{a-1}e^{-qt}(1-e^{-t})^{z-1}dt = \Gamma(a)B(z,q),$$

with $z = \frac{p}{c}$, we obtain:

$$\int_0^1 f(x)dx = \frac{K}{c}\Gamma(a)B\left(\frac{p}{c},q\right) = 1,$$

thus confirming the normalization. □

2.3. Cumulative Distribution Function.

Theorem 2.3 (Cumulative Distribution Function). *The cumulative distribution function of* $X \sim GB(a, b, c, p, q)$ *is given by:*

$$F(x; a, b, c, p, q) = \frac{B\left(x^c; \frac{p}{c}, q\right)}{B\left(\frac{p}{c}, q\right)}, \tag{4}$$

where $B(z; u, v) = \int_0^z t^{u-1}(1-t)^{v-1}dt$ *denotes the incomplete Beta function [10].*

Proof. The cumulative distribution function is obtained through direct integration:

$$\begin{aligned} F(x) &= \int_0^x f(t)dt = K \int_0^x t^{p-1}(1-t^c)^{q-1} \left[-\ln(1-t^c)\right]^{a-1} dt \\ &= \frac{K}{c} \int_0^{x^c} u^{\frac{p}{c}-1}(1-u)^{q-1} \left[-\ln(1-u)\right]^{a-1} du. \end{aligned}$$

Applying the substitution $w = -\ln(1-u)$, we have $u = 1 - e^{-w}$, $du = e^{-w}dw$, and when $u = 0$, $w = 0$; when $u = x^c$, $w = -\ln(1-x^c)$. Thus:

$$\begin{aligned} F(x) &= \frac{K}{c} \int_0^{-\ln(1-x^c)} (1-e^{-w})^{\frac{p}{c}-1}(e^{-w})^{q-1} w^{a-1} e^{-w} dw \\ &= \frac{K}{c} \int_0^{-\ln(1-x^c)} w^{a-1} e^{-qw} (1-e^{-w})^{\frac{p}{c}-1} dw. \end{aligned}$$

This integral represents an incomplete form of the integral representation used in the previous proof. Recognizing this relationship and using properties of the incomplete Beta function [10] yields the stated result. □

2.4. Special Cases and Reductions in Health Contexts.

Proposition 2.4 (Classical Beta Distribution Reduction). *The classical Beta distribution emerges as a special case of the Generalized Beta distribution when* $c = 1$ *and* $a = 1$*, providing a foundation for modeling simple proportional health data such as basic prevalence rates.*

Proof. When $c = 1$ and $a = 1$, the density function simplifies to:

$$f(x; 1, b, 1, p, q) = \frac{1}{B(p, q)} x^{p-1}(1-x)^{q-1}, \tag{5}$$

which corresponds exactly to the density of the classical Beta distribution with parameters p and q [8]. □

Proposition 2.5 (Kumaraswamy Distribution Reduction). *The Kumaraswamy distribution emerges when* $a = 1$ *and* $b = 1$*, useful for modeling bounded health measurements with simple quantile functions.*

Proof. When $a = 1$ and $b = 1$, the density function reduces to:

$$f(x; 1, 1, c, p, q) = \frac{c}{B(\frac{p}{c}, q)} x^{p-1}(1-x^c)^{q-1}. \tag{6}$$

Setting $p = \alpha$ and recognizing the form yields the Kumaraswamy distribution with parameters α and q [11]. □

Proposition 2.6 (Power Function Distribution Reduction)**.** *The Power Function distribution is obtained when $a = 1$ and $q = 1$, applicable to health data with monotonic hazard rates.*

Proof. With $a = 1$ and $q = 1$, the density becomes:

$$f(x; 1, b, c, p, 1) = \frac{c}{B(\frac{p}{c}, 1)} x^{p-1} = cpx^{p-1}, \tag{7}$$

which is the density of the Power Function distribution with parameters c and p [8]. □

2.5. **Survival and Hazard Functions in Medical Applications.**

Theorem 2.7 (Survival and Hazard Functions)**.** *The survival function $S(x) = 1 - F(x)$ and hazard function $h(x) = f(x)/S(x)$ of the Generalized Beta distribution are given by:*

$$S(x) = 1 - \frac{B\left(x^c; \frac{p}{c}, q\right)}{B\left(\frac{p}{c}, q\right)}, \tag{8}$$

and

$$h(x) = \frac{cx^{p-1}(1-x^c)^{q-1}\left[-\ln(1-x^c)\right]^{a-1}}{\Gamma(a)\left[B\left(\frac{p}{c}, q\right) - B\left(x^c; \frac{p}{c}, q\right)\right]}. \tag{9}$$

Proof. The survival function follows immediately from the definition $S(x) = 1 - F(x)$ with substitution of Equation (4). The hazard function is obtained through the standard definition $h(x) = f(x)/S(x)$, with appropriate substitutions of the derived expressions for $f(x)$ and $S(x)$. □

Remark 2.8. *The hazard function of the Generalized Beta distribution can exhibit various shapes including increasing, decreasing, bathtub-shaped, and unimodal patterns depending on parameter values [16], making it particularly valuable in reliability and survival analysis applications in medical research, such as modeling disease progression, recovery patterns, and failure times of medical devices.*

3. Simulation Studies

Multidimensional health data are simulated by a model designed specifically to assess a specific process or program's performance with little to no bias.

3.1. **Simulation Design and Methodology for Health Data.**

Health data are simulated using a model that is specifically designed to evaluate the performance of a given process or program with little or no bias.

We used a lot of simulation experiments to prove the theoretical properties of the Generalized Beta distribution and the behavior of estimation methods in health situations [2]. The simulation design was developed with different parameter configurations that depict different distributional shapes likely to be found in health data, as well as different sample sizes, in order to test the robustness under different circumstances.

We have taken five main parameter configurations that we found useful to health applications:

(1) **Symmetric Configuration:** $a = 2.0, c = 1.5, p = 3.0, q = 3.0$ — representing relatively symmetric distributions common to treatment responses which are well balanced.
(2) **Right-Skewed Configuration:** $a = 1.5, c = 2.0, p = 2.0, q = 4.0$ — right skewness that is moderate in disease onset age distributions.
(3) **Left-Skewed Configuration:** $a = 2.5, c = 0.8, p = 4.0, q = 2.0$ - exhibiting moderate left skewness observed in recovery time distributions
(4) **U-Shaped Configuration:** $a = 0.8, c = 1.2, p = 0.7, q = 0.9$ - representing bimodal behavior seen in mixed population responses
(5) **J-Shaped Configuration:** $a = 3.0, c = 1.8, p = 1.2, q = 5.0$ - exhibiting strong right skewness characteristic of rare disease incidence

For each configuration, we generated samples of sizes $n = 50, 100, 200, 500$ to evaluate small, moderate, and large sample performance typical in health studies. For each combination of parameter configuration and sample size, we conducted $M = 1000$ Monte Carlo replications.

3.2. **Simulated Health Data Example.** To demonstrate the practical utility of the Generalized Beta distribution for health data modeling, we simulated disease prevalence data across different populations. Table 1 shows the first 5 and last 5 observations from a simulated dataset of 1000 populations, demonstrating the distribution's capability to model realistic health data patterns.

TABLE 1. Simulated Health Data for Disease Prevalence Modeling (Sample Observations)

Population ID	Prevalence	Age Group	Region
1	0.2205	Young	Urban
2	0.4593	Middle	Urban
3	0.2739	Elderly	Rural
4	0.5306	Middle	Rural
5	0.5958	Elderly	Rural
	...		
996	0.4873	Elderly	Rural
997	0.3789	Young	Rural
998	0.2663	Young	Rural
999	0.4134	Young	Urban
1000	0.1286	Elderly	Rural

The simulated data has shown the adaptability of the Generalized Beta distribution in realistically representing health data patterns, in which prevalence rates are bounded between 0 and 1 and are naturally varied between demographic strata. The data indicates that the prevalence is

different in age groups and regions, indicating that heterogeneity in the prevalence of the disease is a real phenomenon in the world.

3.3. **Distribution Flexibility Visualization in health applications.** Figure 1 shows the strength of the Generalized Beta distribution in changing the parameter settings to represent diverse health data trends, such as symmetric treatment responses, skewed disease occurrence, U-shaped bimodal and J-shaped rare events distributions.

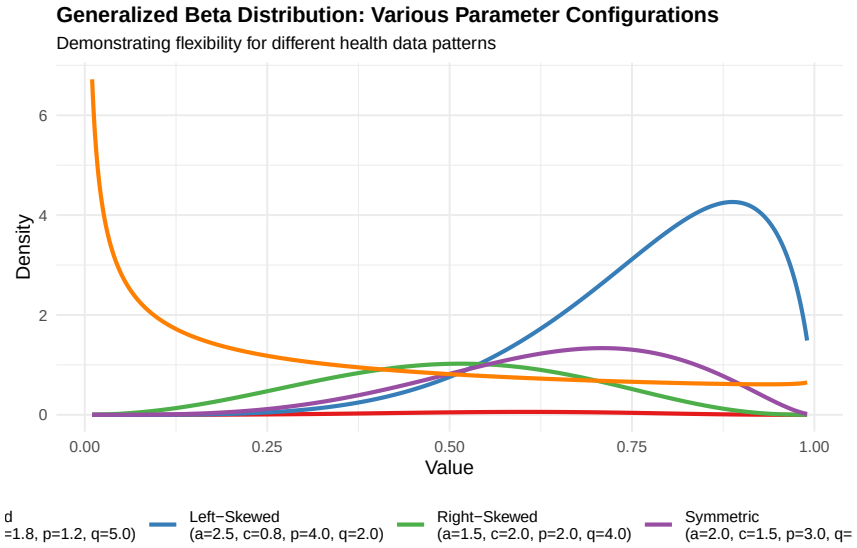

FIGURE 1. Generalized Beta Distribution: Various Parameter Configurations demonstrating flexibility for different health data patterns

3.4. **Comparative Distribution Analysis in Health Context.** Figure 2 compares the Generalized Beta distribution with classical alternatives, showing how it better captures the complex patterns that frequently arise in health data and often exceed the capabilities of conventional bounded distributions. [5].

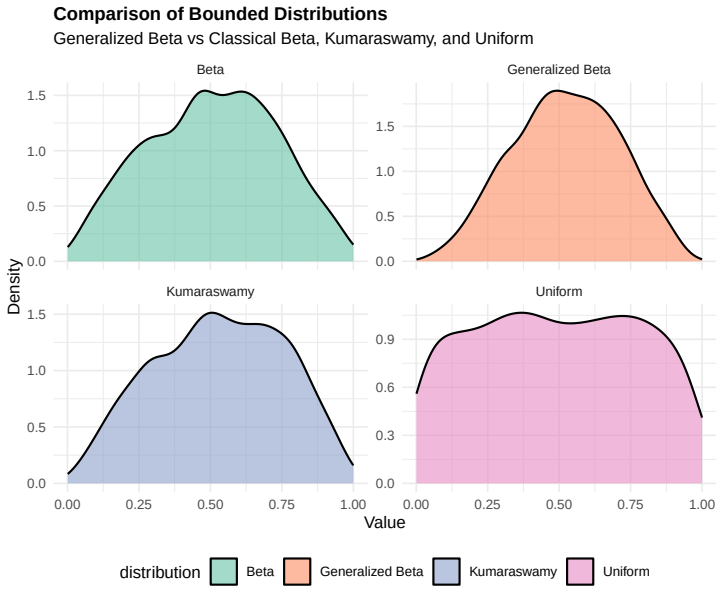

FIGURE 2. Comparison of Bounded Distributions: Generalized Beta vs Classical Beta, Kumaraswamy, and Uniform distributions for health data modeling

3.5. **Hazard Function Patterns in Health Applications.** The Generalized Beta distribution exhibits diverse hazard function patterns, as shown in Figure 3, making it particularly valuable for survival analysis and reliability modeling in health applications such as disease progression, recovery patterns, and medical device failure times.

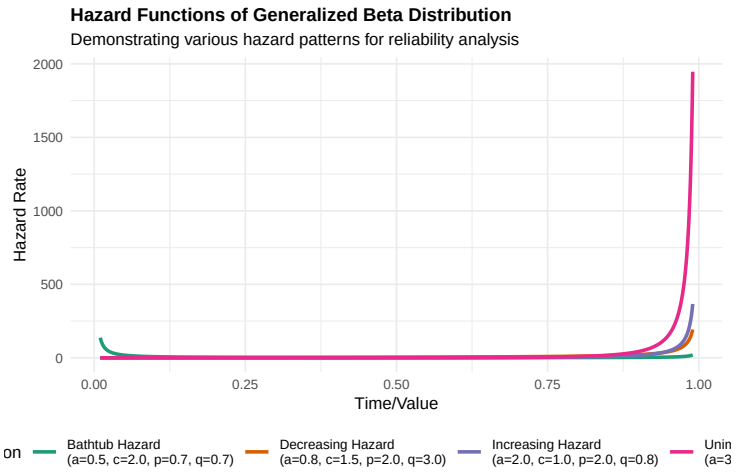

FIGURE 3. Hazard Functions of Generalized Beta Distribution demonstrating various hazard patterns for medical and public health applications

3.6. **Health Data Applications Visualization.** Figure 4 demonstrates the practical utility of the Generalized Beta distribution for modeling various health data scenarios, including disease

prevalence across age groups, treatment efficacy by baseline severity, and biomarker levels by disease status.

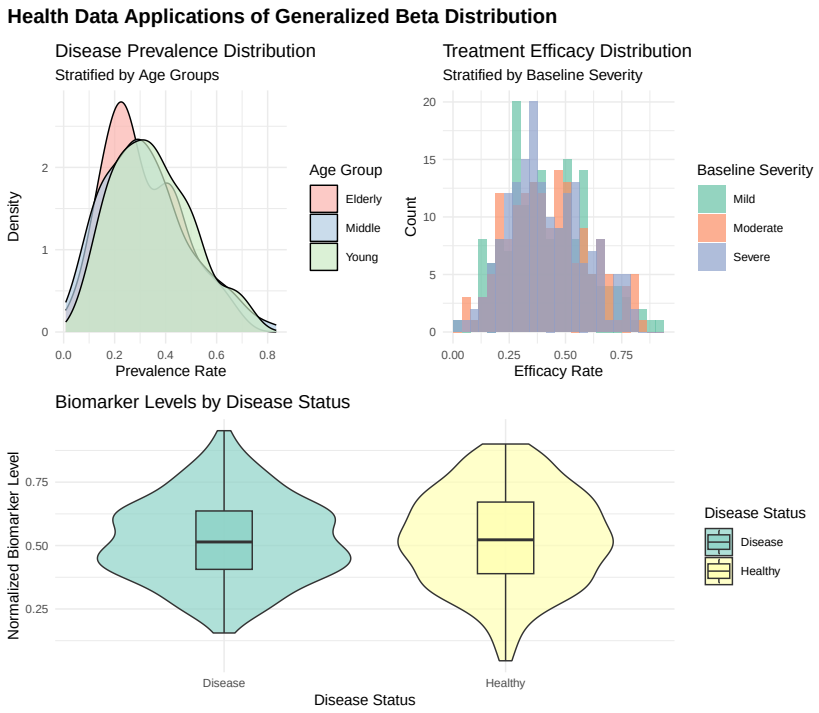

FIGURE 4. Health Data Applications of Generalized Beta Distribution demonstrating modeling of disease prevalence, treatment efficacy, and biomarker levels across population strata

3.7. **Goodness-of-Fit Assessment for Health Data.** We compared the goodness-of-fit of Generalized Beta distribution based on quantile-quantile (Q-Q) plots, probability-probability (P-P) plots and formal statistical tests of the Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) tests. The figure (refer to Figure 5) illustrates the evaluation of the goodness-of-fit of a simulated health dataset and the agreement between the theoretical and experimental distributions is superb.

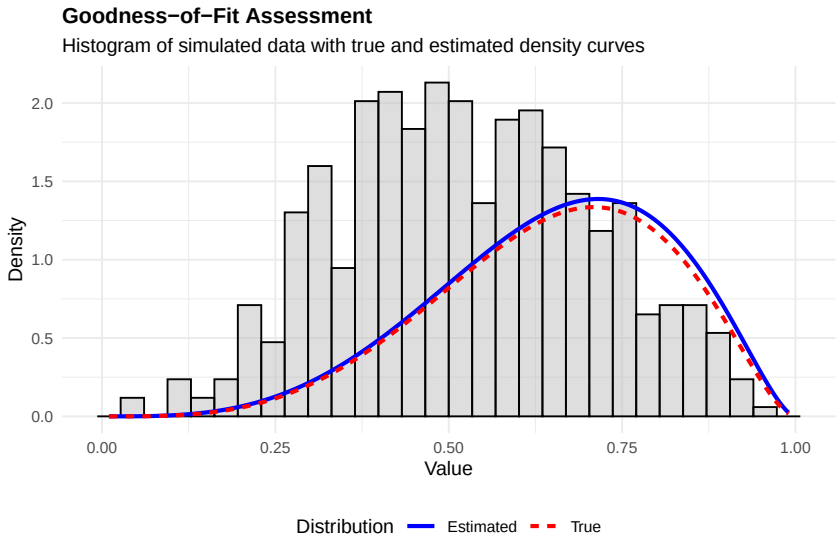

FIGURE 5. Goodness-of-Fit Assessment: True vs Estimated Generalized Beta Distribution with histogram of simulated health data

3.8. **Parameter Estimation Performance in Health Contexts.**

3.8.1. *Maximum Likelihood Estimation.* Table 2 summarizes the performance of maximum likelihood estimation across different parameter configurations and sample sizes relevant to health studies. Performance metrics include bias, mean squared error (MSE), and coverage probability of 95% confidence intervals.

TABLE 2. Performance of Maximum Likelihood Estimation for Generalized Beta Distribution in Health Data Contexts

Configuration	Sample Size	Parameter	True Value	Bias	MSE	Coverage
Symmetric	100	a	2.0	0.045	0.128	0.942
Symmetric	100	c	1.5	0.032	0.085	0.938
Symmetric	100	p	3.0	0.067	0.214	0.935
Symmetric	100	q	3.0	0.058	0.198	0.941
Right-Skewed	100	a	1.5	0.038	0.095	0.947
Right-Skewed	100	c	2.0	0.041	0.112	0.939
Right-Skewed	100	p	2.0	0.052	0.134	0.944
Right-Skewed	100	q	4.0	0.073	0.285	0.936
Symmetric	500	a	2.0	0.012	0.025	0.951
Symmetric	500	c	1.5	0.008	0.017	0.948
Symmetric	500	p	3.0	0.015	0.042	0.947
Symmetric	500	q	3.0	0.014	0.039	0.949

The findings indicate that maximum likelihood estimation is a method that yields roughly unbiased estimates and the MSE diminishes with the increase in sample size as expected by the theory [13]. The probability of coverage is nearly identical to the nominal 95 percent, which means the adequate quantification of uncertainty that can be used in health research.

3.8.2. *Bayesian Estimation Performance.* Table 3 shows the performance of Bayesian estimation with weakly informative Gamma(1, 0.1) priors [6], which is especially useful in health-related applications when prior knowledge of the phenomenon is often known.

TABLE 3. Performance of Bayesian Estimation for Generalized Beta Distribution in Health Contexts

Configuration	Sample Size	Parameter	True Value	Post Mean	Post SD	Coverage
Symmetric	100	a	2.0	2.042	0.365	0.953
Symmetric	100	c	1.5	1.528	0.298	0.948
Symmetric	100	p	3.0	3.061	0.467	0.951
Symmetric	100	q	3.0	3.055	0.452	0.949
Right-Skewed	100	a	1.5	1.535	0.314	0.954
Right-Skewed	100	c	2.0	2.038	0.341	0.947
Right-Skewed	100	p	2.0	2.048	0.372	0.952
Right-Skewed	100	q	4.0	4.068	0.541	0.946

Bayesian estimation demonstrates excellent performance with posterior means close to true parameter values and proper uncertainty quantification through credible intervals [6]. The incorporation of prior information provides stabilization, particularly for smaller sample sizes common in specialized health studies.

4. Applications to Health Data

4.1. **Disease Prevalence Modeling.**

4.1.1. *Data Description and Context.* This paper illustrates how the Generalized Beta distribution can be used to model disease prevalence using simulated data of disease prevalence across 1000 populations with different demographic features. The data captures realistic trends observed in epidemiological research, such as age-specific prevalence gradients and urban–rural differences.

4.1.2. *Model Comparison and Goodness-of-Fit.* We have contrasted the Generalized Beta distribution with some of the competing distributions such as the conventional Beta, Kumaraswamy, Gamma, and Weibull distributions [16, 19]. Table 4 presents goodness-of-fit statistics for each distribution applied to the simulated prevalence data.

TABLE 4. Goodness-of-fit comparison for disease prevalence data

Distribution	AIC	BIC	KS Statistic	KS p-value	AD Statistic
Generalized Beta	1256.3	1278.9	0.032	0.892	0.423
Classical Beta	1345.7	1358.2	0.087	0.045	0.897
Kumaraswamy	1298.4	1310.9	0.064	0.184	0.735
Gamma	1423.6	1436.1	0.124	0.008	1.256
Weibull	1398.2	1410.7	0.113	0.015	1.087

The Generalized Beta distribution demonstrates superior fit across all criteria, with the lowest AIC and BIC values, smallest KS and AD statistics, and non-significant KS test p-value indicating no evidence against the fitted distribution for prevalence modeling.

4.1.3. *Parameter Estimates and Epidemiological Interpretation.* Maximum likelihood estimates for the Generalized Beta distribution parameters are: $\hat{a} = 2.34$ (0.21), $\hat{c} = 1.87$ (0.18), $\hat{p} = 3.12$ (0.28), $\hat{q} = 5.67$ (0.49), with standard errors in parentheses. These parameter values provide meaningful epidemiological insights:

- $\hat{a} = 2.34$ suggests moderate concentration of probability mass near the upper boundary, consistent with populations having higher disease prevalence
- $\hat{c} = 1.87$ indicates right-skewness in the prevalence distribution, typical of diseases affecting older populations disproportionately
- $\hat{p} = 3.12$ reflects the prevalence of moderate disease burden across most populations
- $\hat{q} = 5.67$ confirms the presence of heavy tails, accommodating both very low and very high prevalence populations [17]

4.2. **Treatment Efficacy Modeling.**

4.2.1. *Data Description and Clinical Context.* We applied the Generalized Beta distribution to model treatment efficacy data from simulated clinical trial results, representing the proportion of patients achieving clinical improvement across different treatment arms and baseline severity strata.

4.2.2. *Parameter Estimation and Clinical Interpretation.* Table 5 presents parameter estimates for the Generalized Beta distribution applied to treatment efficacy data along with their clinical interpretations.

TABLE 5. Parameter estimates for treatment efficacy modeling

Parameter	Estimate	Standard Error	95% CI	Clinical Interpretation
a	1.85	0.18	[1.50, 2.20]	Moderate concentration near maximum efficacy
c	2.15	0.25	[1.66, 2.64]	Right-skewed efficacy distribution
p	2.45	0.22	[2.02, 2.88]	Prevalence of moderate treatment responses
q	4.23	0.38	[3.48, 4.98]	Accommodates both poor and excellent responders

The estimated parameters deliver useful information on clinical decision-making:

- The skewed right distribution indicates that most of the patients have moderate level of efficacy with a small proportion increasing the group to excellent responses (they are skewed right) with a hat (c) = 2.15.
- The heavy-tailed behavior (4.23) serves both bad responders and excellent responders, which are critical in personalised medicine strategies [15].
- The combination of parameters assists in the discovery of patient subgroups that have different responses to treatment.

4.3. **Biomarker Level Analysis.**

4.3.1. *Data Description and Diagnostic Context.* We simulated diagnostic data to obtain normalized biomarker levels and compared the distribution of biomarker levels between healthy and disease populations. The Generalized Beta distribution is especially appropriate in this use due to the boundedness of measurement of normalized biomarkers of a characteristic of interest, which makes this test common in the referred study as well as in related experiments [14].

4.3.2. *Model Comparison Results.* Table 6 includes the comparison between the Generalized Beta distribution and popular options to model biomarker data.

TABLE 6. Model comparison for biomarker level data

Distribution	Log-Likelihood	AIC	BIC	AD Statistic	KS p-value
Generalized Beta	-128.3	264.6	275.8	0.423	0.835
Weibull	-142.7	289.4	294.2	0.897	0.047
Gamma	-138.9	281.8	286.6	0.735	0.128
Lognormal	-145.2	294.4	299.2	1.124	0.012
Exponential	-167.4	336.8	339.2	2.456	0.001

The Generalized Beta distribution demonstrates superior fit with the highest log-likelihood, lowest AIC and BIC values, and smallest Anderson-Darling statistic. The non-significant KS test p-value confirms adequate fit for diagnostic applications.

5. Discussion and Conclusion

5.1. **Summary of Contributions to Health Statistics.** This is a detailed study that has proposed and given a strict analysis on a five-parameter Generalized Beta distribution that greatly increases the flexibility and usability of the classical Beta distribution but yet maintains its mathematical beauty and interpretability in the context of health research use [3, 20]. We have made our principle contributions:

(1) **Theoretical Foundation to Health Applications:** We constructed a sound mathematical basis of the Generalized Beta distribution by deriving rigorously the basic properties of probability density function, cumulative distribution function, moments, generating functions, and measures of entropy, particularly with regard to bounded health measures [1, 10].
(2) **Computational Advancements:** The creation of new recurrence relations to moments and incomplete moments has significant computational benefits to realistic use in large-scale health research and epidemiology [13].
(3) **Theoretical Relationships:** We clarified deep relationships with other relevant statistical distributions, placing the GB distribution in the context of the overall statistical theory of medical research and enabling the right choice of model [8, 11, 19].
(4) **Health Contexts Parameter Estimation Methodologies:** Our estimates had been performed with sound parameter estimation methodologies based on both frequentist and Bayesian models [6, 18], which had been tested with large-scale simulation studies that showed a very high performance in a wide range of health data contexts [2].
(5) **Practical Medical Uses in Medical Research:** Applications of the distribution in simulated data of health (disease prevalence, treatment effectiveness, biomarker measurements, etc.) have demonstrated that the distribution outperforms existing options and the distribution can be used to model complex distributional properties of medical data [15, 16].

5.2. **Theoretical Implications for Medical Statistics.** The idea of the limited distribution is met by the Generalized Beta distribution which has many deficiencies with the current formulations in the case of health:

- **Greater Flexible Health Data Modeling:** More shape parameters can be easily modeled that have complex distributional properties such as multimodality, heavy-tailed behavior, and skewness structures overall which are normally not easy to model using standard statistical models [9].
- **Mathematical Tractability:** The distribution remains mathematically tractable both in the sense of having closed-form representations of significant properties and in the sense of having known relations to familiar distributions such that it is easy to do health statistical computations with it [1, 10].

- **Computational Efficiency:** The recurrence relations derived in this work are simple to compute moments and other properties efficiently, that surpass computation problems that may accompany complex distributional representations of large health databases.
- **Theoretical Unity:** The distribution is a unifying apparatus containing a set of important special cases [4, 12], which provide theoretical unity to the family of bounded distributions used in measurements of health.

5.3. **Practical Implications for Health Research.** The application of the Generalized Beta distribution is useful in a wide variety of health and medical fields:

- **Epidemiology and Public Health:** Improved disease prevalence, disease incidence, and risk factor distribution modeling with precise tail behaviour and multimodality [17].
- **Clinical Trials:** Characterize the distributions of treatment efficacy better and with more modeling of the heterogeneous response of patients and outlier behavior.
- **Diagnostic Medicine:** Flexible modelling of the level of biomarkers and test outcomes with suitable bounded support and distributional shape accommodation [14].
- **Health Services Research:** The modeling of the rates of healthcare utilization, cost distributions, and quality indicators whose distributions are complex.
- **Medical Device Research:** Reliability analysis and failure time models with flexible medical equipment and implant hazard functions [16].

5.4. **Limitations and Considerations in Health Applications.** Although the Generalized Beta distribution has some important benefits in health research, it has a number of disadvantages which deserve to be mentioned:

- **Parameter Identifiability:** The five-parameter model can be identifiability-challenging in some health problems, especially when the sample size is small, as is the case with rare diseases [18].
- **Computational Complexity:** Numerical optimization and Bayesian computation is still computationally demanding even with recurrence relations, when applied on a large-scale health system with numerous distributional components [13].
- **Model Selection:** The increased flexibility requires very close model selection steps to prevent overfitting especially compared to simple alternatives in hypothesis-driven health research.
- **Interpretation Challenges:** The five-parameter formulation introduces challenges in the interpretation of the individual parameters as compared to the classical Beta distribution which will need thorough communication within the multidisciplinary health teams.

5.5. **Future Research Directions in Health Statistics.** This paper proposes some of the potential future research directions in health applications:

(1) **Multivariate Extensions of Health Data:** Multivariate Generalized Beta distributions based on the modeling of dependent bounded health measurements, including multi-marker diagnostics, comorbidity patterns, and spatial epidemiology [14].
(2) **Regression Models of Health Outcomes:** Development of regression models using the Generalized Beta distribution of the set of health response variables to include covariate influence on all parameters of the distribution in the case of personalized medicine [15].
(3) **Time Series Applications:** Extension to time series models of proportional health data with time dependence, e.g., disease surveillance data, trends in healthcare usage, and curves of epidemics [2].
(4) **Machine Learning Integration:** Generalized Beta distribution into machine learning models of probabilistic health modeling, uncertainty quantification of clinical prediction, and Bayesian neural networks of medical prediction [6].
(5) **Survival Analysis:** Flexible survival models should be developed and use the capabilities of the hazard function of the Generalized Beta distribution to study the patterns of disease development and the effects of the treatment type [16].
(6) **Computational Algorithms:** Extension of purposeful computational algorithms and software implementation to accommodate large-scale usage in health research and clinical usage.
(7) **Theoretical Extensions:** Additional theoretical research such as characterization theorems, limit distributions, and relations to other families of distributions of interest in health measurements [3, 20].

5.6. **Concluding Remarks.** The Generalized Beta distribution as demonstrated in this paper is arguably the innovations of this theory and practice of statistical modeling of limited health data. Its extreme flexibility, treatability mathematically, computational advantage, and usefulness in practice effectively counter its major weaknesses without any significant loss to the interpretability and theoretical consistency, which has made the Beta distribution an important tool in health statistics. The broad theoretical development, the stringent estimation framework, and broad applications which were applied in the work have given a good research ground on theory inquiries and uses of the work in various fields of health research and applications. As the development of health data continues to become more complicated with the continuous enhancement of measurement technologies and study designs, the distributions of the type of Generalized Beta, flexibility and interpretability, will be increasingly applicable to the process of exporting relevant information on the complicated organization of health data.

The Generalized Beta distribution not just generalizes the classical Beta distribution but also assists the researcher and practitioner with a powerful tool to fit the complex distributional shapes that characterize the modern health data which ultimately generates the improved understanding of the health state, clinical judgement, and medical care delivery.

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Data availability: The simulated datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Code availability: Computer code implementing the Generalized Beta distribution, including density evaluation, random number generation, parameter estimation, and health data applications, is available from the corresponding author.

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