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On the Ostrowski Method for Solving Equations

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Abstract. In this paper, we revisited the Ostrowski's method for solving Banach space valued equations. We developed a technique to determine a subset of the original convergence domain and using this new Lipschitz constants derived. These constants are at least as tight as the earlier ones leading to a finer convergence analysis in both the semi-local and the local convergence case. These techniques are very general, so they can be used to extend the applicability of other methods without additional hypotheses. Numerical experiments complete this study.

1. Introduction

One of the most challenging tasks in computational mathematics is the problem of determining a solution x_* of equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subset B \longrightarrow B_1$ is an operator acting between Banach spaces B and B_1 with $\Omega \neq \emptyset$. The closed form derivation of x_* is possible only in rare cases. This leads practitioners and researchers in developing solution methods that are iterative.

In this work, we consider Ostrowski's method defined for $x_0 \in \Omega$ and each $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{k+1} &= y_n - A_n^{-1}F(y_n), \end{aligned} \tag{1.2}$$

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where $A_n = 2[y_n, x_n; F] - F'(x_n)$. The convergence order is four obtained under certain conditions on the initial data (Ω, F, F', x_0) and Taylor expansion [17,25]. So, the assumptions on the fourth derivative reduce the applicability of these schemes.

For example: Let $B = B_1 = \mathbb{R}$, $\Omega = [-0.5, 1.5]$. Define λ on Ω by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if \ t \neq 0 \\ 0 & if \ t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\lambda'''(t)$ is not bounded on Ω. So, the convergence of scheme (1.2) is not guaranteed by the analyses in [17,24].

We study two types of convergence called local and semi-local. In the first one based on the solution x_* we find the radii of the convergence balls. But in the second one based on the starter x_0 we develop criteria that guarantee convergence of sequence $\{x_n\}$. There is a plethora of this types of results [10,15,16,22,28,38]. But what all these results have in common is that the region of accessibility (or convergence region) is limited in general reducing the applicability of Newton's and other methods [8,20,26,28,31]. Moreover, the error bounds on distances $\|x_{k+1} - x_k\|$ or $\|x_k - x_*\|$ are pessimistic. The same is true for the uniqueness ball of these methods. These problems become more difficult when studying methods of convergence order three or higher [8,17,19,31–33]. We have developed different techniques to addres these problems.

In technique 1, we determine a subset Ω of Ω also containing the iterates. But in this set Ω the Lipschitz-like parameters (or functions) are at least as tight as the original ones, so the resulting convergence is finer. This technique does not depend on the convergence order of the method. But we shall demonstrate it in case of fourth order methods. These methods require the evaluation of the second order Fréchet derivative of operator F. Notice that for a system (nonlinear) of i equations with i unknowns, the first derivative is a matrix with i^2 entries (values), whereas the second Fréchet derivative has i^3 entries. That is why there is a need for avoiding F''.

The rest of the paper is organized as follows: In Section 2 we develop the second technique based on majorizing sequences. The local convergence analysis results appear in Section 3. Numerical examples can be found in Section 4. The paper ends with some concluding remarks.

2. Semi-local Convergence

We base our semi-local convergence analysis on scalar parameters and functions. Let $\eta \geq 0, K_0 > 0, K > 0, K_1 > 0, K_2 > 0, K_3 > 0, L_0 > 0$ with $K_0 \leq K$, $L_0 \leq 2K_1$ and $K_4 = K_2 + K_3$. Define polynomials g_1 and g_2 on the interval $[0, 1)$ by

$$g_1(t) = K_1 t^5 + (2K_1 + K_3)t^4 + K_1 t^3 + (\frac{K}{2} - K_3)t^2 - \frac{K}{2} \tag{2.1}$$

and

$$g_2(t) = L_0 t^4 + (L_0 + K_3)t^3 + K_4 t^2 - K_3 t - K_4. \tag{2.2}$$

We have $g_1(0) = -\frac{K}{2} < 0, g_1(1) = 4K_1 > 0, g_2(0) = -K_4 < 0$ and $g_2(1) = 2L_0 > 0$. It then follows from the intermediate value theorem that polynomials g_1 and g_2 have at least one root in $(0,1)$. Denote by δ_1 and δ_2 the least such roots, respectively. Moreover, it is convenient to define scalar sequences and parameters

$$\begin{aligned} t_0 &= 0,\ s_0 = \eta,\ t_1 = s_0 + \frac{K_0}{2}\frac{(s_0 - t_0)^2}{1 - 2K_1 s_0}, \\ s_{n+1} &= t_{n+1} + \frac{(K_3(t_{n+1} - s_n) + K_4(s_n - t_n))(t_{n+1} - s_n)}{1 - L_0 t_{n+1}} \\ t_{n+2} &= s_{n+1} + \frac{K(s_{n+1} - t_{n+1})^2}{2(1 - (K_1(s_{n+1} + t_{n+1}) + K_3(s_{n+1} - t_{n+1}))}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \alpha_n &= \frac{K(s_n - t_n)}{2(1 - (K_1(s_{n+1} + t_{n+1}) + K_3(s_{n+1} - t_{n+1}))}, \\ \gamma_n &= \frac{K_3(t_n - s_n) + K_4(s_n - t_n)}{1 - L_0 t_{n+1}}, \end{aligned}$$

for all $n = 0, 1, 2, \ldots,$

$$\delta_n = \max\{\alpha_n, \gamma_n\},$$

$$\lambda = \min\{\delta_1, \delta_2\}$$

and

$$\mu = \max\{\delta_1, \delta_2\}.$$

Next, we present a convergence result for sequences $\{t_n\}$ and $\{s_n\}$.

LEMMA 2.1. *Suppose: there exists δ satisfying*

$$0 \le \delta_0 \le \lambda \le \delta \le \mu < 1 - K_1\eta. \tag{2.4}$$

Then, sequences $\{t_n\}, \{s_n\}$ are well defined nondecreasing, bounded from above bt $s_{**} = \frac{\eta}{1-\delta}$ and as such they converge to their unique least upper bound $s_* \in [\eta, s_{**}]$. Moreover, the following error estimates hold for all $n = 1, 2, \ldots$

$$0 \le t_{n+1} - s_n \le \delta(s_n - t_n) \le \delta^{2n+1}\eta, \tag{2.5}$$

$$0 \le s_n - t_n \le \delta(t_n - s_{n-1}) \le \delta^{2n}\eta \tag{2.6}$$

and

$$t_n \le s_n \le t_{n+1}. \tag{2.7}$$

Proof. Items (2.5)-(2.7) hold if

$$0 \leq \alpha_m \leq \delta, \tag{2.8}$$

$$0 \leq \gamma_m \leq \delta \tag{2.9}$$

and

$$t_m \leq s_m \leq t_{m+1} \tag{2.10}$$

are true for all $m = 0, 1, 2, \ldots$. Notice that by the definition of s_0, t_1 and (2.4), $t_1 \geq 0$. We also have (2.8) and (2.9) hold for $m = 0$. Suppose (2.8)-(2.10) hold for $m = 1, 2, \ldots, n$. Then, we can obtain in turn that

$$\begin{aligned} s_m &\leq t_m + \delta^{2m}\eta \leq s_{m-1} + \delta^{2m-1}\eta + \delta^{2m}\eta \\ &\leq \eta + \ldots + \delta\eta + \ldots + \delta^{2m}\eta \\ &= \frac{1-\delta^{2m+1}}{1-\delta}\eta \leq \frac{\eta}{1-\delta} = s_{**}, \end{aligned}$$

and

$$\begin{aligned} t_{m+1} &\leq s_m + \delta^{2m+1}\eta \leq t_m + \delta^{2m}\eta + \delta^{2m+1}\eta \\ &\leq \eta + \delta\eta + \ldots + \delta^{2m+1}\eta \\ &= \frac{1-\delta^{2m+2}}{1-\delta}\eta \leq \frac{\eta}{1-\delta}. \end{aligned}$$

Hence, by (2.7) and the induction hypotheses, we deduce that sequences $\{t_m\}$ and $\{s_m\}$ are nondecreasing. Evidently, (2.8) holds if

$$\begin{aligned} &\frac{K}{2}\delta^{2n}\eta + \delta K_1\left(\frac{1-\delta^{2n+3}}{1-\delta}\eta\right) \\ &+\delta K_1\frac{1-\delta^{2n+2}}{1-\delta}\eta + K_3\delta^{2(n+1)}\eta - \delta \leq 0. \end{aligned} \tag{2.11}$$

Estimate (2.11) motivates us to introduce recurrent functions $h_n^{(1)}(t)$ on the interval $[0, 1)$ by

$$\begin{aligned} h_n^{(1)})t) &= \frac{K}{2}t^{2n-1}\eta + K_1(1 + t + \ldots + t^{2n+2})\eta \\ &\quad +K_1(1 + t + \ldots + t^{2n+1})\eta + K_3t^{2n+3}\eta - 1. \end{aligned} \tag{2.12}$$

We need a relationship between two consecutive functions $f_n^{(1)}(t)$. By this definition, we have in turn that

$$\begin{aligned} h_{n+1}^{(1)}(t) &= \frac{K}{2}t^{2n+1}\eta + K_1(1+t+\ldots+t^{2n+4})\eta \\ &+K_1(1+t+\ldots+t^{2n+3})\eta + K_3t^{2n+3}\eta - 1 \\ &-\frac{K}{2}t^{2n-1}\eta - K_1(1+t+\ldots+t^{2n+2})\eta \\ &-K_1(1+t+\ldots+t^{2n+1})\eta - K_3t^{2n+1}\eta + 1 + h_n^{(1)}(t) \\ &= h_n^{(1)}(t) + \frac{K}{2}t^{2n+1}\eta - \frac{K}{2}t^{2n-1}\eta \\ &+K_1(t^{2n+3}+t^{2n+4})\eta + K_1(t^{2n+2}+t^{2n+3})\eta + K_3t^{2n+3}\eta - K_3t^{2n+1}\eta \\ &= h_n^{(1)}(t) + g_1(t)t^{2n-1}\eta. \end{aligned} \tag{2.13}$$

Notice that by the definition of δ_1

$$h_{n+1}^{(1)}(\delta_1) = h_n^{(1)}(\delta_1).$$

By (2.11)-(2.13), estimate (2.11) shall be true if for $4K_1 < K$

$$h_n^{(1)}(\delta_1) \le 0. \tag{2.14}$$

Let

$$h_\infty^{(1)}(t) = \lim_{n\longrightarrow\infty} h_n^{(1)}(t). \tag{2.15}$$

But then

$$h_\infty^{(1)}(\delta) = \frac{2K_1\eta}{1-\delta} - 1. \tag{2.16}$$

Hence, instead of (2.13) we can show

$$h_\infty^{(1)}(\delta) \le 0, \tag{2.17}$$

which is true by (2.4). If $4K_1 \ge K$ then $f_\infty(t) \ge f_n(t)$, so again $f_\infty(\delta) \le 0$ holds. Similarly, (2.9) holds if

$$K_3\delta^{2n+1}\eta + K_4\delta^{2n}\eta + \delta L_0\frac{1-\delta^{2n+2}}{1-\delta}\eta - \delta \le 0 \tag{2.18}$$

or

$$h_n^{(2)}(\delta) \le 0, \tag{2.19}$$

where

$$h_n^{(2)}(t) = K_3t^{2n}\eta + K_4t^{2n-1}\eta + L_0(1+t+\ldots+t^{2n+1})\eta - 1. \tag{2.20}$$

This time we have

$$\begin{aligned} h^{(2)}_{n+1}(t) &= K_3t^{2n+2}\eta + K_4t^{2n+1}\eta + L_0(1+t+\ldots+t^{2n+3})\eta \\ &\quad -1 - K_3t^{2n}\eta - K_4t^{2n-1}\eta - L_0(1+t+\ldots+t^{2n+1})\eta + 1 + h^{(2)}_n(t) \\ &= h^{(2)}_n(t) + K_3t^{2n+2}\eta - K_3t^{2n}\eta + K_4t^{2n+1} - K_4t^{2n+1}\eta \\ &\quad +L_0(t^{2n+2}+t^{2n+3})\eta \\ &= h^{(2)}_n(t) + [K_3t^3 - K_3t + K_4t^2 - K_4 + L_0t^2 + L - 0t^4]t^{2n-1}\delta \\ &= h^{(2)}_n(t) + g_2(t)t^{2n-1}\eta. \end{aligned} \tag{2.21}$$

By the definition of δ_2

$$h^{(2)}_{n+1}(\delta_2) = h^{(2)}_n(\delta).$$

Let $h^{(2)}_\infty(t) = \lim_{n\longrightarrow\infty} h^{(2)}_n(t)$. Then, we get

$$h^{(2)}_\infty(\delta) = \frac{L_0\eta}{1-\delta} - 1.$$

Hence, instead of (2.19), we can show

$$h^{(2)}_\infty(\delta) \le 0,$$

which is true by (2.4). The induction for (2.8)-(2.10) is completed. Therefore, sequences $\{t_n\}, \{s_n\}$ are nondecreasing, bounded from above by s_{**} and as such they converge to s_*.

□

The semi-local convergence analysis shall be based on conditions (A).

Suppose:

(A1) There exists $x_0 \in \Omega$, $\eta \ge 0$ such that $F'(x_0)^{-1} \in L(E_1, E)$ and

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta.$$

(A2) For each $x \in \Omega$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x - x_0\|.$$

set $\Omega_0 = U[x_0, \frac{1}{L_0}] \cap \Omega$.

(A3) For each $x, y \in \Omega_0$

$$\|F'(x_0)^{-1}(F'(y) - F'(x))\| \le K\|y - x\|,$$

$$\|F'(x_0)^{-1}([y, x; F] - F'(x_0))\| \le K_1(\|y - x_0\| + \|x - x_0\|),$$

$$\|F'(x_0)^{-1}([z, y; F] - [y, x; F])\| \le K_2(\|z - y\| + \|y - x\|)$$

and

$$\|F'(x_0)^{-1}([z, y; F] - F'(y))\| \le K_3\|z - y\|.$$

(A4) $U[x_0, s_*] \subset \Omega$ and

(A5) Conditions of Lemma 2.1 hold.

Next, we present the semi-local convergence of method (1.2).

THEOREM 2.2. *Suppose that conditions (A) hold. Then, sequences* $\{y_n\}, \{x_n\}$ *generated by method (1.2) are well defined in* $U[x_0, s_*]$, *remain in* $U[x_0, s_*]$ *for each* $n = 0, 1, 2, \ldots$ *and converge to a solution* $x_* \in U[x_0, s_*]$ *of equation* $F(x) = 0$. *Moreover, the following assertion holds*

$$\|x_n - x_*\| \leq s_* - t_n. \tag{2.22}$$

Proof. Mathematical induction on m shall be used to show

(I_m) $\|y_m - x_m\| \leq s_m - t_m$

and

(II_m) $\|x_{m+1} - y_m\| \leq t_{m+1} - s_m$.

By the first substep of method (1.2) we have

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 = s_0 \leq s_*.$$

So (I_0) holds and $y_0 \in U[x_0, s_*]$. By the first substep of method (1.2) we can write

$$F(y_0) = F(y_0) - F(x_0) - F'(x_0)(y_0 - x_0). \tag{2.23}$$

Using (A2) and (2.23), we have

$$\|F'(x_0)^{-1}F(y_0)\| \leq \frac{K_0}{2}\|y_0 - x_0\|^2 \leq \frac{K_0}{2}(s_0 - t_0)^2. \tag{2.24}$$

We need to show the invertability of linear operator A. We have by (A3)

$$\begin{aligned}
\|F'(x_0)^{-1}(A_0 - F'(x_0))\| &\leq 2\|F'(x_0)^{-1}([y_0, x_0; F] - F'(x_0))\| \\
&\leq 2K_1(\|y_0 - x_0\| + \|x_0 - x_0\|) \\
&\leq 2K_1(s_0 + t_0) < 1,
\end{aligned}$$

so

$$\|A_0^{-1}F'(x_0)\| \leq \frac{1}{1 - 2K_1(s_0 + t_0)}, \tag{2.25}$$

by the Banach lemma on linear invertible operators [19, 25]. Then, iterate x_1 exists by the second substep of method (1.2), and we can write

$$x_1 - y_0 = (A_0^{-1}F'(x_0))(F'(x_0)^{-1}F(y_0)). \tag{2.26}$$

By (2.24)-(2.26), we get

$$\begin{aligned}
\|x_1 - y_0\| &\leq \|A_0^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_0)\| \\
&\leq \frac{K_0(s_0 - t_0)^2}{2(1 - 2K_1(s_0 + t_0))} = t_1 - s_0,
\end{aligned}$$

showing (II_0). Moreover, we have

$$\begin{aligned}\|x_1 - x_0\| &\leq \|x_1 - y_0\| + \|y_0 - x_0\| \\ &\leq t_1 - s_0 + s_0 - t_0 = t_1 \leq s_*,\end{aligned}$$

so $x_1 \in U[x_0, s_*]$. Suppose that (I_m) and (II_m) hold, $y_m, x_{m+1} \in U[x_0, s_*]$ and $F'(x_m)^{-1}, A_m^{-1}$ exist for each $m = 1, 2, \ldots, n$. We shall prove they hold for $m = n + 1$. Using the second substep of method (1.2), we get

$$\begin{aligned}\|F(x_0)^{-1}F(x_{n+1})\| &= \|F'(x_0)^{-1}(F(x_{n+1}) - F(y_n)) - A_n(x_{n+1} - y_n))\| \\ &= \|F'(x_0)^{-1}([x_{n+1}, y_n; F] - A_n)(x_{n+1} - y_n)\| \\ &\leq F'(x_0)^{-1}([x_{n+1}, y_n; F] - [y_n, x_n; F])\| \\ &\quad + \|F'(x_0)^{-1}([y_n, x_n; F] - F'(x_n))\| \\ &\leq (K_2(\|x_{n+1} - y_n\| + \|y_n - x_n\|) + K_3\|y_n - x_n\|)\|x_{n+1} - y_n\|\end{aligned} \tag{2.27}$$

We need to show $F'(x_{n+1})$ is invertible. By (A2) and the induction hypotheses we obtain

$$\begin{aligned}\|F'(x_0)^{-1}(F'(x_{n+1}) - F'(x_0))\| &\leq L_0\|x_{n+1} - x_0\| \\ &\leq L_0(t_{n+1} - t_0) = L_0 t_{n=1} < 1,\end{aligned}$$

so

$$\|F'(x_{n+1})^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0 t_{n+1}}. \tag{2.28}$$

Hence, we get by (2.27), (2.28) and the first substep of method (1.20 that

$$\begin{aligned}\|y_{n+1} - x_{n+1}\| &= \|F'(x_{n+1})^{-1}F(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq \frac{(K_2(t_{n+1} - s_n) + (s_n - t_n)) + K_3(s_n - t_n))(t_{n+1} - s_n)}{1 - L_0 t_{n+1}} \\ &= s_{n+1} - t_{n+1},\end{aligned} \tag{2.29}$$

since $K_4 = K_2 + K_3$, showing (I_m) for $m = n + 1$. Then, we also have

$$\begin{aligned}\|y_{n+1} - x_0\| &\leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_0\| \\ &\leq s_{n+1} - t_{n+1} + t_{n+1} - s_0 = s_{n+1} \leq s_*,\end{aligned}$$

so $y_{n+1} \in U[x_0, s_*]$. Operator A_{n+1}^{-1} shall be shown to exist

$$\begin{aligned}
\|F'(x_0)^{-1}(A_{n+1} - F'(x_0))\| &\leq \|F'(x_0)^{-1}([y_{n+1}, x_{n+1}; F] - F'(x_0))\| \\
&\quad + \|F'(x_0)^{-1}([y_{n+1}; x_{n+1}; F] - F'(x_{n+1}))\| \\
&\leq K_1(\|y_{n+1} - x_0\| + \|x_{n+1} - x_0\|) + K_3\|y_{n+1} - x_{n+1}\| \\
&\leq K_1(s_{n+1} + t_{n+1}) + K_3(s_{n+1} - t_{n+1}) < 1,
\end{aligned}$$

so

$$\|A_{n+1}^{-1}F'(x_0)\| \leq \frac{1}{1 - (K_1(s_{n+1} + t_{n+1}) + K_3(s_{n+1} - t_{n+1}))}. \tag{2.30}$$

By the first substep of method (1.2), we can write

$$F(y_{n+1}) = F(y_{n+1}) - F(x_{n+1}) - F'(x_{n+1})(y_{n+1} - x_{n+1}),$$

so

$$\|F'(x_0)^{-1}F(y_{n+1})\| \leq \frac{K}{2}\|y_{n+1} - x_{n+1}\|^2 \leq \frac{K}{2}(s_{n+1} - t_{n+1})^2,$$

so

$$\begin{aligned}
\|x_{n+2} - y_{n+1}\| &\leq \|A_{n+1}^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_{n+1})\| \\
&\leq \frac{K(s_{n+1} - t_{n+1})^2}{2(1 - (K_1(s_{n+1} + t_{n+1}) + K_3(s_{n+1} - t_{n+1})))} \\
&= t_{n+2} - s_{n+1},
\end{aligned}$$

showing (II_m) for $m = n + 1$. We can get

$$\begin{aligned}
\|x_{n+2} - x_0\| &\leq \|x_{n+2} - y_{n+1}\| + \|y_{n+1} - x_0\| \\
&\leq t_{n+2} - s_{n+1} + s_{n+1} - t_0 = t_{n+2} \leq s_*,
\end{aligned}$$

so $x_{n+2} \in U[x_0, s_*]$. Furthermore, we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\
&= t_{n+1} - s_n + s_n - t_n = t_{n+1} - t_n,
\end{aligned}$$

so sequence $\{x_n\}$ is fundamental in a Banach space B, so it converges to some $x_* \in U[x_0, s_*]$. By letting $n \longrightarrow \infty$ in (2.27), we obtain

$$\|F'(x_0)^{-1}F(x_{k+1})\| \leq (K_2(t_{n+1} - s_n) + K_4(s_n - t_n))(s_{n+1} - s_n) \longrightarrow 0,$$

so $F(x_*) = 0$ by the continuity of F.

□

A uniqueness of the solution result is given next.

PROPOSITION 2.3. *Suppose:*

(i) *There exists a simple solution* x_* *of equation* $F(x) = 0$.

(ii) *There exists* $\bar{s} \geq s_*$ *such that*

$$L_0(\bar{s} + s_*) < 2.$$

Set $\Omega_1 = U[x_0, \bar{s}] \cap \Omega$. *Then, the only solution of equation* $F(x) = 0$ *in the region* Ω_1 *is* x_*.

Proof. Let $\bar{x} \in \Omega_1$ with $F(\bar{x}) = 0$. Let $M = \int_0^1 F'(\bar{x} + \theta(x_* - \bar{x}))d\theta$. Then, in view of (A2) and (ii), we obtain

$$\begin{aligned} \|F'(x_0)^{-1}(M - F'(x_0))\| &\leq L_0 \int_0^1 [(1-\theta)\|\bar{x} - x_0\| + \theta\|x_* - x_0\|]d\theta \\ &\leq \frac{L_0}{2}(\bar{s} + s_*) < 1, \end{aligned}$$

so $\bar{x} = x_*$ since M^{-1} exists and $M(x_* - \bar{x}) = F(x_*) - F(\bar{x}) = 0 - 0 = 0$.

□

REMARK 2.4. *Notice that* s_{**} *given in closed form can repalce* s_* *in the conditions of Theorem 2.2.*

3. Local Convergence

As in Section 2 we develop some functions and parameters. Let $L_i, i = 0, 1, 2, 3, 4$ be given parameters. Define function φ_1 on the interval $T = [0, \frac{1}{L_0})$ by

$$\varphi_1(t) = \frac{Lt}{2(1 - L_0 t)}.$$

Notice that parameter

$$r_A = \frac{2}{2L_0 + L} < \frac{1}{L_0} \tag{3.1}$$

solves equation

$$\varphi_1(t) = 1.$$

Define functions on the interval T by

$$q(t) = L_0\varphi_1(t)t - 1 \text{ and } p(t) = (2L_1(1 + \varphi_1(t)) + L)t.$$

Suppose that these functions have smallest zeros r_q and r_p in $(0, \frac{1}{L_0})$, respectively. Let $r_1 = \min\{r_q, r_p\}$ and $T_0 = [0, r_1)$. Define function φ_2 on T_0 by

$$\begin{aligned} \varphi_2(t) &= \Bigg[\frac{L\varphi_1(t)}{2(1 - L_0\varphi_1(t)t)} \\ &\quad + \frac{L_4(L_2 + L_3)(1 + \varphi_1(t))\varphi_1(t)}{(1 - L_0\varphi_1(t)t)(1 - p(t))}\Bigg] t. \end{aligned}$$

Suppose that function

$$\varphi_2(t) - 1$$

has smallest zero $r_2 \in (0, r_1)$. We shall show that parameter

$$r = \min\{r_A, r_2\} \tag{3.2}$$

is a convergence radius for method (1.2). Let $T_1 = [0, r)$. Then, it follows by these definitions that for each $t \in T_1$

$$L_0 t < 1 \tag{3.3}$$

$$0 \le \varphi_1(t) < 1, \tag{3.4}$$

$$0 \le \varphi_1(t)t < 1 \tag{3.5}$$

$$0 \le p(t) < 1 \tag{3.6}$$

and

$$0 \le \varphi_2(t)t < 1 \tag{3.7}$$

hold.

The conditions (H) to be used in the local convergence of method (1.2) are as follows.

Suppose:

(H1) There exists a simple solution $x_* \in \Omega$ of equation $F(x) = 0$.

(H2) For each $x \in \Omega$

$$\|F'(x)^{-1}(F'(x) - F'(x_*))\| \le L_0\|x - x_*\|.$$

Set $\Omega_0 = U[x_*, \frac{1}{L_0}] \cap \Omega$.

(H3) For each $x, y \in \Omega_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \le L\|y - x\|,$$

$$\|F'(x_*)^{-1}(F'(y) - F'(x_*))\| \le L_1(\|y - x_*\| + \|x - x_*\|),$$

$$\|F'(x_*)^{-1}([y, x; F] - F'(x))\| \le L_2\|y - x\|,$$

$$\|F'(x_*)^{-1}([y, x; F] - F'(y))\| \le L_3\|y - x\|$$

and

$$\|F'(x_*)^{-1}F'(x)\| \le L_4\|x - x_*\|.$$

and

(H4) $U[x_*, r] \subset \Omega$.

In view of conditions (H) and the developed notation we can show the local convergence result for method (1.2).

THEOREM 3.1. *Under the conditions (H), further suppose that $x_0 \in U(x_*, r) - \{x_*\}$. Then, sequence $\{x_k\}, \{y_n\}$ generated by method (1.2) is well defined in $U(x_*, r)$, remains in $U(x_*, r)$ for each $k = 0, 1, 2, \ldots$ and converges to x_*.*

Proof. Let $u \in U(x_*, r) - \{x_*\}$. By (H1) and (H2), we get in turn that

$$\|F'(x_*)^{-1}(F'(u) - F'(x_*))\| \leq L_0\|u - x_*\| \leq L_0 r < 1,$$

so $F'(u)$ is invertibale and

$$\|F'(u)^{-1}F'(x_*)\| \leq \frac{1}{1 - L_0\|u - x_*\|}. \tag{3.8}$$

Iterate y_0 is well defined by the first substep of method (1.2) and (3.8) for $u = x_0$. Then, we can write

$$\begin{aligned} y_0 - x_* &= x_0 - x_* - F'(x_0)^{-1}F(x_0) \\ &= (F'(x_0)^{-1}F'(x_*)) \\ &\quad \times (\int_0^1 F'(x_*)^{-1}(F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta(x_0 - x_*). \end{aligned} \tag{3.9}$$

By (3.2), (3.4), (H3), (3.8) and (3.9), we have in turn that

$$\begin{aligned} \|y_0 - x_*\| &\leq \frac{L_0\|x_0 - x_*\|^2}{2(1 - L_0\|x_0 - x_*\|} \\ &\leq \frac{L\|x_0 - x_*\|^2}{2(1 - L_0\|x_0 - x_*\|)} \\ &\leq \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r \end{aligned} \tag{3.10}$$

so $y_0 \in U(x_*, r)$. Next, we show linear operator $A_)$ is invertible. Indeed, using (3.2), (3.6), (H3) and (3.10), we get in turn that

$$\begin{aligned} \|F'(x_*)^{-1}(A_0 - F'(x_*))\| &\leq \|F'(x_*)^{-1}([y_0, x_0; F] - F'(x_*))\| \\ &\quad + \|F'(x_*)^{-1}([y_0, x_0; F] - F'(x_*))\| \\ &\quad + \|F'(x_*)^{-1}(F'(x_0) - F'(x_*))\| \\ &\leq 2L_1(\|y_0 - x_*\| + \|x_0 - x_*\|) + L_0\|x_0 - x_*\| \\ &\leq 2L_1(1 + \varphi_1(\|x_0 - x_*\|))\|x_0 - x_*\| + L\|x_0 - x_*\| \\ &\leq p(\|x_0 - x_*\|) \leq p(r) < 1, \end{aligned}$$

so

$$\|A_0^{-1}F'(x_*)\| \leq \frac{1}{1 - p(\|x_0 - x_*\|)} \tag{3.11}$$

and iterate x_1 is well defined by the second substep of method (1.2) for $n = 0$. Then, we can write

$$x_1 - x_* = (y_0 - x_* - F'(y_0)^{-1}F(y_0)) + F'(y_0)^{-1}(A_) - F'(y_0))A_0^{-1}F(y_0). \tag{3.12}$$

Using (3.2), (3.7), (H3), (3.8) (for $u = x_0, y_0$) and (3.10)–(3.13), we obtain in turn that

$$\begin{aligned}
\|x_1 - x_*\| &\leq \|y_0 - x_* - F'(y_0)^{-1}F(y_0)\| \\
&\quad + \|F'(y_0)^{-1}F'(x_*)\|\|F'(x_*)^{-1}(A_0 - F'(y_0))\|\|A_0^{-1}F'(x_*)\|\|F'(x_*)^{-1}F(y_0)\| \\
&\leq \frac{L\|y_0 - x_*\|^2}{2(1 - L_0\|x_0 - x_*\|} + \frac{(L_2 + L_3)\|y_0 - x_0\|L_4\|y_0 - x_*\|}{(1 - L_0\|y_0 - x_*\|)(1 - p(\|x_0 - x_*\|))} \\
&\leq \varphi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r,
\end{aligned}$$

so $x_1 \in U(x_*, r)$, where we also used

$$\begin{aligned}
\|F'(x_*)^{-1}(A_0 - F'(y_0))\| &\leq \|F'(x_*)^{-1}([y_0, x_0; F] - F'(x_0))\| \\
&\quad + \|F'(x_*)^{-1}([y_0, x_0; F] - F'(y_0))\| \\
&\leq (L_2 + L_3)\|y_0 - x_0\| \leq (L_2 + L_3)(\|y_0 - x_*\| + \|x_0 - x_*\|) \\
&\leq (L_2 + L_3)(1 + \varphi_1(\|x_0 - x_*\|))\|x_0 - x_*\|,
\end{aligned}$$

and

$$\begin{aligned}
\|F'(x_*)^{-1}F(y_0)\| &= \|\int_0^1 F'(x_*)^{-1}F'(x_* + \theta(y_0 - x_*))d\theta(y_0 - x_*)\| \\
&\leq L_4\|y_0 - x_*\|^2.
\end{aligned}$$

So, far showed

$$\|y_0 - x_*\| \leq \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\| < r$$

and

$$\|x_1 - x_*\| \leq \varphi_2(\|x_0 - x_*\|)\|x_0 - x_*\| < r.$$

By simply replacing x_0, y_0, x_1 by x_m, y_m, x_{m+1} in the preceding calculations, we get

$$\|y_m - x_*\| \leq \varphi_1(\|x_m - x_*\|)\|x_m - x_*\| < r$$

and

$$\|x_{m+1} - x_*\| \leq \varphi_2(\|x_m - x_*\|)\|x_m - x_*\| < r.$$

Then, from the estimation

$$\|x_{m+1} - x_*\| \leq \alpha\|x_m - x_*\| < r, \tag{3.13}$$

where $\alpha = \varphi_2(\|x_0 - x_*\|) \in [0, 1)$, $\lim_{m \longrightarrow \infty} x_m = x_*$ and $y_m, x_{m+1} \in U(x_*, r)$.

□

REMARK 3.2. *By the definition of r, we see that*

$$r \leq r_A. \tag{3.14}$$

Parameter r_A was shown in [4] to be a convergence radius for Newton's method. Notice the radius of convergence for Newton's method given independently by Traub [35] and Rheinbold [29] is

$$r_{TR} = \frac{2}{3M_1},$$

where L_1 is the Lipschitz constant on Ω. So, we have

$$r_{TR} \leq r_A,$$

since $L \leq L_1$ and $L_0 \leq M_1$.

4. Numerical Experiments

We provide some examples, showing that the old convergence criteria are not verified but ours are.

EXAMPLE 4.1. *Define function*

$$f(t) = \theta_0 t + \theta_1 + \theta_2 \sin \theta_3 t, \; t_0 = 0,$$

where θ_j, $j = 0, 1, 2, 3$ are parameters. Then, clearly for θ_3 large and θ_2 small, $\frac{L_0}{K}$ can be small (arbitrarily).

EXAMPLE 4.2. *Let $B = B_1 = U[0, 1]$ the domain of functions given on $[0, 1]$ which are continuous. We consider the max-norm. Choose $\Omega = B(0, d)$, $d > 1$. Define F on Ω be*

$$F(x)(s) = x(s) - w(s) - \xi \int_0^1 K(s, t) x^3(t) dt, \tag{4.1}$$

$x \in B, s \in [0, 1], w \in B$ is given, ξ is a parameter and K is the Green's kernel given by

$$K(s_2, s_1) = \begin{cases} (1 - s_2)s_1, & s_1 \leq s_2 \\ s_2(1 - s_1), & s_2 \leq s_1. \end{cases}$$

By (4.1), we have

$$(F'(x)(z))(s) = z(s) - 3\xi \int_0^1 K(s, t) x^2(t) z(t) dt,$$

$t \in Bs \in [0, 1]$. Consider $x_0(s) = w(s) = 1$ and $|\xi| < \frac{8}{3}$. We get

$$\|I - F'(x_0)\| < \frac{3}{8}|\xi|, \; F'(x_0)^{-1} \in L(B_1 B),$$

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\xi|}, \; \eta = \frac{|\xi|}{8 - 3|\xi|}, \; L_0 = \frac{12|\xi|}{8 - 3|\xi|},$$

$K = \frac{6d|\xi|}{8-3|\xi|}$, $K_1 = \frac{L_0}{2}$ and $K_2 = \frac{K}{2} = K_3$.

EXAMPLE 4.3. *Let $B = B_1 = \mathbb{R}^3$ and Ω be as in the Example 4.2. It is well known that the boundary value problem [16]*

$$\psi(0) = 0, \psi(1) = 1,$$

$$\psi'' = -\psi - \tau\psi^2$$

can be given as a Hammerstein-like nonlinear integral equation

$$\psi(s) = s + \int_0^1 K(s,t)(\psi^3(t) + \tau\psi^2(t))dt$$

where τ is a parameter. Then, define $F : \Omega \longrightarrow T_2$ by

$$[F(x)](s) = x(s) - s - \int_0^1 K(s,t)(x^3(t) + \tau x^2(t))dt.$$

Choose $x_0(s) = s$ and $\Omega = U(x_0, r_0)$. Then, clearly $U(x_0, r_0) \subset U(0, r_0 + 1)$, since $\|x_0\| = 1$. Suppose $2\tau < 5$. Then, by conditions (A) are satisfied for $L_0 = \frac{2\tau+3r_0+6}{8}$, $K = \frac{\tau+6r_0+3}{4}$, $K_1 = \frac{L_0}{2}$ and $K_2 = \frac{K}{2} = K_3$. and $\eta = \frac{1+\tau}{5-2\tau}$. Notice that $L_0 < K$.

The rest of the examples are given for the local convergence study of Newton's method.

EXAMPLE 4.4. *Let $B = B_1 = \mathbb{R}^3$, $\Omega = U[0,1]$ and $x^* = (0,0,0)^{tr}$. Define mapping E on Ω for $\lambda = (\lambda_1, \lambda_2, \lambda_3)^{tr}$ as*

$$E(\lambda) = (e^{\lambda_1} - 1, \frac{e-1}{2}\lambda_2^2 + \lambda_1, \lambda_3)^{tr}.$$

Then, conditions (H) hold provided that $L_0 = e - 1, L = e^{\frac{1}{L_0}}$ and $M_1 = e$, since $F'(x^*)^{-1} = F'(x^*) = diag\{1, 1, 1, \}$. Notice that

$$L_0 < L < M_1,$$

$$L_1 = \frac{L_0}{2},\ L_4 = L,\ L_2 = L_3 = \frac{L}{2}.$$

$$r_{TR} = 0.2453 < r_A = 0.3827, r = 0.2124.$$

Hence, our radius of convergence is larger.

EXAMPLE 4.5. *Let $B = B_1$ and Ω be as in Example 4.2. Define F on Ω as*

$$F(\varphi_1)(x) = \varphi_1(x) - \int_0^1 x\varphi_1(j)^3 dj.$$

Then, we obtain

$$F'(\varphi_1(\psi_1))(x) = \psi_1(x) - 3\int_0^1 xj\varphi_1(j)^2\psi_1(j)dj$$

for all $\psi_1 \in \Omega$. So, we can choose $L_0 = 1.5, L = M_1 = 3$.

$$L_1 = \frac{L_0}{2},\ L_4 = L,\ L_2 = L_3 = \frac{L}{2}.$$

But then, we get again

$$r_{TR} = 0.2222 < r_A = 0.3333,\ r = 0.2663.$$

5. Conclusion

Ostrowski's method was revisited and its applicability was extended in both the semi-local and local convergence case. In particular, the benefits in the semi-local convergence case include: weaker sufficient convergence criteria (i.e. more starters x_0 become available); tighter upper bounds on $\|x_{k+1} - x_k\|$, $\|x_k - x_*\|$ (i.e., fewer iterates are computed to reach a predecided error accuracy) and the information on the location of x_* is more precise.

The results are based on generalized continuity which is more general than Lipschitz continuity used before. Our two techniques are very general and can be used to extend the applicability of other methods.

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