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Orthogonal F-Quasi-Contractions on O-Complete Metric Spaces with Applications

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Abstract. In this paper, we introduce the notion of *orthogonal F-quasi-contractions* on orthogonal metric spaces. We establish two fixed point theorems for such mappings: one requiring orthogonal continuity and another in which this assumption is relaxed. When the orthogonality relation is taken to be universal, the results yield *new* fixed point theorems for *F-quasi-contractions* on complete metric spaces. We construct examples to show that the newly introduced notion overlaps with and extends the F-weak contractions and F-contractions notions to handle more complex, non-continuous, or asymmetrical mapping behaviors.

As an application, we prove the existence and uniqueness of a solution to an initial value problem for a general Volterra-type integro-differential equation. Such equations arise in various fields, including epidemic modeling, control theory, and physics.

1. Introduction

The Banach contraction principle [1] is one of the fundamental tools in nonlinear analysis and its applications [14]. Since its introduction in 1922, numerous generalizations have been proposed [15], among which the F-contraction introduced by Wardowski [3] in 2012 and the subsequent F-weak contraction introduced by Wardowski and Dung [4] in 2014 occupy a prominent position. These mappings replace the Lipschitz constant with a suitable real-valued function F belonging to a family $\mathcal{W}$ (see Definition 2.8), thereby allowing a broader class of contractive conditions while still guaranteeing the existence and uniqueness of a fixed point in complete metric spaces. The notions of F-contraction and F-weak contraction continue to play an essential role in the development of metric fixed point theory and its applications (see, e.g., [7–10] and the references therein).

In 2017, Gordji *et al.* [2] initiated a parallel line of research by introducing the concept of an orthogonal set (O-set) and the corresponding orthogonal metric space (O-metric space). In such spaces, conditions on the space or on mappings defined on it are required only along O-sets. Consequently, the framework of O-metric spaces is more general than that of metric spaces.

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This observation has stimulated considerable research activity in fixed point theory within this setting (see, e.g., [5, 6, 11–13] and the references therein).

Several authors have successfully combined these two directions in the context of O-complete metric spaces (see, e.g., [5,6,12]). For instance, Sawangsup *et al.* (2020) and Kumar and Malhotra (2024) studied orthogonal F-contractions and orthogonal F-weak contractions, respectively. These studies demonstrate the effectiveness of merging orthogonality structures with nonlinear F-contractive conditions and provide genuine applications of the corresponding fixed point results.

Motivated by these developments, the aim of this paper is to introduce and study a natural class of *orthogonal F-quasi-contractions* on O-complete metric spaces. Examples are constructed to show that the introduced class contains mappings that are not included in previously studied classes, such as F-contractions, F-weak contractions, orthogonal F-contractions, and orthogonal F-weak contractions (Proposition 3.4). Moreover, they show that the notion of F-quasi-contractions overlaps with and extends that of F-weak contractions and F-contractions to handle more complex, non-continuous, or asymmetrical mapping behaviors (Section 3.1.1).

We establish two fixed point theorems for such mappings on O-complete spaces. The first (Theorem 3.6) assumes orthogonal continuity, whereas the second (Theorem 3.8) replaces this assumption with a weaker condition. When the orthogonality relation is taken to be the universal relation, both theorems reduce to *new* fixed point results for *F-quasi-contractions* on complete metric spaces (Corollaries 3.10 and 3.11). Finally, we illustrate the applicability of our results by establishing the existence and uniqueness of a solution to a Volterra-type integro-differential equation (Theorem 4.1).

The paper is organized as follows. Section 2 collects the necessary preliminaries. The main theorems are stated and proved in Section 3. An application to Volterra integro-differential equations is given in Section 4, and the paper ends with a short conclusion in Section 5.

2. Preliminaries

In this section, we collect all facts about orthogonal metric spaces and F-contractions needed throughout the paper.

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^+ \subset \mathbb{R}$ the set of all positive real numbers. Let $\mathbb{Z}^+$ denote the set of positive integers.

The next *six* definitions are due to Gordji *et al* [2]. We refer the reader to the same for illustrative examples.

Definition 2.1. Let X be a nonempty set, and $\wedge \subset \mathsf{X} \times \mathsf{X}$ a binary relation. The pair $(\mathsf{X}, \wedge)$ is called an *orthogonal* set (*O-set*) if

$$\text{there exists } s_0 \in \mathsf{X} \text{ such that for all } s \in \mathsf{X} \text{ we have either } (s_0, s) \in \wedge \text{ or } (s, s_0) \in \wedge.$$

Such an element s_0 is called an *orthogonal element*.

Definition 2.2. A self-map $\mathcal{T}$ on an O-set $(\mathsf{X}, \wedge)$ is said to be $\wedge$-*preserving* if $(\mathcal{T}s_1, \mathcal{T}s_2) \in \wedge$ whenever $(s_1, s_2) \in \wedge$.

Definition 2.3. Let $(\mathsf{X}, \wedge)$ be an O-set, and $\mathfrak{q} : \mathsf{X} \times \mathsf{X} \to [0, \infty)$ be a metric. The triple $(\mathsf{X}, \wedge, \mathfrak{q})$ is called an *orthogonal metric space*.

Definition 2.4. Let $(\mathsf{X}, \wedge)$ be an O-set. A sequence $\{s_n : n \in \mathbb{Z}^+\} \subset \mathsf{X}$ is called an *orthogonal* sequence (*O-sequence*) if

$$(s_n, s_{n+1}) \in \wedge \quad \text{or} \quad (s_{n+1}, s_n) \in \wedge \quad \text{for all} \quad n \in \mathbb{Z}^+.$$

Definition 2.5. A self-map $\mathcal{T}$ on an orthogonal metric space $(\mathsf{X}, \wedge, \mathfrak{q})$ is said to be *orthogonally* continuous ($\wedge$-*continuous*) at $t_0 \in \mathsf{X}$ if for each O-sequence $\{s_n\} \subset \mathsf{X}$ such that $s_n \to t_0 \in \mathsf{X}$, we have $\mathcal{T}s_n \to \mathcal{T}t_0$. The map $\mathcal{T}$ is said to be $\wedge$-continuous on X if it is $\wedge$-continuous at each $t \in \mathsf{X}$.

Definition 2.6. An O-metric space $(\mathsf{X}, \wedge, \mathfrak{q})$ is said to be *orthogonally* complete (*O-complete*) if every Cauchy O-sequence in X is convergent in X.

Remark 2.7. As demonstrated in [2, Example 3.3] and [2, Example 3.6], respectively:

- every continuous mapping $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ is $\wedge$-continuous, and the converse does not hold in general.
- every complete metric space is O-complete, and the converse statement is false.

Definition 2.8. [3] Let $\mathcal{W}$ denote the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying:

(W_1) For all $s, t \in \mathbb{R}^+$ with $s < t$, we have $F(s) < F(t)$;

(W_2) For each sequence $\{s_n : n \in \mathbb{Z}^+\} \subset \mathbb{R}^+$ we have

$$\lim_{n\to\infty} s_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(s_n) = -\infty;$$

(W_3) There exists a real number $0 < \sigma < 1$ such that $\lim_{s\to 0+} s^\sigma F(s) = 0$.

For example, the function

$$F : \mathbb{R}^+ \to \mathbb{R} \quad \text{defined by} \quad F(p) = \ln(p) \tag{1}$$

belongs to $\mathcal{W}$. Other examples can be found in [3].

Definition 2.9. [6] A self-map $\mathcal{T}$ on an orthogonal metric space $(\mathsf{X}, \wedge, \mathfrak{q})$ is called an *orthogonal* F-contraction ($\wedge_F$-*contraction*) if there exist $\tau \in \mathbb{R}+$ and $F \in \mathcal{W}$ such that

$$\tau + F(\mathfrak{q}(\mathcal{T}s, \mathcal{T}t)) \leq F(\mathfrak{q}(s, t))\ \forall s, t \in \mathsf{X} \text{ with } (s, t) \in \wedge \text{ and } \mathcal{T}s \neq \mathcal{T}t.$$

Definition 2.10. [5] A self-map $\mathcal{T}$ on an orthogonal metric space $(\mathsf{X}, \wedge, \mathfrak{q})$ is called an *orthogonal* F-weak contraction ($\wedge_F$-*weak contraction*) if there exist $\tau \in \mathbb{R}_+$ and $F \in \mathcal{W}$ such that

$$\tau + F(\mathfrak{q}(\mathcal{T}s, \mathcal{T}t)) \leq F(\max \mathfrak{U}(s, t))\ \forall s, t \in \mathsf{X} \text{ with } (s, t) \in \wedge \text{ and } \mathcal{T}s \neq \mathcal{T}t,$$

where

$$\mathfrak{U}(s,t)=\left\{\mathfrak{q}(s,t),\mathfrak{q}(s,\mathcal{T}s),\mathfrak{q}(t,\mathcal{T}t),\frac{\mathfrak{q}(s,\mathcal{T}t)+\mathfrak{q}(\mathcal{T}s,t)}{2}\right\}. \tag{2}$$

For the binary relation $\wedge=\mathsf{X}\times\mathsf{X}$, the notions above reduce to the notions of F-*contraction* [3] and F-*weak contraction* [4], respectively. We recall the definitions below.

Definition 2.11. A self-map $\mathcal{T}$ on a metric space X is called an F-*contraction* if there exist $\tau\in\mathbb{R}+$ and $F\in\mathcal{W}$ such that

$$\tau+F(\mathfrak{q}(\mathcal{T}s,\mathcal{T}t))\leq F(\mathfrak{q}(s,t))\ \forall s,t\in\mathsf{X}\text{ with }\mathcal{T}s\neq\mathcal{T}t. \tag{3}$$

Definition 2.12. A self-map $\mathcal{T}$ on a metric space X is called an F-*weak contraction* if there exist $\tau\in\mathbb{R}^+$ and $F\in\mathcal{W}$ such that

$$\tau+F(\mathfrak{q}(\mathcal{T}s,\mathcal{T}t))\leq F(\max\mathfrak{U}(s,t))\ \forall s,t\in\mathsf{X}\text{ with }\mathcal{T}s\neq\mathcal{T}t. \tag{4}$$

Remark 2.13. As is demonstrated in

- [3, Remark 2.1], an F-contraction is necessarily a *continuous* map. In fact, combining (3) and Definition 2.8 (W_1) gives

$$\mathfrak{q}(\mathcal{T}s,\mathcal{T}t)<\mathfrak{q}(s,t)\quad\text{for all distinct}\quad s,t\in\mathsf{X}.$$

  Hence, the claim.
- [4, Example 2.3], an F-weak contraction need not be an F-contraction.
- [5, Example 3.3], $\wedge_F$-weak contractions are genuine extensions of $\wedge_F$-contractions.

3. Main Results : Orthogonal F-quasi-contractions and Fixed Point Theorems

In this section, we define and study the notion of orthogonal F-quasi-contractions on O-metric spaces. Then, we formulate and prove fixed point theorems for the mappings.

3.1. Orthogonal F-quasi-contractions.

Definition 3.1. Let $(\mathsf{X},\wedge,\mathfrak{q})$ be an orthogonal metric space. A map $\mathcal{T}:\mathsf{X}\to\mathsf{X}$ is called an *orthogonal F-quasi-contraction* (*$\wedge_F$-quasi-contraction*) if there exist $\tau\in\mathbb{R}^+$ and $F\in\mathcal{W}$ such that

$$\tau+F(\mathfrak{q}(\mathcal{T}s,\mathcal{T}t))\leq F(\max\mathfrak{M}(s,t))\ \forall s,t\in\mathsf{X}\text{ with }(s,t)\in\wedge\text{ and }\mathcal{T}s\neq\mathcal{T}t, \tag{5}$$

where

$$\mathfrak{M}(s,t)=\left\{\mathfrak{q}(s,t),\mathfrak{q}(s,\mathcal{T}s),\mathfrak{q}(t,\mathcal{T}t),\frac{1}{2}\mathfrak{q}(s,\mathcal{T}t),\mathfrak{q}(t,\mathcal{T}s)\right\}. \tag{6}$$

By considering the relation $\wedge=\mathsf{X}\times\mathsf{X}$, we obtain the following notion on metric spaces.

Definition 3.2. Let $(\mathsf{X},\mathfrak{q})$ be a metric space. A map $\mathcal{T}:\mathsf{X}\to\mathsf{X}$ is called an F-*quasi-contraction* if there exist $\tau\in\mathbb{R}^+$ and $F\in\mathcal{W}$ such that

$$\tau+F(\mathfrak{q}(\mathcal{T}s,\mathcal{T}t))\leq F(\max\mathfrak{M}(s,t))\ \forall s,t\in\mathsf{X}\text{ with }\mathcal{T}s\neq\mathcal{T}t. \tag{7}$$

Thus, every F-quasi-contraction is $\wedge_F$-quasi-contraction. But, the converse is not true in general as demonstrated by the example below.

Example 1. Let $\mathsf{X} = \{a, b, c, d\}$ and define the orthogonal relation $\wedge$ by

$$\wedge = \{(a, z) : z \in \mathsf{X}\} \subset \mathsf{X} \times \mathsf{X}.$$

Equip X by the metric $\mathfrak{q}$ defined by

$$\mathfrak{q}(a, b) = 1, \quad \mathfrak{q}(a, c) = 1, \quad \mathfrak{q}(a, d) = 2, \quad \mathfrak{q}(b, c) = 2, \quad \mathfrak{q}(b, d) = 3 \quad \text{and} \quad \mathfrak{q}(c, d) = 3. \tag{8}$$

Then, the map $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ defined by

$$\mathcal{T}(a) = a, \quad \mathcal{T}(b) = c, \quad \mathcal{T}(c) = d, \quad \mathcal{T}(d) = c$$

is an $\wedge_F$-quasi-contraction that is not F-quasi-contraction as described below.

- *$\mathcal{T}$ is not an F-quasi-contraction.* For the non-orthogonal pair (b, c), we have

$$\mathfrak{q}(\mathcal{T}b, \mathcal{T}c) = \mathfrak{q}(c, d) = 3, \quad \text{and} \quad \max \mathfrak{M}(b, c) = 3,$$

so the F-quasi-contractive inequality (7) implies

$$\tau \leq F(3) - F(3) = 0,$$

for any $F \in \mathcal{W}$. This is a contradiction. Hence, the claim.

- *$\mathcal{T}$ is an $\wedge_F$-quasi-contraction.* For the orthogonal pairs (a, z) with $\mathcal{T}a \neq \mathcal{T}z$ we have

$$\mathfrak{q}(\mathcal{T}a, \mathcal{T}b) = \mathfrak{q}(a, c) = 1 \quad \text{and} \quad \max \mathfrak{M}(a, b) = 2.$$
$$\mathfrak{q}(\mathcal{T}a, \mathcal{T}c) = \mathfrak{q}(a, d) = 2 \quad \text{and} \quad \max \mathfrak{M}(a, c) = 3.$$
$$\mathfrak{q}(\mathcal{T}a, \mathcal{T}d) = \mathfrak{q}(a, c) = 1 \quad \text{and} \quad \max \mathfrak{M}(a, d) = 3.$$

So, for $F(p) = \ln(p) \in \mathcal{W}$ and any $0 < \tau \leq \ln\left(\frac{3}{2}\right)$ the condition (5) holds, since

$$\tau + \ln(\mathfrak{q}(\mathcal{T}a, \mathcal{T}z)) \leq \ln(\max \mathfrak{M}(a, z)).$$

Hence the claim follows.

3.1.1. *F-quasi-contractions vs. F-weak contractions.* We notice that any F-contraction (which by Definition 2.11 satisfies $\tau + F(\mathfrak{q}(\mathcal{T}s, \mathcal{T}t)) \leq F(\mathfrak{q}(s, t)))$ is *simultaneously* an F-weak contraction and an F-quasi contraction since

$$\mathfrak{q}(s, t) \leq \max \mathfrak{U}(s, t), \quad \mathfrak{q}(s, t) \leq \max \mathfrak{M}(s, t), \quad \text{and} \quad F \in \mathcal{W} \text{ is an increasing function.}$$

However, the following example demonstrates that an F-quasi-contraction need not be an F-weak contraction or F-contraction.

Example 2. Consider the complete metric space $(\mathsf{X}, \mathfrak{q})$, where $\mathsf{X} = [0,1]$ and $\mathfrak{q}(s,t) = |s-t|$, and define the map $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ by

$$\mathcal{T}t = \begin{cases} \frac{1}{4}, & t \in [0,1); \\ \frac{3}{4}, & t = 1. \end{cases}$$

Then for $s \in [0,1)$ and $t = 1$ we have

$$\begin{aligned} &\mathfrak{q}(\mathcal{T}s, \mathcal{T}t) = \frac{1}{2}, \qquad \mathfrak{q}(s,t) = 1-s, \qquad \mathfrak{q}(s, \mathcal{T}s) = \left|s - \frac{1}{4}\right|, \qquad \mathfrak{q}(t, \mathcal{T}t) = \frac{1}{4}, \\ &\mathfrak{q}(s, \mathcal{T}t) = \left|s - \frac{3}{4}\right| \quad \text{and} \quad \mathfrak{q}(t, \mathcal{T}s) = \frac{3}{4}. \end{aligned} \tag{9}$$

So,

$$\max \mathfrak{M}(s,t) = \max\left\{1-s, \left|s-\frac{1}{4}\right|, \frac{1}{4}, \frac{1}{2}\left|s-\frac{3}{4}\right|, \frac{3}{4}\right\} = 1.$$

Letting $F(p) = \ln(p)$ (see (1)), then (7) gives

$$\tau \leq \ln(1) - \ln\left(\frac{1}{2}\right) = \ln(2).$$

Thus, for this F and $0 < \tau < \ln(2)$, the condition (7) is satisfied for all $s, t \in \mathsf{X}$ with $\mathcal{T}s \neq \mathcal{T}t$. Hence, $\mathcal{T}$ is an F-quasi-contraction. But, $\mathcal{T}$ is *neither* an F-contraction *nor* F-weak contraction, as explained below.

- By Remark 2.13, $\mathcal{T}$ is not an F-contraction because it is not a continuous function.
- Using (9) and $\mathfrak{U}(s,t)$ (2) we get

  $$\max \mathfrak{U}(s,t) = \max\left\{1-s, \left|s-\frac{1}{4}\right|, \frac{1}{4}, \frac{\left|s-\frac{3}{4}\right| + \frac{3}{4}}{2}\right\}.$$

  So, when $s = \frac{3}{4}$ we have $\max \mathfrak{U}(s,t) = \frac{1}{2}$. Consequently, for any $F \in \mathcal{W}$ the condition (4) gives

  $$\tau \leq F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}\right) = 0,$$

  which is a contradiction. Thus, $\mathcal{T}$ is not an F-weak contraction.

The following example demonstrates that the class of F-weak contractions is *not* contained within the class of F-quasi-contractions.

Example 3. Let $\mathsf{X} = \{x, y, z, u\}$ be a set equipped with a metric $\mathfrak{q}$ defined by

$$\begin{aligned} &\mathfrak{q}(x,y) = 10, \qquad &\mathfrak{q}(x,u) = 10, \qquad &\mathfrak{q}(y,z) = 10.2, \\ &\mathfrak{q}(x,z) = 20, \qquad &\mathfrak{q}(u,z) = 10.2, \qquad &\mathfrak{q}(y,u) = 1. \end{aligned}$$

Define the map $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ by

$$\mathcal{T}x = u \quad \text{and} \quad \mathcal{T}y = \mathcal{T}z = \mathcal{T}u = z.$$

Next, we analyze the conditions (4) and (7) for the pair $(s,t)=(x,y)$, noting that

$$\mathcal{T}x = u \neq \mathcal{T}y = z \quad \text{and} \quad \mathfrak{q}(\mathcal{T}x,\mathcal{T}y)=\mathfrak{q}(u,z)=10.2.$$

(1) $\mathcal{T}$ *is an F-weak contraction:* Recall the set $\mathfrak{U}(s,t)$ from (2). For the pair (x,y), we have:

$$\begin{aligned}\max \mathfrak{U}(x,y) &= \max\left\{\mathfrak{q}(x,y),\mathfrak{q}(x,u),\mathfrak{q}(y,z),\frac{\mathfrak{q}(x,z)+\mathfrak{q}(y,u)}{2}\right\},\\ &= \max\left\{10,10,10.2,\frac{20+1}{2}\right\} = \max\{10.2,10.5\}=10.5.\end{aligned}$$

If $F(p)=\ln(p)\in\mathcal{W}$, then the F-weak condition (4) becomes

$$\tau+\ln(10.2)\le\ln(10.5),$$

which holds for any $0<\tau\le\ln(10.5/10.2)$. Similar checks for other pairs (where $\mathcal{T}s\neq\mathcal{T}t$) confirm that $\mathcal{T}$ satisfies Definition 2.12.

(2) $\mathcal{T}$ *is not an F-quasi contraction:* Recall the set $\mathfrak{M}(s,t)$ from (6). For the same pair (x,y), we have:

$$\begin{aligned}\max \mathfrak{M}(x,y) \max&\left\{\mathfrak{q}(x,y),\mathfrak{q}(x,u),\mathfrak{q}(y,z),\frac{\mathfrak{q}(x,z)}{2},\mathfrak{q}(y,u)\right\},\\ &= \max\left\{10,10,10.2,\frac{20}{2},1\right\}=10.2.\end{aligned}$$

Consequently, the F-quasi condition (7) yields:

$$\tau+F(10.2)\le F(10.2),$$

for any $F\in\mathcal{W}$, implying $\tau\le 0$. This contradicts the condition $\tau>0$.

Whence, $\mathcal{T}$ is an F-weak contraction but not an F-quasi-contraction.

Remark 3.3. Example 2, in particular, shows that an $\wedge_F$-weak contraction need not be $\wedge_F$-quasi-contraction.

By modifying the distance between the images in the previous setup, we now show that the intersection of the classes of F-weak contractions and F-quasi-contractions is non-empty and contains mappings that are not F-contractions.

Example 4. Let $\mathsf{X}=\{x,y,z,u\}$ be a metric space with the distance function $\mathfrak{q}$ defined as follows:

$$\begin{array}{lll}\mathfrak{q}(x,y)=4, & \mathfrak{q}(x,u)=10, & \mathfrak{q}(y,z)=10,\\ \mathfrak{q}(x,z)=12, & \mathfrak{q}(u,z)=5, & \mathfrak{q}(y,u)=8.\end{array}$$

As previously, define the mapping $\mathcal{T}:\mathsf{X}\to\mathsf{X}$ by

$$\mathcal{T}x=u,\quad \text{and} \quad \mathcal{T}y=\mathcal{T}z=\mathcal{T}u=z.$$

We use the function $F(p)=\ln(p)\in\mathcal{W}$ and analyze the mapping for the pair (x,y), where $\mathfrak{q}(\mathcal{T}x,\mathcal{T}y)=\mathfrak{q}(u,z)=5$.

(1) $\mathcal{T}$ *is not an F-contraction:* By Remark 2.13, a mapping $\mathcal{T}$ is an F-contraction only if

$$\mathfrak{q}(\mathcal{T}s, \mathcal{T}t) < \mathfrak{q}(s,t) \quad \text{for all distinct} \quad s,t \in \mathsf{X} \quad \text{with} \quad \mathcal{T}s \neq \mathcal{T}t.$$

But, for the pair (x,y), we have

$$\mathfrak{q}(\mathcal{T}x, \mathcal{T}y) = 5 > 4 = \mathfrak{q}(x,y).$$

Hence, the claim.

(2) $\mathcal{T}$ *is an F-weak contraction:* We have

$$\max \mathfrak{U}(x,y) = \max\{4, 10, 10, 10\} = 10.$$

The condition (4) gives

$$\tau + \ln(5) \leq \ln(10) \quad \implies \tau \leq \ln(2).$$

(3) $\mathcal{T}$ *is an F-quasi-contraction:* We have

$$\max \mathfrak{M}(x,y) = \max\{4, 10, 10, 6, 8\} = 10.$$

The condition (7) gives

$$\tau + \ln(5) \leq \ln(10) \quad \implies \quad \tau \leq \ln(2).$$

So, the mapping $\mathcal{T}$ satisfies both the conditions for any $0 < \tau \leq \ln(2)$. Similar checks for other pairs (where $\mathcal{T}s \neq \mathcal{T}t$) confirm that $\mathcal{T}$ satisfies both Definitions 2.12 and 3.2.

Proposition 3.4. *Summarizing our discussions above, we get the following results:*

$\{F\text{-contractions}\} \subsetneq \{F\text{-quasi-contractions}\}$,

$\{F\text{-weak contractions}\} \not\subset \{F\text{-quasi-contractions}\}$, $\quad \{F\text{-quasi-contractions}\} \not\subset \{F\text{-weak contractions}\}$

and

$$\{F\text{-quasi-contractions}\} \cap (\{F\text{-weak contractions}\} - \{F\text{-contractions}\}) \neq \emptyset.$$

3.2. **Fixed point theorems.** We will use the following Lemma to prove our main results.

Lemma 3.5. *Let $(\mathsf{X}, \wedge, \mathfrak{q})$ be an orthogonal metric space, and $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ be an $\wedge$-preserving $\wedge_F$-quasi-contraction. Let $s_0 \in \mathsf{X}$ be an orthogonal element in $(\mathsf{X}, \wedge)$. Then the Picard sequence $\{s_n = \mathcal{T}^n s_0 : n \in \mathbb{Z}^+\}$ is a Cauchy O-sequence.*

Proof. Clearly, $\{s_n\}$ is an O-sequence since $\mathcal{T}$ is $\wedge$-preserving.

i. Note that if there exists an $N_0 \in Z^+$ such that $s_{N_0+1} = s_{N_0}$, then $\{s_n\}$ is Cauchy.

ii. Suppose $s_{n+1} \neq s_n$ for all $n \in \mathbb{Z}^+$. Since $\mathcal{T}$ is an $\wedge_F$-quasi-contraction, then using Definition 3.1 we have

$$F(\mathfrak{q}(s_n, s_{n+1})) = F(\mathfrak{q}(\mathcal{T}s_{n-1}, \mathcal{T}s_n)) \leq F(\max \mathfrak{M}(s_{n-1}, s_n)) - \tau \tag{10}$$

for each n, where

$$\mathfrak{M}(s_{n-1}, s_n) = \left\{\mathfrak{q}(s_{n-1}, s_n), \mathfrak{q}(s_{n-1}, s_n), \mathfrak{q}(s_n, s_{n+1}), \frac{1}{2}\mathfrak{q}(s_{n-1}, s_{n+1}), \mathfrak{q}(s_n, s_n)\right\}.$$

So, using Definition 2.8 ($\mathcal{W}_1$) and the triangle inequality we have

$$\begin{aligned} F(\mathfrak{q}(s_n, s_{n+1})) &\leq F(\max \mathfrak{M}(s_{n-1}, s_n)) - \tau, \\ &\leq F\left(\max\left\{\mathfrak{q}(s_{n-1}, s_n), \mathfrak{q}(s_n, s_{n+1}), \frac{\mathfrak{q}(s_{n-1}, s_n) + \mathfrak{q}(s_n, s_{n+1})}{2}\right\}\right) - \tau, \\ &= F\left(\max\left\{\mathfrak{q}(s_{n-1}, s_n), \mathfrak{q}(s_n, s_{n+1})\right\}\right) - \tau. \end{aligned}$$

This leads to two possibilities considered below.

(1) When $\max \mathfrak{M}(s_{n-1}, s_n) = \mathfrak{q}(s_n, s_{n+1})$. Then, (10) gives

$$F(\mathfrak{q}(s_n, s_{n+1})) \leq F(\mathfrak{q}(s_n, s_{n+1})) - \tau,$$

which is a contradiction since $\tau > 0$.

(2) When $\max \mathfrak{M}(s_{n-1}, s_n) = \mathfrak{q}(s_{n-1}, s_n)$. Then, (10) yields

$$F(\mathfrak{q}(s_n, s_{n+1})) \leq F(\mathfrak{q}(s_{n-1}, s_n)) - \tau \leq F(\mathfrak{q}(s_0, s_1)) - n\tau. \tag{11}$$

Consequently,

$$\lim_{n\to\infty} F(\mathfrak{q}(s_n, s_{n+1})) = -\infty.$$

Then, using Definition 2.8 we get

$$\lim_{n\to\infty} \mathfrak{q}(s_n, s_{n+1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} (\mathfrak{q}(s_n, s_{n+1}))^\sigma F(\mathfrak{q}(s_n, s_{n+1})) = 0. \tag{12}$$

By (11) and (12), we deduce that

$$\lim_{n\to\infty} n(\mathfrak{q}(s_n, s_{n+1}))^\sigma = 0.$$

This implies that there exists $n_0 \in \mathbb{Z}^+$ such that

$$\mathfrak{q}(s_n, s_{n+1}) \leq \frac{1}{n^{\frac{1}{\sigma}}} \quad \text{for all} \quad n \geq n_0. \tag{13}$$

Using (13) and the triangle inequality, we get

$$\mathfrak{q}(s_m, s_n) \leq \mathfrak{q}(s_m, s_{m-1}) + \ldots + \mathfrak{q}(s_{n+1}, s_n) < \sum_{j=n}^{\infty} \mathfrak{q}(s_j, s_{j+1}) \leq \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{\sigma}}},$$

for all $m, n \in \mathbb{Z}^+$ with $m > n \geq n_0$. Since for $0 < \sigma < 1$ we have $\sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{\sigma}}} < \infty$, then

$$\lim_{n,m\to\infty} \mathfrak{q}(s_m, s_n) = 0.$$

Hence, the sequence $\{s_n\}$ is a Cauchy O-sequence. □

Theorem 3.6. *Let* $(\mathsf{X}, \wedge, \mathfrak{q})$ *be an O-complete metric space, and* $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ *be an* $\wedge_F$*-quasi-contraction. If* $\mathcal{T}$ *is* $\wedge$*-preserving and* $\wedge$*-continuous, then* $\mathcal{T}$ *has a unique fixed point.*

Proof. Let $s_0 \in \mathsf{X}$ be an orthogonal element in $(\mathsf{X}, \wedge)$. Then

(1) $\mathcal{T}$ *admits a fixed point.* By Lemma 3.5, the Picard sequence $\{s_n = \mathcal{T}^n s_0 : n \in \mathbb{Z}^+\} \subset \mathsf{X}$ is Cauchy O-sequence. Then by the O-completeness of X, there exists a $\widehat{s} \in \mathsf{X}$ such that

$$\lim_{n\to\infty} s_n = \widehat{s}.$$

Since $\mathcal{T}$ is $\wedge$-continuous, then we have

$$\widehat{s} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \mathcal{T} s_{n-1} = \mathcal{T} \lim_{n\to\infty} s_{n-1} = \mathcal{T}\widehat{s}.$$

Thus, $\widehat{s}$ is a fixed point of $\mathcal{T}$.

(2) $\widehat{s}$ *is the only fixed point of* $\mathcal{T}$. On the contrary, suppose that $s^* \in \mathsf{X}$ is another fixed point of $\mathcal{T}$. Since $\mathcal{T}$ is $\wedge$-preserving, then we have

$$(\mathcal{T}^n s_0, \mathcal{T}^n s^*) = (s_n, s^*) \in \wedge \subset \mathsf{X} \times \mathsf{X}. \tag{14}$$

Using (5), we get

$$F(\mathfrak{q}(s_n, s^*)) = F(\mathfrak{q}(\mathcal{T} s_{n-1}, \mathcal{T} s^*)) \leq F(\max \mathfrak{M}(s_{n-1}, s^*)) - \tau, \tag{15}$$

where

$$\mathfrak{M}(s_{n-1}, s^*) = \left\{\mathfrak{q}(s_{n-1}, s^*), \mathfrak{q}(s_{n-1}, s_n), \mathfrak{q}(s^*, s^*), \frac{1}{2}\mathfrak{q}(s_{n-1}, s^*), \mathfrak{q}(s_n, s^*)\right\}.$$

So, using Definition 2.8 (W_1) and the triangle inequality, (15) becomes

$$F(\mathfrak{q}(s_n, s^*)) = F(\mathfrak{q}(\mathcal{T} s_{n-1}, \mathcal{T} s^*)) \leq F\left(\max\{\mathfrak{q}(s_{n-1}, s^*), \mathfrak{q}(s_{n-1}, s_n), \mathfrak{q}(s_n, s^*)\}\right) - \tau. \tag{16}$$

To proceed further, there are three cases to consider.

i. If $\max \mathfrak{M}(s_{n-1}, s^*) = \mathfrak{q}(s_n, s^*)$ for all $n \in \mathbb{Z}^+$, then (16) becomes

$$F(\mathfrak{q}(s_n, s^*)) \leq F(\mathfrak{q}(s_n, s^*)) - \tau,$$

which contradicts the condition $\tau > 0$.

ii. If $\max \mathfrak{M}(s_{n-1}, s^*) = \mathfrak{q}(s_n, s_{n-1})$ for all $n \in \mathbb{Z}^+$, then (16) becomes

$$F(\mathfrak{q}(s_n, s^*)) \leq F(\mathfrak{q}(s_n, s_{n-1})) - \tau \leq F(\mathfrak{q}(s_1, s_0)) - n\tau, \tag{17}$$

which implies

$$\lim_{n\to\infty} F(s_n, s^*) = -\infty.$$

Then, by Definition 2.8 (W_2), we have

$$0 = \lim_{n\to\infty} \mathfrak{q}(s_n, s^*) = \mathfrak{q}\left(\lim_{n\to\infty} s_n, s^*\right) = \mathfrak{q}(\widehat{s}, s^*).$$

Thus, $\widehat{s} = s^*$.

iii. If $\max \mathfrak{M}(s_{n-1}, s^*) = \mathfrak{q}(s_{n-1}, s^*)$ for all $n \in \mathbb{Z}^+$, then (16) becomes

$$F(\mathfrak{q}(s_n, s^*)) \leq F(\mathfrak{q}(s_{n-1}, s^*)) - \tau \leq F(\mathfrak{q}(s_0, s^*)) - n\tau,$$

which implies

$$\lim_{n\to\infty} F(s_n, s^*) = -\infty.$$

So, proceeding as in the case above, we conclude that $\widehat{s} = s^*$.

$\square$

Example 5. The map $\mathcal{T}$ in each of Examples 1, 2 and 4 satisfies all the hypotheses of Theorem 3.6, and therefore has a unique fixed point.

Definition 3.7. An O-sequence $\{t_n : n \in \mathbb{Z}^+\}$ in an orthogonal metric space $(\mathsf{X}, \wedge, \mathfrak{q})$ is said to be orthogonally convergent (*O-convergent*) if there exists $\widehat{t} \in \mathsf{X}$ such that

$$\lim_{n\to\infty} t_n = \widehat{t} \quad \text{implies that} \quad (t_n, \widehat{t}) \in \wedge \quad \text{or} \quad (\widehat{t}, t_n) \in \wedge \quad \forall n \in \mathbb{Z}^+.$$

Using the above notion, we formulate our second main result:

Theorem 3.8. *Let* $(\mathsf{X}, \wedge, \mathfrak{q})$ *be an O-complete metric space. Let* $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ *be an* $\wedge$*-preserving and an* $\wedge_F$*-quasi-contraction with a* continuous *function* $F \in \mathcal{W}$*. If the Picard sequence*

$$\{s_n = \mathcal{T}^n s_0 : s_0 \textit{ is an orthogonal element}\} \subset \mathsf{X}$$

is O-convergent whose limit is $\widehat{s}$*, then* $\widehat{s}$ *is the only fixed point of* $\mathcal{T}$*.*

Proof. As explained at the beginning of the proof for Theorem 3.6, for the Picard sequence $\{s_n = \mathcal{T}^n s_0\}$, where $s_0 \in \mathsf{X}$ is an orthogonal element, there exists $\widehat{s} \in \mathsf{X}$ such that

$$\lim_{n\to\infty} s_n = \widehat{s}.$$

Assuming that $\{s_n : n \in \mathbb{Z}^+\}$ is O-convergent, we next prove that $\widehat{s}$ is a fixed point. The uniqueness of $\widehat{s}$ can be established by following the corresponding part in the proof of Theorem 3.6. We proceed in two cases.

(1) For each $n \in \mathbb{Z}^+$, there exists $k_n \in \mathbb{Z}^+$ such that

$$s_{k_n+1} = \mathcal{T}\widehat{s} \quad \text{with} \quad k_n > k_{n-1} \quad \text{and} \quad k_0 = 1.$$

This gives

$$\widehat{s} = \lim_{n\to\infty} s_{k_n+1} = \lim_{n\to\infty} \mathcal{T}\widehat{s} = \mathcal{T}\widehat{s}.$$

Hence, $\widehat{s}$ is a fixed point of $\mathcal{T}$.

(2) There exists $j_0 \in \mathbb{Z}^+$ such that

$$s_{n+1} \neq \mathcal{T}\widehat{s} \quad \forall n \geq j_0.$$

Since $(s_n, \widehat{s}) \in \wedge$ for all n and $\mathcal{T}$ is $\wedge$-preserving, then $(s_{n+1}, \mathcal{T}\widehat{s}) \in \wedge$. Using (5),we get

$$F(\mathfrak{q}(\mathcal{T}s_n, \mathcal{T}\widehat{s})) = F(\mathfrak{q}(s_{n+1}, \mathcal{T}\widehat{s})) \leq F(\max \mathfrak{M}(s_n, \widehat{s})) - \tau, \tag{18}$$

where

$$\mathfrak{M}(s_n, \widehat{s}) = \left\{\mathfrak{q}(s_n, \widehat{s}), \mathfrak{q}(s_n, s_{n+1}), \mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s}), \frac{1}{2}\mathfrak{q}(s_n, \mathcal{T}\widehat{s}), \mathfrak{q}(s_{n+1}, \widehat{s})\right\}.$$

Notice that

$$\lim_{n\to\infty} \mathfrak{q}(s_n, \widehat{s}) = \lim_{n\to\infty} \mathfrak{q}(s_n, s_{n+1}) = \lim_{n\to\infty} \mathfrak{q}(s_{n+1}, \widehat{s}) = 0.$$

So, if we suppose that $\mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s}) > 0$, then there exists $j_1 \in \mathbb{Z}^+$ such that

$$\max \mathfrak{M}(s_n, \widehat{s}) = \mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s}) \quad \text{for all} \quad n \geq j_1.$$

Then (18) gives

$$F(\mathfrak{q}(s_{n+1}, \mathcal{T}\widehat{s})) \leq F(\mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s})) - \tau \quad \text{for all} \quad n \geq \max\{j_0, j_1\}. \tag{19}$$

Taking the limit $n \to \infty$ in (19) gives

$$F(\mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s})) \leq F(\mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s})) - \tau,$$

since F is continuous. This contradicts the condition $\tau > 0$. So, $\mathfrak{q}(\widehat{s}, \mathcal{T}\widehat{s}) = 0$. Hence, $\widehat{s}$ is a fixed point.

□

Remark 3.9. Unlike in Theorem 3.6, the requirement that an $\wedge_F$-quasi-contraction is $\wedge$-continuous is relaxed in Theorem 3.8.

By considering the relation $\wedge = \mathsf{X} \times \mathsf{X}$ in Theorems 3.6 and 3.8, we obtain the following results on metric spaces that appear to be new in the literature.

Corollary 3.10. *Let X be a complete metric space. If $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ be a continuous F-quasi-contraction, then $\mathcal{T}$ has a unique fixed point.*

Corollary 3.11. *Let X be a complete metric space. If $\mathcal{T} : \mathsf{X} \to \mathsf{X}$ be an F-quasi-contraction with a* continuous *function $F \in \mathcal{W}$, then T admits a unique fixed point.*

4. Application to Volterra-Type Integro-Differential Equations

Volterra-type integro-differential equations arise in various fields, including epidemic modeling [17, 18], control theory [19] and physics [20]. These equations combine differential and integral terms to capture memory effects in dynamic systems [16].

In this section, we demonstrate how our results can be applied to establish the existence and uniqueness of solutions to such equations.

We consider the initial value problem (IVP):

$$\begin{cases} u'(t) = f(t, u(t)) + \int_0^t K(t,s)u(s)\,ds, \quad t \in [0, a], \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{20}$$

where

$$f:[0,a]\times\mathbb{R}\to\mathbb{R} \quad \text{and} \quad K:[0,a]\times[0,a]\to\mathbb{R}$$

are given continuous functions.

Theorem 4.1. *There exists a unique continuous function $\widehat{u}:[0,a]\to\mathbb{R}$ that solves the IVP (20) provided that*

$$|f(t,x)-f(t,y)|\leq k_1|x-y|e^{-\delta|x-y|} \quad \textit{for} \quad \delta>0,\ k_1>0, \tag{21}$$

$$|K(t,s)|\leq k_2 \quad \textit{for} \quad k_2>0, \quad \textit{and} \tag{22}$$

$$a\left(k_1+\frac{k_2a}{2}\right)<1. \tag{23}$$

Proof. Let $\mathsf{X}=C([0,a]):=\{g:[0,a]\to\mathbb{R}: g$ is a continuous function$\}$, and on X define the metric $\mathfrak{q}$ by

$$\mathfrak{q}(x,y)=\sup_{t\in[0,a]}|x(t)-y(t)|.$$

Then $(\mathsf{X},\mathfrak{q})$ is a complete metric space [14].

Note that integrating (20) gives

$$u(t)=u_0+\int_0^t f(s,u(s))\,ds+\int_0^t\left(\int_0^v K(v,s)\,u(s)\,ds\right)dv.$$

Now, define the operator $\mathcal{T}:\mathsf{X}\to\mathsf{X}$ by

$$(\mathcal{T}u)(t)=u_0+\int_0^t f(s,u(s))\,ds+\int_0^t\int_0^v K(v,s)u(s)\,ds\,dv. \tag{24}$$

Next, we verify the hypotheses of Corollary 3.10.

Clearly, $\mathcal{T}$ is a continuous map. So, it remains to show that $\mathcal{T}$ is an F-quasi-contraction.

For any $x,y\in\mathsf{X}$ with $\mathcal{T}x\neq\mathcal{T}y$, assuming (21) and (22), we have

$$\begin{aligned}|(\mathcal{T}x)(t)-(\mathcal{T}y)(t)| &\leq \int_0^t|f(s,x(s))-f(s,y(s))|ds+\int_0^t\int_0^v|K(v,s)||x(s)-y(s)|dsdv,\\ &\leq k_1\mathfrak{q}(x,y)\int_0^t e^{-\delta\mathfrak{q}(x,y)}\,ds+k_2\mathfrak{q}(x,y)\int_0^t\int_0^v ds\,dv,\\ &\leq\left(k_1ae^{-\delta\mathfrak{q}(x,y)}+\frac{k_2a^2}{2}\right)\mathfrak{q}(x,y)\leq\left(k_1a+\frac{k_2a^2}{2}\right)\mathfrak{q}(x,y),\end{aligned}$$

since $e^{-\delta\mathfrak{q}(x,y)}\leq 1$.

Thus,

$$\mathfrak{q}(\mathcal{T}x,\mathcal{T}y)\leq a\left(k_1+\frac{k_2a}{2}\right)\mathfrak{q}(x,y).$$

Let $\beta=a\left(k_1+\frac{k_2a}{2}\right)$. Then we have

$$\mathfrak{q}(\mathcal{T}x,\mathcal{T}y)\leq\beta\mathfrak{q}(x,y)\leq\beta\max\mathfrak{M}(x,y).$$

So, for $F(p) = \ln(p) \in \mathcal{W}$ we have

$$\tau + F(\mathfrak{q}(\mathcal{T}x, \mathcal{T}y)) \leq F(\max \mathfrak{M}(x, y)) \quad \text{where} \quad \tau = -\ln(\beta) > 0,$$

since by (23) we have $\beta < 1$.

Hence, (7) is satisfied, and the operator $\mathcal{T}$ (24) is a continuous F-quasi-contraction on X. Consequently, by Corollary 3.10, $\mathcal{T}$ has a unique fixed point, which is the only solution to the IVP (20). □

5. Conclusion

In this paper, we have introduced the notion of orthogonal F-quasi-contraction on orthogonal metric spaces (Definition 3.1) and established fixed point results for such mappings (Theorems 3.6 and 3.8). By specializing the orthogonality relation to the universal one, we also obtained new fixed point results for F-quasi-contractions on complete metric spaces (Corollaries 3.10 and 3.11). We have constructed examples showing that the newly introduced notion overlaps with and extends the F-weak contractions and F-contractions notions to handle more complex, non-continuous, or asymmetrical mapping behaviors. (Proposition 3.4).

As a concrete application, we proved the existence and uniqueness of a continuous solution to a general Volterra-type integro-differential equation (Theorem 4.1) by means of Corollary 3.10.

The results extend and unify several lines of research in fixed point theory and illustrate the usefulness of the orthogonal framework even when the orthogonality relation is taken to be universal. Future work may include the study of multivalued versions, common fixed point theorems for pairs of such maps, etc.

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