Eur. J. Math. Anal. 6 (2026) 14
doi: 10.28924/ada/ma.6.14

Boas-Type Theorems for the Mehler-Fock-Clifford Transform

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Abstract. In this paper, we give necessary and sufficient conditions in terms of $M(f)$, the Mehler-Fock-Clifford transform of f , to ensure that f belongs either to one of the generalized Lipschitz classes H_α^m and h_α^m for $\alpha > 0$.

1. Introduction and preliminaries

Boas results are one of the classical topics in harmonic analysis and approximation theory which consists in finding necessary and sufficient conditions on the Fourier coefficients of a function to belong to a generalised Lipschitz class.
In 1967, Boas proved the first characterisation of this type, see [5]. Since then, this theory has been widely studied by several authors.
In [5] and [6], the author has studied the continuity and smoothness properties of a function f with absolutely convergent Fourier series. He gave the best possible sufficient conditions in terms of the Fourier coefficients of f which ensure the belonging of f either to one of the Lipschitz classes $Lip(\alpha)$ and $lip(\alpha)$ for some $0 < \alpha \leq 1$.
Tikhonov, in [8] and [9], considered the cases of cosine and sine series separately. Recently,Volosivets have published two papers.
In this paper, we will see an analogue of this theory for the Mehler Fock-Clifford.
We will try to complete the work done in our research laboratory on this theory, see [10] and [11].
To do this, we will start by recalling the necessary results of this transformation that we need to study this theory.
Let $L^p(\mathbb{R}^+, d\mu)$ the space of measurable function on $[0; +\infty[$ satisfying:

$$\|f\|_p = \left(\int_0^{+\infty} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}} < +\infty,$$

Received: 29 Mar 2026.
Key words and phrases. Mehler-Fock-Clifford transform, Generalized translation operator, Boas theorems.

if $1 \le p < +\infty$ and

$$\|f\|_\infty = esssup_{x\in\mathbb{R}^+}|f(x)|,$$

if $p = +\infty$ and μ is a positive measure.

$Loc(\mathbb{R}^+)$ the space of locally integrable functions.

Definition 1.1. *$P_{i\sqrt{\lambda}-\frac{1}{2}}(x)$ is cone function (associated Legendre function of zero order) and it is represented in terms of Gaussian hypergeometric function ${}_2F_1$ as:*

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(x) ={}_2 F_1\Big(\frac{1}{2}+i\sqrt{\lambda}, \frac{1}{2}-i\sqrt{\lambda}; 1; \frac{(1-x)}{2}\Big).$$

We are now going to see some properties of kernel $P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})$ cited in [4]:

Lemma 1.2. *For every positive integer m there exists $M > 0$ such that:*

$$\Big|\frac{d^m}{dx^m}P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big| \le M\cosh(\pi\sqrt{\lambda}).$$

Lemma 1.3.

$$\Big|P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big| \le P_{-\frac{1}{2}}(2\sqrt{x}).$$

Using asymptotic behaviours of $P_{-\frac{1}{2}}(2\sqrt{x})$, we have:

$$\Big|P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big| \le C.$$

The theory and properties of Hankel-Clifford transform have already been studied by the several researchers via [1] [2] and [3]. As per this argument the Legendre-Clifford function according to [?] is defined as:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) = \frac{\sqrt{2}}{\pi}\cosh(\pi\sqrt{\lambda})\int_0^\infty \frac{\cos(\sqrt{\lambda}t)}{\sqrt{2\sqrt{x}+\cosh t}}dt.$$

Now we define the Mehler-Fock-Clifford (MFC) transform as:

$$F(\lambda) = (Mf)(\lambda) = \int_{\frac{1}{4}}^\infty f(x)P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\frac{dx}{\sqrt{x}}\ , \quad \lambda > 0\ ,$$

and its inversion:

$$f(x) = (M^{-1}F)(x) = \frac{1}{2}\int_0^\infty \tanh(\pi\lambda)P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})F(\lambda)d\lambda\ , \quad x > \frac{1}{4}\ .$$

We will now move on to defining the operator that will play the role of the translation operator. We define the symmetric function $D(x,y,z) \ge 0$ as:

$$D(x,y,z) = \int_0^\infty \tanh(\pi\sqrt{\lambda})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{y})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{z})d\lambda,$$

where $D(x,y,y) = \frac{1}{\pi}(16\sqrt{xyz}+1-4x-4y-4z)^{-\frac{1}{2}}$ for $z \in I_{x,y}$ and $D(x,y,z) = 0$ otherwise with

$$I_{x,y} = \Big]4\sqrt{xy} - \Big[(4x-1)(4y-1)\Big]^{\frac{1}{2}}; 4\sqrt{xy} + \Big[(4x-1)(4y-1)\Big]^{\frac{1}{2}}\Big[.$$

Now using the inversion of MFC-transform, then we have:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{y}) = \int_{\frac{1}{4}}^{\infty} D(x,y,z)P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{z})\frac{dz}{\sqrt{z}},$$

and

$$\int_{\frac{1}{4}}^{\infty} D(x,y,z)\frac{dz}{\sqrt{z}} = 1.$$

The translation operator is defined as:

$$(T_h f)(y) = \int_{\frac{1}{4}}^{\infty} D(x,y,z)f(z)\frac{dz}{\sqrt{z}},$$

and we see that:

$$P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{y}) = \Big(T_h(P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{z}))\Big)(y).$$

Theorem 1.4. *[4] If $f \in L^1(]\frac{1}{4};+\infty;x^{-\frac{1}{2}}dx)$, then the MFC-transform of the translation operator checks the following property:*

$$M(T_h f)(\lambda) = P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})(Mf)(\lambda).$$

If $f \in L^1(\mathbb{R}^+, d\mu)$ then f verifies the propreties of Theorem 1.4

2. Main result

We define differences of the order m, $m \in \mathbb{N}$ by

$$\Delta_h^m f(x) = \Big[\Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}\Big)I + T_h\Big]^m f(x).$$

We have that:

$$M(\Delta_h^m f)(\lambda) = \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda).$$

By using de formula of inversion of Mehler-Fock-Clifford we get:

$$\Delta_h^m f(x) = \frac{1}{2}\int_0^{+\infty} \tanh(\pi\sqrt{\lambda})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda)d\lambda.$$

So:

$$|\Delta_h^m f(x)| \leq \frac{1}{2}\int_0^{+\infty} \Big|\tanh(\pi\sqrt{\lambda})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda)\Big|d\lambda.$$

In what follows, to simplify the writing of equalities, we will denote:

$$dv_x(\lambda) = \tanh(\pi\sqrt{\lambda})|P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})|d\lambda.$$

Since $\tanh(\lambda) \geq 0$ for every $\lambda \in \mathbb{R}^+$, then v_x is a positive measure. Then:

$$|\Delta_h^m f(x)| \leq \frac{1}{2}\int_0^{+\infty} \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m |M(f)(\lambda)|dv_x(\lambda).$$

Now, we define the generalized Lipschitz H_α^m and h_α^m.

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said belong to H_α^m for $\alpha > 0$ if:

$$\delta_h^m f(x) = \int_0^{+\infty} \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m |M(f)(\lambda)|dv_x(\lambda) < +\infty,$$

and

$$\delta_h^m f(x) = O(h^\alpha),$$

as $h \to 0$, and is said belong to h_α^m for $\alpha > 0$ if:

$$\delta_h^m f(x) = \int_0^{+\infty} \left(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\right)^m |M(f)(\lambda)| dv_x(\lambda) < +\infty,$$

and

$$\delta_h^m f(x) = o(h^\alpha),$$

as $h \to 0$.
It's clear to see that if $f \in H_\alpha^m$, then:

$$|\Delta_h^m f(x)| = O(h^\alpha),$$

and if $f \in h_\alpha^m$, then:

$$|\Delta_h^m f(x)| = o(h^\alpha).$$

We know that:

$$|P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})| \leq P_{-\frac{1}{2}}(2\sqrt{h}).$$

So

$$-P_{-\frac{1}{2}}(2\sqrt{h}) \leq P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) \leq P_{-\frac{1}{2}}(2\sqrt{h}).$$

Then:

$$1 \leq 1 + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h}) + P_{-\frac{1}{2}}(2\sqrt{h}) \leq 1 + 2P_{-\frac{1}{2}}(2\sqrt{h}).$$

Now, we will prove a result that will be used to prove the first result in this paper.

Lemma 2.1. *Let g be a non-negative, measurable function defined on $\mathbb{R}^+$, and $m \in \mathbb{N}$. Then:*

(1) *if $0 < \alpha \leq m$ is given number, $x^m g(x) \in L^1(\mathbb{R}^+, d\mu) \cap L_{loc}(\mathbb{R}^+)$ such that μ is a positive measure and*

$$\int_0^y x^m g(x) d\mu(x) = O(y^{m-\alpha}), \tag{2.1}$$

for every $y > 0$. Then $g \in L^1(]y; +\infty[, d\mu)$ and

$$\int_y^{+\infty} g(x) d\mu(x) = O(y^{-\alpha}), \tag{2.2}$$

for every $y > 0$.

(2) *Conversely, if $0 \leq \alpha < m$ and (2.2) holds, then (2.1) also holds.*

Proof.

(1) By (2.1) we have that:

$$\int_{2^k}^{2^{k+1}} x^m g(x)d\mu(x) \leq \int_0^{2^{k+1}} x^m g(x)d\mu(x) \leq C(2^{k+1})^{m-\alpha}.$$

So

$$\int_{2^k}^{2^{k+1}} x^m g(x)d\mu(x) \leq C2^{m-\alpha-k\alpha+mk}.$$

Since

$$2^{km}\int_{2^k}^{2^{k+1}} g(x)d\mu(x) \leq \int_{2^k}^{2^{k+1}} x^m g(x)d\mu(x).$$

We get

$$\int_{2^k}^{2^{k+1}} g(x)d\mu(x) \leq C2^{m-\alpha-k\alpha}$$

$$\begin{aligned}\int_{2^k}^{+\infty} g(x)d\mu(x) &= C2^{m-\alpha}\Big(2^{-\alpha k} + 2^{-\alpha(k+1)} + ...\Big),\\ &= C2^{m-\alpha}\frac{1}{1-2^{-\alpha}}2^{-\alpha k},\\ &= O(2^{-\alpha k}).\end{aligned}$$

If $0 < y < +\infty$, let $k \in \mathbb{Z}$ such that $2^k \leq y \leq 2^{k+1}$, so

$$\begin{aligned}\int_y^{+\infty} g(x)d\mu(x) &\leq \int_{2^k}^{+\infty} g(x)d\mu(x),\\ &\leq C_1 2^{-\alpha k},\\ &= 2^\alpha C_1 2^{-\alpha(k+1)},\\ &\leq 2^\alpha C_1 y^{-\alpha}.\end{aligned}$$

Then

$$\int_y^{+\infty} g(x)d\mu(x) = O(y^{-\alpha}),$$

and

$$g \in L^1(]y;+\infty[, d\mu).$$

(2) By (2.2), there exists $c > 0$ such that:

$$\int_{2^{k-1}}^{2^k} g(x)d\mu(x) \leq \int_{2^{k-1}}^{+\infty} g(x)d\mu(x) \leq C(2^{k-1})^{-\alpha}.$$

So:

$$\int_{2^{k-1}}^{2^k} x^m g(x)d\mu(x) \leq 2^{km}\int_{2^{k-1}}^{+\infty} g(x)d\mu(x) \leq C(2^{k-1})^{-\alpha}.$$

Then:

$$\begin{aligned}\int_{2^{k-1}}^{2^k} g(x)d\mu(x) &\leq C_2\Big(2^{k(m-\alpha)} + 2^{(k+1)(m-\alpha)} + ...\Big),\\ &\leq C_2 2^{(m-\alpha)k}\frac{1}{1-2^{m-\alpha}}2^{\alpha},\\ &= 2^{\alpha}C_1 2^{-\alpha(k+1)},\\ &= O\Big(2^{(m-\alpha)k}\Big).\end{aligned}$$

If $0 < y < +\infty$, let $k \in \mathbb{Z}$ such that $2^k \leq y \leq 2^{k+1}$, so

$$\begin{aligned}\int_0^y x^m g(x)d\mu(x) &\leq \int_0^{2^{k+}} x^m g(x)d\mu(x),\\ &\leq C_2 2^{(k+1)(m-\alpha)},\\ &= C_2 2^{(m-\alpha)}2^{k(m-\alpha)},\\ &\leq C_2 2^{(m-\alpha)}y^{m-\alpha}.\end{aligned}$$

Which shows that:

$$\int_0^y x^m g(x)d\mu(x) = O(y^{m-\alpha}).$$

■ We will see the first main result of this paper.

Theorem 2.2. *Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $m \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^+, d\mu)$ such that μ is a positive measure.*

(1) *Suppose that $f \in L^1(\mathbb{R}^+, d\mu) \cap C(\mathbb{R}^+)$. If*

$$\int_0^y |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) = O(y^{2m-\alpha}), \tag{2.3}$$

for every $y > 0$. Then $M(f) \in L^1(]y; +\infty[, dv_x(\lambda))$ and $f \in H^m_\alpha$

(2) *Conversely, suppose that $f \in L^1(\mathbb{R}^+, d\mu)$, $M(f) \in L^1(]y; +\infty[, dv_x(\lambda))$. If $f \in H^\alpha_m$ then (2.3) holds.*

Remark 2.3. *From Lemma 2.1 it follows that:*
The statement (2.3) implies

$$\int_y^{+\infty} |M(f)(\lambda)|dv_x(\lambda) = O(y^{-\alpha}),$$

and

$$\delta_h^m f(x) < +\infty$$

Proof.

(1) by Remark 2.3 we have

$$\int_y^{+\infty} |M(f)(\lambda)|dv_x(\lambda) = O(y^{-\alpha}),$$

and

$$M(f) \in L^1(]y; +\infty[, dv_x(\lambda)).$$

We have that:

$$\delta_h^m f(x) = \int_0^{+\infty} \Big| \tanh(\pi\sqrt{\lambda}) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda) \Big| d\lambda.$$

Then

$$\delta_h^m f(x) = I_1 + I_2,$$

such that:

$$I_1 = \int_0^{\frac{1}{h}} \Big| \tanh(\pi\sqrt{\lambda}) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda) \Big| d\lambda,$$

and

$$I_2 = \int_{\frac{1}{h}}^{+\infty} \Big| \tanh(\pi\sqrt{\lambda}) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda) \Big| d\lambda.$$

By propreties of $P_{i\sqrt{\lambda}-\frac{1}{2}}$ we have that:

$$I_2 \leq (1+2M)^m \int_{\frac{1}{h}}^{+\infty} \Big| \tanh(\pi\sqrt{\lambda}) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) M(f)(\lambda) \Big| d\lambda.$$

i.e

$$I_2 \leq (1+2M)^m \int_{\frac{1}{h}}^{+\infty} |M(f)(\lambda)| dv_x(\lambda).$$

By hypothesis we have

$$\int_0^{\frac{1}{h}} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) = O(\frac{1}{h^{2m-\alpha}}).$$

By Lemma 2.1 we get:

$$\int_{\frac{1}{h}}^{+\infty} |M(f)(\lambda)| dv_x(\lambda) = O(h^\alpha).$$

To estimate I_1, set

$$\phi(x) = \int_x^{\infty} |M(f)(\lambda)| dv_x(\lambda).$$

We have that

$$\int_0^x \lambda\phi'(\lambda) d\lambda \leq x \int_0^x \phi'(\lambda) d\lambda.$$

So:

$$\int_0^x \lambda\phi'(\lambda) d\lambda \leq x\Big(\phi(x) - \phi(0)\Big).$$

It follows that:

$$\int_0^x \lambda\phi'(\lambda) d\lambda \leq -x \int_0^x |M(f)(\lambda)| dv_x(\lambda).$$

Then:

$$\int_0^x |M(f)(\lambda)| dv_x(\lambda) \leq \frac{-1}{x} \int_0^x \lambda\phi'(\lambda) d\lambda.$$

Upon an integration by parts, we obtain:

$$\int_0^x \lambda\phi'(\lambda) d\lambda = x\phi(x) - \int_0^x \phi(\lambda) d\lambda.$$

So:

$$\frac{-1}{x} \int_0^x \lambda\phi'(\lambda) d\lambda = \frac{1}{x} \int_0^x \phi(\lambda) d\lambda - \phi(x).$$

Then

$$\frac{-1}{x}\int_0^x \lambda\phi'(\lambda)d\lambda \leq \frac{1}{x}\int_0^x \phi(\lambda)d\lambda.$$

So, if $x \to +\infty$

$$\frac{-1}{x}\int_0^x \lambda\phi'(\lambda)d\lambda \leq \frac{1}{x}\int_0^x O(\lambda^{-\alpha})d\lambda = O(x^{-\alpha}).$$

Then

$$\int_0^x |M(f)(\lambda)|dv_x(\lambda) = O(x^{-\alpha}).$$

So

$$\int_0^{\frac{1}{h}} |M(f)(\lambda)|dv_x(\lambda) = O(h^{\alpha}).$$

It follows that

$$I_1 = O(h^{\alpha}).$$

Then

$$\delta_h^m f(x) = O(h^{\alpha}) + O(h^{\alpha}).$$

So

$$\delta_h^m f(x) = O(h^{\alpha}).$$

Which show that

$$f \in H_\alpha^m.$$

(2) We have that:

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) \leq \left(\frac{\beta}{h}\right)^{2m}\int_0^{\frac{\beta}{h}} |M(f)(\lambda)|dv_x(\lambda).$$

So

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) \leq \left(\frac{\beta}{h}\right)^{2m}\int_0^{+\infty} |M(f)(\lambda)|dv_x(\lambda).$$

Since

$$1 \leq \left(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\right)^m.$$

Then

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) \leq \left(\frac{\beta}{h}\right)^{2m}\int_0^{+\infty} \left(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\right)^m |M(f)(\lambda)|dv_x(\lambda).$$

Since $f \in H_\alpha^m$, so

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) = \left(\frac{\beta}{h}\right)^{2m} O(h^{\alpha}).$$

Hence, for $y = \frac{\beta}{h}$, we get:

$$\int_0^{y} |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) = O\left(\left(\frac{\beta}{y}\right)^{\alpha-2m}\right).$$

So:

$$\int_0^{y} |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) = O(y^{2m-\alpha}).$$

Thus, (2.3) is holds.

■ Now, to see the second main result of this paper, we will state and prove the following result:

Lemma 2.4. *Let g be a non-negative, measurable function defined on $\mathbb{R}^+$, and $m \in \mathbb{N}$. Then:*

(1) *if $0 < \alpha < m$ is given number, $x^m g(x) \in L^1(\mathbb{R}^+, d\mu) \cap L_{loc}(\mathbb{R}^+)$ such that μ is a positive measure and*
$$\int_0^y x^m g(x) d\mu(x) = o(y^{m-\alpha}), \tag{2.4}$$
as $y \to +\infty$. Then $g \in L^1(]y; +\infty[, d\mu)$ for y large and
$$\int_y^{+\infty} g(x) d\mu(x) = o(y^{-\alpha}), \tag{2.5}$$
as $y \to +\infty$.

(2) *Conversely, if $0 < \alpha \leq m$ and (2.2),(2.5) holds, then (2.4) also holds.*

Proof.

(1) By (2.4) we have for every $\epsilon > 0$, $\exists k_0 \in \mathbb{Z}$ such that:
$$\int_{2^k}^{2^{k+1}} x^m g(x) d\mu(x) \leq \int_0^{2^{k+1}} x^m g(x) d\mu(x) \leq \epsilon (2^{k+1})^{m-\alpha}.$$
For every $k \geq k_0$. Let $k \geq k_0$, we have
$$\int_{2^k}^{2^{k+1}} x^m g(x) d\mu(x) \leq \epsilon 2^{m-\alpha-k\alpha+mk}.$$
Since
$$2^{km} \int_{2^k}^{2^{k+1}} g(x) d\mu(x) \leq \int_{2^k}^{2^{k+1}} x^m g(x) d\mu(x).$$
We get
$$\int_{2^k}^{2^{k+1}} g(x) d\mu(x) \leq \epsilon 2^{m-\alpha-k\alpha}.$$
So
$$\begin{aligned}\int_{2^k}^{+\infty} g(x) d\mu(x) &\leq \epsilon 2^{m-\alpha}\Big(2^{-\alpha k} + 2^{-\alpha(k+1)} + ...\Big),\\ &= \epsilon 2^{m-\alpha} \frac{1}{1-2^{-\alpha}} 2^{-\alpha k},\\ &= C_1 \epsilon 2^{-\alpha k},\end{aligned}$$
with
$$C_1 = \frac{2^{m-\alpha}}{1-2^{-\alpha}}.$$
Since $\epsilon > 0$ is arbitrary, then
$$\int_{2^k}^{+\infty} g(x) d\mu(x) = o\Big((2^k)^{-\alpha}\Big).$$

For $y \to +\infty$ we have that $y \geq 2^k$, so

$$\int_y^{+\infty} g(x)d\mu(x) \leq \int_{2^k}^{+\infty} g(x)d\mu(x).$$

it follows that

$$\int_y^{+\infty} g(x)d\mu(x) = o(y^{-\alpha}),$$

and

$$g \in L^1(]y;+\infty[, d\mu),$$

for large y.

(2) By (2.5), for every $\epsilon > 0$, there exists $k_0 \in \mathbb{Z}$ such that:

$$\int_{2^{k-1}}^{2^k} g(x)d\mu(x) \leq \int_{2^{k-1}}^{+\infty} g(x)d\mu(x) \leq \epsilon(2^{-k\alpha}).$$

So:

$$\int_{2^{k-1}}^{2^k} x^m g(x)d\mu(x) \leq 2^{km} \int_{2^{k-1}}^{2^k} g(x)d\mu(x) \leq \epsilon 2^{k(m-\alpha)}.$$

Then:

$$\begin{aligned}\int_{2^{k-1}}^{2^k} g(x)d\mu(x) &\leq 2^{km} \int_{2^{k-1}}^{2^k} g(x)d\mu(x),\\ &\leq 2^{km}\epsilon 2^{-\alpha k},\\ &= \epsilon 2^{k(m-\alpha)}.\end{aligned}$$

So

$$\begin{aligned}\int_{2^{k_0}}^{2^k} x^m g(x)d\mu(x) &\leq \epsilon 2^{(k_0+1)(m-\alpha)} + ... + \epsilon 2^{(k+1)(m-\alpha)},\\ &= \epsilon 2^{(k_0+1)(m-\alpha)} \frac{1-2^{(m-\alpha)(k-k_0+1)}}{1-2^{m-\alpha}},\\ &= C_2\epsilon 2^{k(m-\alpha)},\end{aligned}$$

with

$$C_2 = \frac{1-2^{(m-\alpha)(k-k_0+1)}}{1-2^{m-\alpha}}.$$

On the other hand, by the condition (2.3), we have

$$\int_0^{2^{k_0}} x^m g(x)d\mu(x) \leq C_3 2^{(m-\alpha)k_0}.$$

So

$$\begin{aligned}\int_0^{2^k} x^m g(x)d\mu(x) &= \int_0^{2^{k_0}} x^m g(x)d\mu(x) + \int_{2^{k_0}}^{2^k} x^m g(x)d\mu(x),\\ &\leq C_3 2^{(m-\alpha)k_0} + C_2 2^{(m-k)k}\epsilon\frac{1}{2^\alpha - 1}.\end{aligned}$$

For large k we get:

$$\int_0^{2^k} x^m g(x)d\mu(x) \leq \epsilon\Big(C_2 + C_3 2^{(m-k)k}\frac{1}{2^\alpha - 1}\Big)2^{k(m-\alpha)}.$$

For $y \to +\infty$ we have that $y \geq 2^k$, so

$$\int_0^{y} x^m g(x)d\mu(x) \leq \epsilon\Big(C_2 + C_3 2^{(m-k)k}\frac{1}{2^\alpha - 1}\Big)y^{m-\alpha}.$$

Since ϵ is arbitrary, then (2.4) holds.

The proof is completed.

■ And here is the second main result of this paper:

Theorem 2.5. *Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $m \in \mathbb{N}$ and $0 < \alpha < 2m$ such that μ is a positive measure.*

(1) *Suppose that $f \in L^1(\mathbb{R}^+, d\mu) \cap C(\mathbb{R}^+)$. If*

$$\int_0^y |\lambda|^{2m}|M(f)(\lambda)|dv_x(\lambda) = o(y^{2m-\alpha}), \tag{2.6}$$

as $y \to +\infty$. Then $M(f) \in L^1(]y; +\infty[, dv_x(\lambda))$ and $f \in h^m_\alpha$.

(2) *Conversely, suppose that $f \in L^1(\mathbb{R}^+, d\mu)$, $M(f) \in L^1(]y; +\infty[, dv_x(\lambda))$. If $f \in h^\alpha_m$ then (2.3) holds.*

Remark 2.6. *From Lemma 2.4 it follows that:*
The statement (2.6) implies

$$\int_y^{+\infty} |M(f)(\lambda)|dv_x(\lambda) = o(y^{-\alpha}),$$

and

$$\delta^m_h f(x) < +\infty.$$

Proof.

(1) by Remark 2.6 we have

$$\int_y^{+\infty} |M(f)(\lambda)|dv_x(\lambda) = o(y^{-\alpha}),$$

and

$$M(f) \in L^1(]y; +\infty[, dv_x(\lambda)).$$

We have that:

$$\delta^m_h f(x) = \int_0^{+\infty} \Big|\tanh(\pi\sqrt{\lambda})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda)\Big|d\lambda.$$

then

$$\delta^m_h f(x) = I_1 + I_2,$$

such that:

$$I_1 = \int_0^{\frac{1}{h}} \Big|\tanh(\pi\sqrt{\lambda})P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x})\Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda)\Big|d\lambda,$$

and

$$I_2 = \int_{\frac{1}{h}}^{+\infty} \left| \tanh(\pi\sqrt{\lambda}) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) \Big(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\Big)^m M(f)(\lambda) \right| d\lambda.$$

By propreties of $P_{i\sqrt{\lambda}-\frac{1}{2}}$ we have that:

$$I_2 \leq (1+2M)^m \int_{\frac{1}{h}}^{+\infty} \left| \tanh(\pi\sqrt{\lambda}) P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{x}) M(f)(\lambda) \right| d\lambda.$$

i.e

$$I_2 \leq (1+2M)^m \int_{\frac{1}{h}}^{+\infty} |M(f)(\lambda)| dv_x(\lambda).$$

By hypothesis we have

$$\int_0^{\frac{1}{h}} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) = o(\frac{1}{h^{2m-\alpha}}).$$

B Lemma 2.4 we get:

$$\int_{\frac{1}{h}}^{+\infty} |M(f)(\lambda)| dv_x(\lambda) = o(h^{\alpha}).$$

To estimate I_1, set

$$\phi(x) = \int_x^{\infty} |M(f)(\lambda)| dv_x(\lambda).$$

We have that

$$\int_0^x \lambda\phi'(\lambda) d\lambda \leq x \int_0^x \phi'(\lambda) d\lambda.$$

So:

$$\int_0^x \lambda\phi'(\lambda) d\lambda \leq x\Big(\phi(x) - \phi(0)\Big).$$

It follows that:

$$\int_0^x \lambda\phi'(\lambda) d\lambda \leq -x \int_0^x |M(f)(\lambda)| dv_x(\lambda).$$

Then:

$$\int_0^x |M(f)(\lambda)| dv_x(\lambda) \leq \frac{-1}{x} \int_0^x \lambda\phi'(\lambda) d\lambda.$$

Upon an integration by parts, we obtain:

$$\int_0^x \lambda\phi'(\lambda) dv\lambda = x\phi(x) - \int_0^x \phi(\lambda) d\lambda.$$

So:

$$\frac{-1}{x} \int_0^x \lambda\phi'(\lambda) d\lambda = \frac{1}{x} \int_0^x \phi(\lambda) d\lambda - \phi(x).$$

Then

$$\frac{-1}{x} \int_0^x \lambda\phi'(\lambda) d\lambda \leq \frac{1}{x} \int_0^x \phi(\lambda) d\lambda.$$

So if $x \to +\infty$

$$\frac{-1}{x} \int_0^x \lambda\phi'(\lambda) d\lambda \leq \frac{1}{x} \int_0^x o(\lambda^{-\alpha}) d\lambda = o(x^{-\alpha}).$$

Then

$$\int_0^x |M(f)(\lambda)| dv_x(\lambda) = o(x^{-\alpha}).$$

So

$$\int_0^{\frac{1}{h}} |M(f)(\lambda)| dv_x(\lambda) = o(h^\alpha).$$

It follows that

$$I_1 = o(h^\alpha).$$

Then

$$\delta_h^m f(x) = o(h^\alpha) + o(h^\alpha).$$

It follows that

$$\delta_h^m f(x) = o(h^\alpha).$$

Which show that

$$f \in h_\alpha^m.$$

(2) We have that:

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) \leq \left(\frac{\beta}{h}\right)^{2m} \int_0^{\frac{\beta}{h}} |M(f)(\lambda)| dv_x(\lambda).$$

So

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) \leq \left(\frac{\beta}{h}\right)^{2m} \int_0^{+\infty} |M(f)(\lambda)| dv_x(\lambda).$$

Since

$$1 \leq \left(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\right)^m.$$

Then

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) \leq \left(\frac{\beta}{h}\right)^{2m} \int_0^{+\infty} \left(1 + P_{\frac{-1}{2}}(2\sqrt{h}) + P_{i\sqrt{\lambda}-\frac{1}{2}}(2\sqrt{h})\right)^m |M(f)(\lambda)| dv_x(\lambda).$$

Since $f \in h_\alpha^m$, so

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) = \left(\frac{\beta}{h}\right)^{2m} o(h^\alpha).$$

Hence, for $y = \frac{\beta}{h}$, we get:

$$\int_0^{y} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) = o\left(\left(\frac{\beta}{y}\right)^{\alpha-2m}\right).$$

So:

$$\int_0^{y} |\lambda|^{2m} |M(f)(\lambda)| dv_x(\lambda) = o(y^{2m-\alpha}).$$

Thus, (2.3) is holds.

■

Acknowledgment: The authors would like to thank the referee for his valuable comments and suggestions.

Competing interests: The authors declare that there is no conflict of interest regarding the publication of this paper.

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