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Unilateral Problem for a Viscoelastic Beam Equation Type p-Laplacian with Strong Damping and Logarithmic Source

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Abstract. In this manuscript, we investigate the unilateral problem for a viscoelastic beam equation of p-Laplacian type. The competition of the strong damping versus the logarithmic source term is considered. We use the potential well theory. Taking into account the initial data is in the stability set created by the Nehari surface, we prove the existence and uniqueness of global solutions by using the penalization method and Faedo-Galerkin's approximation.

1. Introduction

We denote the p-Laplacian operator by $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$, which can be extended to a monotone, bounded, hemicontinuos and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega), \qquad \langle -\Delta_p u, v\rangle_p = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx.$$

In [3] the authors establish existence of global solution to the problem

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t + f(u) = 0 \text{ in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = \Delta u = 0 \text{ on } \Gamma \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0, \;\; u_t(x,0) = u_1 \text{ in } \Omega, \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$.

Equations of the type (1.1) are related to models of elastoplastic microstructure flows. As considered by An and Peirce [1, 2], they are essentially of the form

$$u_{tt} + u_{xxxx} - a(u_x^2)_x = 0.$$

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A more general equation,

$$u_{tt} + \Delta^2 u - div(\sigma(|\nabla u|^2)\nabla u) - \Delta u_t + h_1(u_t) + h_2(u) = h_3(x),$$

was considered by Yang et al [22–24]. They studied de existence of attractors and their Hausdorff dimensions. Another related equation is

$$u_{tt} + \Delta^2 u - div(f_0(\nabla u)) + ku_t = \Delta(f_1(u)) - f_2(u),$$

which was considered by Chueshov and Lasiecka [12]

The problem (1.1), with its memory term $\int_0^t g(t-s)\Delta u(s)ds$, can be regarded as a fourth-order viscoelastic plate equation with a lower order perturbation of the p-Laplacian type. This kind of problem can be also regarded as an elastoplastic flow equation with some kind of memory effect.

We observe that for viscoelastic plate equation, it is usual consider a memory of the form

$$\int_0^t g(t-s)\Delta^2 u(s)ds,$$

see for instance [10]. However, because the main dissipation of the system (1.1) is given by strong damping $-\Delta u_t$, here we consider a weaker memory, acting only on Δu. There is a large literature about stability in viscoelasticity. We refer the reader to [11, 13].

A nonlinear perturbation of problem (1.1) is given by

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t + f(u) \geq 0. \tag{1.4}$$

Variational inequality theory was introduced by Hartman and Stampacchia (1966) [14] as a tool for the study of partial differential equations with applications principally in mechanics.

In [7] the authors investigated the unilateral problem associated with this perturbation, that is, a variational inequality given for (1.4) (see [16]). Making use of the penalization method and Galerkin's approximations, they established existence and the uniqueness of strong solutions.

The unilateral problem is very interesting because, in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. Variational inequality theory was introduced by Hartman and Stampacchia (1966) [14] as a tool for the study of partial differential equations with applications principally in mechanics. Bensoussan and Lions (1982) [9] used variational inequalities initially in the study of stochastic control. In [5] was obtained a variational inequality for the Navier-Stokes operator with variable viscosity. In [6] was studied the contact problem on the Oldroyd model of viscoelastic fluids. By using results from the theory of monotone operators, was established the existence of weak solutions. In [8] was studied the problem for parabolic variational inequalities with Volterra type operators. The authors proved the existence and the uniqueness of the solution. For contact problems on elasticity and finite element method, see Kikuchi-Oden [15] and reference therein. In [18] was studied the unilateral problem for the

Klein-Gordon operator with the nonlinearity of Kirchhoff-Carrier type. By using an appropriate penalization was shown the existence and uniqueness of solutions for the perturbed equation. In [19] was considered the unilateral problem for a nonlinear wave equation with p-Laplacian operator and source term. By using an appropriate penalization, authors obtained a variational inequality for the equation perturbed and then the existence of solutions was proved.

In this work, we propose to investigate the existence and uniqueness of solutions for the variational inequality associated with the problem (1.4) with the source term $f(u) = -|u|^{r-2}u\ln|u|$. More precisely, we investigate the existence and uniqueness of solutions for the unilateral problem

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t \geq |u|^{r-2}u\ln|u| \text{ in } \Omega\times\mathbb{R}^+, \tag{1.5}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \Omega, \tag{1.6}$$

$$u(x,t) = \Delta u(x,t) = 0 \text{ on } \Gamma\times\mathbb{R}^+. \tag{1.7}$$

This work is organized as follows: In section 2 we introduce the notation and some well-known results. In section 3 we introduce the potential theory suitable for our problem. In section 4 define strong solution to the boundary value problem (1.5)-(1.7) and present the theorem of existence of strong solution. In section 5 we apply the penalization method. The existence of global solutions is given by using Faedo-Galerkin approximation. Finally, in Section 6 we prove the result of uniqueness.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary Γ of class C^2. For $T > 0$, we denote by Q the cylinder $\Omega\times(0,T)$, with lateral boundary $\Sigma = \Gamma\times(0,T)$. By $\langle\cdot,\cdot\rangle$ we will represent the duality pairing between a Banach space X and X', X' being the topological dual of the space X, and by C we denote various positive constants. The inner product in $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively, will be denoted by $(\nabla\cdot,\nabla\cdot)$, $(\cdot,\cdot)$. The norm in $L^p(\Omega)$ will be denoted by $|\cdot|_p$.

The inequality (1.5) must be satisfied in the following sense. Let

$$K = \{v \in H_0^1(\Omega); v \geq 0 \text{ a.e. in } \Omega\}$$

be a closed and convex subset of $H_0^1(\Omega)$, the unilateral problem consists to find a solution $u(x,t)$ satisfying

$$\int_Q (u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t - |u|^{r-2}u\ln|u|)(v-u_t) \geq 0, \tag{2.1}$$

for all $v \in K$ with $u_t(x,t) \in K$ a.e. on $[0,T]$ and the initial and boundary data

$$u = \Delta u = 0 \text{ in } \Gamma\times(0,T), \tag{2.2}$$

$$u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega. \tag{2.3}$$

To study the existence and uniqueness of the problem (1.5)–(1.7), let us consider the following hypotheses:

H1. Suppose that

$$\begin{cases} 2 \leq p & , \quad \text{if } n = 1,2 \\ 2 \leq p \leq \dfrac{2n-2}{n-2} & , \quad \text{if } n \geq 3. \end{cases}$$

H2. With respect to the power r, let us suppose that

$$\begin{cases} 2 < r < +\infty & , \quad \text{if } n = 1,2 \\ 2 < r < \dfrac{2n}{n-2} & , \quad \text{if } n \geq 3. \end{cases}$$

H3. With respect to the function $g : [0, +\infty) \to \mathbb{R}$, we will assume that $g \in C^1[0, T]$ and

$$g(0) > 0, \quad l = 1 - \mu \int_0^\infty g(s)ds > 0,$$

where $\mu > 0$ is the embedding constant for $|\nabla u| \leq \sqrt{\mu}\,|\Delta u|$, for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

H4. There exists a constant $k_1 > 0$ such that

$$g'(t) \leq -k_1 g(t), \ \forall t \geq 0.$$

By $H1$ we have

$$H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

The Lemmas below will be a important role in this manuscript.

Lemma 2.1. *(Sobolev Poincaré inequality) Let p be a number with $2 < p < \infty$ if $n = 1,2$ or $2 \leq p \leq \dfrac{2n}{n-2}$ if $n \geq 3$, then there exists a constant $C > 0$ such that*

$$|u|_p \leq C|\nabla u|, \forall u \in H_0^1(\Omega)$$

Lemma 2.2. *(Technical lemma) For $v \in C^1(0, T; H_0^1(\Omega))$, we have*

$$\int_\Omega \int_0^t g(t-s)\nabla v \cdot \nabla v_t ds dx = \frac{1}{2}(g' \diamond \nabla v)(t) - \frac{1}{2}g(t)|\nabla v(t)|^2 - \frac{1}{2}\frac{d}{dt}\left[(g \diamond \nabla v)(t) - \left(\int_0^t g(s)ds\right)|\nabla v(t)|^2\right],$$

where $(g \diamond \nabla u)(t) = \displaystyle\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|^2 ds.$

Proof. Differentiating the term $(g \diamond \nabla u)(t)$ we arrive to the above inequality. □

3. Potential well

In this section, we use the potential well theory, a power full tool in the study of the global existence of solution in partial differential equation. See Payne-Sattinger [17]. It is well-known that the energy of a PDE system, in some sense, splits into kinetic and potential energy. The source term induces potential energy in the system that acts in opposition to the effect of the stabilizing mechanism. In this sense, it is possible that the energy from the source term destabilizes all the system and produces a blow-up in a finite time. To provide a global solution, we are able to construct a stability set corresponding to the source term created from the Nehari Manifold, see Y. Ye [20]. In the stability set, there exists a valley or a well of the depth d created in the potential energy. If d is strictly positive, then we find that, for solutions with the initial data in the good part of the potential well, the potential energy of the solution can never escape the potential well. In general, the energy from the source term causes the blow-up in a finite time. However, the good part of the potential well is an invariant set where it remains bounded. As a result, the total energy of the solution remains finite for any time interval $[0, T]$, providing the global existence of the solution.

For the model considered here, the total energy is given by

$$\begin{aligned} E(t) = \frac{1}{2}\Bigg[&|u_t(t)|^2 + |\Delta u(t)|^2 + \frac{2}{p}|\nabla u(t)|_p^p + (g \diamond \nabla u)(t) \\ &- \left(\int_0^t g(s)ds\right) |\nabla u(t)|^2 + \frac{2}{r^2}|u(t)|_r^r \\ &- \frac{2}{r}\int_\Omega |u(t)|^r \ln |u(t)| dx \Bigg] \end{aligned} \tag{3.1}$$

and satisfies

$$\frac{d}{dt}E(t) \leq -|\nabla u_t(t)|^2. \tag{3.2}$$

From (H_3) we get

$$\begin{aligned} I(t) = \frac{1}{2}\Bigg[&|u_t(t)|^2 + \left(1 - \mu\int_0^t g(s)ds\right) |\Delta u(t)|^2 + (g \diamond \nabla u(t) + \frac{2}{p}|\nabla u(t)|_p^p \\ &+ \frac{2}{r^2}|u(t)|_r^r - \frac{2}{r}\int_\Omega |u(t)|^r \ln |u(t)| dx\Bigg]. \end{aligned} \tag{3.3}$$

Then, we introduce the functional $J : H_0^1(\Omega) \cap H^2(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} J(u) = \frac{1}{2}\Bigg[&\left(1 - \mu\int_0^t g(s)ds\right) |\Delta u|^2 + \frac{2}{p}|\nabla u(t)|_p^p + (g \diamond \nabla u)(t) + \frac{2}{r^2}|u(t)|_r^r \\ &- \frac{2}{r}\int_\Omega |u(t)|^r \ln |u(t)| dx\Bigg]. \end{aligned} \tag{3.4}$$

For $u \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$\begin{aligned} J(\lambda u) &= \frac{\lambda^2}{2}\left(1-\mu\int_0^t g(s)ds\right)|\Delta u|^2 + \frac{\lambda^p}{p}|\nabla u(t)|_p^p + \frac{\lambda}{2}(g\diamond\nabla u)(t) \\ &+ \frac{\lambda^r}{r^2}|u(t)|_r^r - \frac{\lambda^r}{r}\int_\Omega |u(t)|^r \ln|u(t)|dx. \end{aligned} \tag{3.5}$$

Associated with J, we have the well-known Nehari Manifold given by

$$\mathcal{N} \overset{def}{=} \left\{u \in H_0^1(\Omega)\cap H^2(\Omega)\setminus\{0\}; \left[\frac{d}{d\lambda}J(\lambda u)\right]_{\lambda=1} = 0\right\} \tag{3.6}$$

or equivalently,

$$\begin{aligned} \mathcal{N} &= \left\{u \in H_0^1(\Omega)\cap H^2(\Omega))\setminus\{0\}; \left(1-\mu\int_0^t g(s)ds\right)|\Delta u|^2\right. \\ &+ \left.|\nabla u(t)|_p^p + \frac{1}{2}(g\diamond\nabla u)(t) = \int_\Omega |u(t)|^r\ln|u(t)|dx\right\}. \end{aligned} \tag{3.7}$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [4]

$$d \overset{def}{=} \inf_{u\in(H_0^1(\Omega)\cap H^2(\Omega)\setminus\{0\}}\sup_{\lambda\geq 0} J(\lambda u).$$

similar to the result in [21] one has $0 < d = \inf_{u\in\mathcal{N}} J(u)$.

Now, we introduce

$$W = \{u \in H_0^1(\Omega)\cap H^2(\Omega); J(u) < d\}\cup\{0\}$$

and partition it into two sets $W = W_1 \cup W_2$ as follows

$$\begin{aligned} W_1 &= \left\{u \in W;\left(1-\mu\int_0^t g(s)ds\right)|\Delta u|^2\right. \\ &+ \left.\frac{2}{p}|\nabla u(t)|_p^p + (g\diamond\nabla u)(t) + \frac{2}{r^2}|u(t)|_r^r > \frac{2}{r}\int_\Omega |u(t)|^r\ln|u(t)|dx\right\}\cup\{0\} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} W_2 &= \left\{u \in W;\left(1-\mu\int_0^t g(s)ds\right)|\Delta u|^2\right. \\ &+ \left.\frac{2}{p}|\nabla u(t)|_p^p + (g\diamond\nabla u)(t) + \frac{2}{r^2}|u(t)|_r^r < \frac{2}{r}\int_\Omega |u(t)|^r\ln|u(t)|dx\right\}. \end{aligned} \tag{3.9}$$

So, we define by W_1 the set of stability for the problem (1.5)-(1.7), and before starting the section of existence and uniqueness of solution, we will prove that $\mathcal{W}_1$ is invariant set for sub-critical initial energy.

Proposition 1. *Let $u_0 \in W_1$ and $u_1 \in H_0^1(\Omega)$. If $E(0) < d$ then $u(t) \in \mathcal{W}_1$.*

Proof. Let $T > 0$ be the maximum existence time. From (3.2) we get

$$E(t) \leq E(0) < d, \text{ for all } t \in [0, T).$$

and then,

$$\frac{1}{2}\int_\Omega |u_t(t)|^2 \, dx + J(u(t)) < d, \text{ for all } t \in [0, T). \tag{3.10}$$

Arguing by contradiction, we suppose that there exists a first $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) > 0$ for all $0 \leq t < t_0$, that is,

$$\left(1 - \mu \int_0^t g(s)ds\right) |\Delta u(t_0)|^2 + \frac{2}{p}|\nabla u(t_0)|_p^p + (g \diamond \nabla u)(t_0) + \frac{2}{r^2}|u(t_0)|_r^r$$
$$= \frac{2}{r}\int_\Omega |u(t_0)|^r \ln |u(t_0)| dx$$

From the definition of $\mathcal{N}$, we have that $u(t_0) \in \mathcal{N}$, which leads to

$$J(u(t_0)) \geq \inf_{u(t) \in \mathcal{N}} J(u(t)) = d.$$

We deduce

$$\frac{1}{2}\int_\Omega |u_t(t_0)|^2 \, dx + J(u(t_0)) \geq d,$$

which contradicts with (3.10). Then $u(t) \in \mathcal{W}_1$ for all $t \in [0, T)$. □

4. Existence of strong solutions

Next, we shall state the main results of this paper.

Theorem 4.1. *Consider the space*

$$H_\Gamma^3(\Omega) = \{u \in H^3(\Omega) | u = \Delta u = 0 \text{ on } \Gamma\}.$$

If $u_0 \in W_1 \cap H_\Gamma^3(\Omega)$, $J(u_0) < d$, $u_1 \in H_0^1(\Omega)$ and the hypothesis (H_1)-(H_4) holds, then there exists a function $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$u \in L^\infty(0, T; (H_0^1(\Omega) \cap H^2(\Omega))) \cap L^\infty(0, T; H_\Gamma^3(\Omega)), \tag{4.1}$$

$$u_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \tag{4.2}$$

$$u_{tt} \in L^\infty(0, T; H^{-1}(\Omega)), \tag{4.3}$$

$$u_t(t) \in K \;\; a.e. \text{ in } [0, T], \tag{4.4}$$

$$\begin{aligned}\int_0^T & \big[\langle u_{tt}, v - u_t\rangle + (\Delta^2 u, v - u_t) - (\Delta_p u, v - u_t) \\ &+ \left(\int_0^t g(t-s)\Delta u(s)ds, v - u_t\right) - (\Delta u_t, v - u_t) \\ &- (|u|^{r-2} u \ln |u|, v - u_t)\big] \geq 0,\end{aligned} \tag{4.5}$$

for all $v \in L^2(0,T;H_0^1(\Omega))$, $v(t) \in K$ *a.e. in* t *and initial data*

$$u(0) = u_0, \quad u_t(0) = u_1.$$

The proof of Theorem 4.1 is given in Section 5 by the penalization method. It consists in considering a perturbation of the problem (1.5) adding a singular term called penalization, depending on a parameter $\epsilon > 0$. We solve the mixed problem in Q for the penalization operator and the estimates obtained for the local solution of the penalized equation, allow to pass to limits, when ϵ goes to zero, in order to obtain a function u which is the solution of our problem. First of all, let us consider the penalization operator

$$\beta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$$

associated to the closed convex set K, cf. Lions [16], p. 370. The operator β is monotonous, hemicontinuous, takes bounded sets of $H_0^1(\Omega)$ into bounded sets of $H^{-1}(\Omega)$, its kernel is K and

$$\beta : L^2(0,T;H_0^1(\Omega)) \longrightarrow L^2(0,T;(H^{-1}(\Omega))$$

is monotone and hemicontinous. The penalized problem associated with the variational inequality (1.5)-(1.7), consists in given $0 < \epsilon < 1$, find u^ϵ satisfying

$$\begin{aligned} u_{tt}^\epsilon + \Delta^2 u^\epsilon - \Delta_p u^\epsilon + \int_0^t g(t-s)\Delta u^\epsilon(s)ds - \Delta u_t^\epsilon \\ + \frac{1}{\epsilon}(\beta(u_t^\epsilon)) - |u^\epsilon|^{r-2}u^\epsilon \ln|u| = 0, \text{ in } Q \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &u^\epsilon(x,0) = u_{\epsilon 0}(x), u_t^\epsilon(x,0) = u_{\epsilon 1}(x) \text{ in } \Omega. \\ &u^\epsilon(x.t) = \Delta u^\epsilon(x,t) = 0 \text{ on } \partial\Omega \times \mathbb{R}^+. \end{aligned} \tag{4.7}$$

Definition 4.2. *Suppose that* $u_{\epsilon 0} \in W_1$, $J(u_{\epsilon 0}) < d$, $u_{\epsilon 1} \in H_0^1(\Omega)$ *and hypothesis* $(H_1) - (H_4)$ *holds. A strong solution to the boundary value problem (4.6)-(4.7) is a function* u^ϵ *such that*

$$\begin{aligned} u^\epsilon &\in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)), \\ u_t^\epsilon &\in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)), \\ u_{tt}^\epsilon &\in L^2(0,T;(H_0^1(\Omega) \cap H^2(\Omega))') \end{aligned}$$

satisfying for all $w \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\begin{aligned} \frac{d}{dt}(u_t^\epsilon(t),w) + (\Delta u^\epsilon(t),\Delta w) + (-\Delta_p u^\epsilon(t),w) + \int_0^t g(t-s)(\Delta u^\epsilon(s),w)ds \\ + (\nabla u_t^\epsilon(t),\nabla w) + \frac{1}{\epsilon}(\beta(u_t^\epsilon(t)),w) - (|u^\epsilon(t)|^{r-2}u^\epsilon(t)\ln|u^\epsilon(t)|),w) = 0 \end{aligned}$$

and initial data

$$u^\epsilon(0) = u_{\epsilon 0},\ u^\epsilon_t(0) = u_{\epsilon 1}.$$

The solution of problem (4.6)-(4.7) is given by the following theorem:

Theorem 4.3. *Assume that hypotheses $(H_1) - (H_4)$ holds,*

$$u_{\epsilon 0} \in W_1,\ J(u_{\epsilon 0}) < d \text{ and } u_{\epsilon 1} \in H^1_0(\Omega), \tag{4.8}$$

then, for each $0 < \epsilon < 1$, there exists a function u^ϵ strong solution of (4.6)-(4.7).

5. Penalization method

In order to prove Theorem 4.1, we first prove the penalized Theorem 4.3. The existence of global solutions will be given by using Faedo-Galerkin method. First we consider the approximate problem. Then we obtain the a priori estimates needed to passage to the limit in the approximate solutions.

5.1. **Approximate problem.** Let $\{w_j\}$ be the Galerkin basis given by eigenfunctions of Δ^2 with boundary condition $u = \Delta u = 0$ on $\Gamma \times \mathbb{R}^+$ and let $V_m \subset \mathcal{N}$ be the subspace spanned by the vectors $w_1, w_2, ..., w_m$.. Consider

$$u^{\epsilon m}(t) = \sum_{j=1}^{m} g_{\epsilon j m}(t) w_j$$

solution of approximate problem

$$(u^{\epsilon m}_{tt}(t), w) + (\Delta u^{\epsilon m}(t), \Delta w) + (-\Delta_p u^{\epsilon m}(t), w) + \int_0^t g(t-s)(\Delta u^{\epsilon m}(t), w) ds$$
$$-(|u^{\epsilon m}(t)|^{r-2} u^{\epsilon m}(t) \ln |u^{\epsilon m}|, w) + (\nabla u^{\epsilon m}(t), \nabla w) + \frac{1}{\epsilon}(\beta(u^{\epsilon m}_t)(t), w) = 0 \tag{5.1}$$

with initial conditions

$$u^{\epsilon m}(0) = u_{\epsilon 0 m} \to u_{\epsilon 0} \text{ strongly in } H^2(\Omega) \cap H^1_0(\Omega), \tag{5.2}$$
$$u^{\epsilon m}_t(0) = u_{\epsilon 1 m} \to u_{\epsilon 1} \text{ strongly in } L^2(\Omega). \tag{5.3}$$

The system of ordinary differential equation (5.1) in the variable t has a local solution $u^{\epsilon m}(t)$ defined in $[0, t_m[,\ 0 < t_m \leq T$. In the next step obtain priori estimates for the solution $u^{\epsilon m}(t)$ that permits us to extend this solution to the whole interval $[0, T]$.

5.2. **First estimate.** We consider $w = u_t^{\epsilon m}$ in (5.1) to obtain

$$\frac{d}{dt}\left[\frac{1}{2}|u_t^{\epsilon m}(t)|^2 + \frac{1}{2}|\Delta u^{\epsilon m}(t)|^2 + \frac{1}{p}|\nabla u^{\epsilon m}(t)|_p^p + \frac{1}{r^2}|u^{\epsilon m}(t)|_p^p\right.$$
$$\left. -\frac{1}{r}\int_\Omega |u^{\epsilon m}(t)|^r \ln|u^{\epsilon m}(t)|dx\right] + |\nabla u_t^{\epsilon m}(t)|^2 + \frac{1}{\epsilon}(\beta(u_t^{\epsilon m}(t)), u_t^{\epsilon m}(t)) \tag{5.4}$$
$$= \int_0^t g(t-s)(\nabla u^{\epsilon m}(s), \nabla u_t^{\epsilon m}(t))ds.$$

We have $(\beta(u_t^{\epsilon m}(t)), u_t^{\epsilon m}(t)) \geq 0$. Then from Lemma 2.2 and (H_4)

$$\frac{1}{2}\frac{d}{dt}\left[|u_t^{\epsilon m}(t)|^2 + |\Delta u^{\epsilon m}(t)|^2 + \frac{2}{p}|\nabla u^{\epsilon m}(t)|_p^p + (g\diamond\nabla u^{\epsilon m})(t)\right.$$
$$-\left(\int_0^t g(s)ds\right)|\nabla u^{\epsilon m}(t)|^2 + \frac{2}{r^2}|u^{\epsilon m}(t)|_r^r$$
$$\left.-\frac{2}{r}\int_\Omega |u^{\epsilon m}(t)|^r \ln|u^{\epsilon m}(t)|dx\right] + |\nabla u_t^{\epsilon m}(t)|^2 + \tag{5.5}$$
$$\leq \frac{1}{2}(g'\diamond\nabla u^{\epsilon m})(t) - \frac{1}{2}g(t)|\nabla u^{\epsilon m}(t)|^2 \leq 0.$$

Let

$$E_{\epsilon m}(t) = \frac{1}{2}\left[|u_t^{\epsilon m}(t)|^2 + |\Delta u^{\epsilon m}(t)|^2 + \frac{2}{p}|\nabla u^{\epsilon m}(t)|_p^p + (g\diamond\nabla u^{\epsilon m})(t)\right.$$
$$-\left(\int_0^t g(s)ds\right)|\nabla u^{\epsilon m}(t)|^2 + \frac{2}{r^2}|u^{\epsilon m}(t)|_r^r \tag{5.6}$$
$$\left.-\frac{2}{r}\int_\Omega |u^{\epsilon m}(t)|^r \ln|u^{\epsilon m}(t)|dx\right].$$

So, by (5.5) and (5.8), we have

$$\frac{d}{dt}E_{\epsilon m}(t) \leq -|\nabla u_t^{\epsilon m}(t)|^2.$$

Integrating from 0 to t, $t \leq t_m$, we obtain

$$E_{\epsilon m}(t) + \int_0^t |\nabla u_t^{\epsilon m}(t)|^2 \leq E_{\epsilon m}(0). \tag{5.7}$$

By (H_3), it follows

$$\frac{1}{2}\left[|u_t^{\epsilon m}(t)|^2 + \left(1-\mu\int_0^t g(s)ds\right)|\Delta u^{\epsilon m}(t)|^2 + (g\diamond\nabla u^{\epsilon m})(t) + \frac{2}{p}|\nabla u^{\epsilon m}(t)|_p^p\right.$$
$$\left.+\frac{2}{r^2}|u^{\epsilon m}(t)|_r^r - \frac{2}{r}\int_\Omega |u^{\epsilon m}(t)|^r \ln|u^{\epsilon m}(t)|dx\right] \tag{5.8}$$
$$+\int_0^t |\nabla u_t^{\epsilon m}|^2 ds \leq E_{\epsilon m}(t) \leq E_{\epsilon m}(0) = \frac{1}{2}|u_{\epsilon 1m}|^2 + C_1 J(u_{\epsilon 0m}),$$

where $C_1 > 0$ is a positive constant, independent of m and t.

We have $J(u_{0\epsilon m}) < d$ and by (5.3), there exists a constant $C_2 > 0$ such that

$$|u_t^{\epsilon m}(t)|^2 + \left(1 - \mu \int_0^t g(s)ds\right) |\Delta u^{\epsilon m}(t)|^2 + (g \diamond \nabla u^{\epsilon m})(t) + \frac{2}{p}|\nabla u^{\epsilon m}(t)|_p^p$$
$$+ \frac{2}{r^2}|u^{\epsilon m}(t)|_r^r - \frac{2}{r}\int_\Omega |u^{\epsilon m}(t)|^r \ln|u^{\epsilon m}(t)|dx \tag{5.9}$$
$$+ \int_0^t |\nabla u_t^{\epsilon m}|^2 ds \le C_2.$$

From (3.7) and (5.9) we get

$$\Delta u^{\epsilon m} \rightharpoonup \Delta u^\epsilon \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{5.10}$$

$$u^{\epsilon m} \rightharpoonup u^\epsilon \text{ in } L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{5.11}$$

$$-\Delta_p u^{\epsilon m} \rightharpoonup \chi \text{ in } L^2(0,T;H^{-1}(\Omega)), \tag{5.12}$$

$$u_t^{\epsilon m} \rightharpoonup u_t^\epsilon \text{ in } L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega)), \tag{5.13}$$

$$\beta(u_t^{\epsilon m}) \rightharpoonup \psi \text{ in } L^2(0,T;H^{-1}(\Omega)). \tag{5.14}$$

Follows from(5.11), (5.13) and Aubin-Lions Theorem, for any $T > 0$,

$$u^{\epsilon m} \to u^\epsilon \text{ in } L^2(0,T;H_0^1(\Omega)), \text{ strong and a.e. in } Q. \tag{5.15}$$

Now, we prove that $\chi(t) = -\Delta_p u^\epsilon(t)$. We consider $x, y \in \mathbb{R}, p \ge 2$. Then the elementary inequality

$$\left||x|^{p-2}x - |y|^{p-2}y\right| \le C\left(|x|^{p-2} + |y|^{p-2}\right)|x-y| \tag{5.16}$$

is a consequence of the mean value theorem. Using (5.16) and Hölder generalized inequality with

$$\frac{p-2}{2(p-1)} + \frac{1}{2} + \frac{1}{2(p-1)} = 1,$$

we deduce for $\theta \in \mathcal{D}(0,T)$ and $v \in V_m$,

$$\begin{aligned}
&\left|\int_0^T \langle(-\Delta u_p^{\epsilon m}(t)) - (-\Delta u_p^\epsilon(t)), v\rangle_p \theta(t)dt\right| \\
&= \left|\int_0^T \int_\Omega \left(|\nabla u^{\epsilon m}(t)|^{p-2}\nabla u^{\epsilon m}(t) - |\nabla u^\epsilon(t)|^{p-2}\nabla u^\epsilon(t)\right)\nabla v\, dx\, \theta(t)\, dt\right| \\
&\le C|\theta|_\infty \int_0^T \int_\Omega \left(|\nabla u^{\epsilon m}(t)|^{p-2} + |\nabla u^\epsilon(t)|^{p-2}\right)|\nabla u^{\epsilon m}(t) - \nabla u^\epsilon(t)||\nabla v|\, dx\, dt \\
&\le C_1 \int_0^T \left(|\nabla u^{\epsilon m}(t)|_{2(p-1)}^{p-2} + |\nabla u^\epsilon(t)|_{2(p-1)}^{p-2}\right)|\nabla u^{\epsilon m}(t) - \nabla u^\epsilon(t)||\nabla v|_{2(p-1)}\, dt \\
&\le C_2 \int_0^T |\nabla u^{\epsilon m}(t) - \nabla u^\epsilon(t)|\, dt
\end{aligned} \tag{5.17}$$

where C_1 and C_2 are positive constants independent of m and t.

Now, from estimate (5.10) and (5.11), we have

$$\frac{d}{dt}|\nabla u^{\epsilon m}(t)-\nabla u^{\epsilon}(t)|^2 \leq 2|\Delta(u^{\epsilon m}(t)-u^{\epsilon}(t))||\nabla(u_t^{\epsilon m}(t)-u_t^{\epsilon}(t))| \leq C_3,$$

where C_3 is a constant independent of m and t.

So,

$$|u^{\epsilon m}(t)-u^{\epsilon}(t)|_{H_0^1(\Omega)} \in H^1[0,T] \hookrightarrow C[0,T],$$

whence $\nabla u^{\epsilon m}(t) \to \nabla u^{\epsilon}(t)$ a. e. in $[0,T]$. Therefore, $\chi = -\Delta_p u^{\epsilon}$.

Now, we observe that Sobolev inequality

$$\begin{aligned}\int_{\Omega} ||u^{\epsilon m}(t)|^{r-2}u^{\epsilon m}(t)\ln|u^{\epsilon m}(t)||^2 dx &\leq |u^{\epsilon m}(t)|_{2r}^{2r} \leq C^{2r}|\nabla u^{\epsilon m}(t)|^{2r} \\ &\leq \mu^r C^{2r}|\Delta u^{\epsilon m}(t)|^r \leq C_4,\end{aligned}$$

where C_4 is a constant independent of m and t.

Then

$$(|u^{\epsilon m}|^{r-2}u^{\epsilon m}\ln|u^{\epsilon m}|) \text{ is bounded in } L^2(0,T;L^2(\Omega)) = L^2(Q). \tag{5.18}$$

Using continuity of function $s \to |s|^{r-2}s\ln|s|$ and (5.15) we have

$$|u^{\epsilon m}|^{r-2}u^{\epsilon m}\ln|u^{\epsilon m}| \to |u^{\epsilon}|^{r-2}u^{\epsilon}\ln|u^{\epsilon}| \text{ a.e. in } Q. \tag{5.19}$$

By (5.18), (5.19) and applying Lions Lemma (Lemma 1.3, page 12, [16]), we get

$$|u^{\epsilon m}|^{r-2}u^{\epsilon m}\ln|u^{\epsilon m}| \rightharpoonup |u^{\epsilon}|^{r-2}u^{\epsilon}\ln|u^{\epsilon}| \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{5.20}$$

5.3. **Second estimate.** Let us consider the initial data

$$u_{\epsilon 0} \in H_{\Gamma}^3(\Omega),\ u_{\epsilon 1} \in H_0^1(\Omega) \text{ and } u_0^{\epsilon m} = \Delta u_0^{\epsilon m} = 0 \text{ on } \Gamma. \tag{5.21}$$

We consider $w = -\Delta u_t^{\epsilon m}$ in approximate equation (5.1).

Then we have

$$\begin{aligned}\frac{d}{dt}\left\{\frac{1}{2}|\nabla u_t^{\epsilon m}(t)|^2+\frac{1}{2}|\nabla\Delta u^{\epsilon m}(t)|^2\right\} &+ \langle\Delta_p u_t^{\epsilon m}(t),\Delta u_t^{\epsilon m}(t)\rangle \\ &+|\Delta u_t^{\epsilon m}(t)|^2+\frac{1}{\epsilon}(\beta(u_t^{\epsilon m}(t),-\Delta u_t^{\epsilon m}(t)) \\ &= (|u^{\epsilon m}(t)|^{r-2}u^{\epsilon m}\ln|u^{\epsilon m}(t)|,-\Delta u_t^{\epsilon m}(t)) \\ &+\int_0^t g(t-s)(\Delta u^{\epsilon m}(s),\Delta u_t^{\epsilon m}(t))ds.\end{aligned}$$

Now,

$$\langle\Delta_p u^{\epsilon m}(t),\Delta u_t^{\epsilon m}(t)\rangle = \frac{d}{dt}\langle\Delta_p u^{\epsilon m}(t),\Delta u^{\epsilon m}(t)\rangle - J_1,$$

where

$$J_1 = \int_\Omega \big\{(p-2)|\nabla u^{\epsilon m}(t)|^{p-4}(\nabla u^{\epsilon m}(t)\cdot\nabla u_t^{\epsilon m}(t))\nabla u^{\epsilon m}(t)$$
$$+|\nabla u^{\epsilon m}(t)|^{p-2}\nabla u_t^{\epsilon m}(t)\big\}\cdot\nabla\Delta u^{\epsilon m}(t)dx.$$

Then

$$\frac{d}{dt}\left\{\frac{1}{2}|\nabla u_t^{\epsilon m}(t)|^2+\frac{1}{2}|\nabla\Delta u^{\epsilon m}(t)|^2+\langle\Delta_p u^{\epsilon m}(t),\Delta u^{\epsilon m}(t)\rangle\right\}$$
$$+|\Delta u_t^{\epsilon m}(t)|^2+\frac{1}{\epsilon}(\beta(u_t^{\epsilon m}(t)),-\Delta u_t^{\epsilon m}(t))=J_1+J_2+J_3. \tag{5.22}$$

where

$$J_2=\int_\Omega |u^{\epsilon m}(t)|^{r-2}u^{\epsilon m}\ln|u^{\epsilon m}(t)|\Delta u_t^{\epsilon m}(t)$$

and

$$J_3=\int_0^t g(t-s)(\Delta u^{\epsilon m}(s),\Delta u_t^{\epsilon m}(t))ds.$$

Let us the right hand side of (5.22). We denote by C a generic positive constant not depending on m,t. By estimate (5.9) and

$$\frac{p-2}{2(p-1)}+\frac{1}{2(p-1)}+\frac{1}{2}=1,$$

$$|J_1|\le(p-1)\int_\Omega|\nabla u^{\epsilon m}(t)|^{p-2}|\nabla u_t^{\epsilon m}(t)||\nabla\Delta u^{\epsilon m}(t)|dx$$
$$\le(p-1)|\nabla u^{\epsilon m}(t)|_{2(p-1)}^{p-2}|\nabla u_t^{\epsilon m}(t)|_{2(p-1)}|\nabla\Delta u^{\epsilon m}(t)|$$
$$\le C|\nabla u_t^{\epsilon m}(t)|_{2(p-1)}|\nabla\Delta u^{\epsilon m}(t)|.$$

How $H_0^1(\Omega)\cap H^2(\Omega)\hookrightarrow W_0^{1,2}(\Omega)$, we have

$$|\nabla u_t^{\epsilon m}(t)|_{2(p-1)}^2\le\mu_2|\Delta u_t^{\epsilon m}(t)|^2,$$

where $\mu_2>0$ is the corresponding embedding constant. Then

$$|J_1|\le\frac{1}{2}|\Delta u_t^{\epsilon m}(t)|^2+C|\nabla\Delta u^{\epsilon m}(t)|^2. \tag{5.23}$$

Let $\Omega_1 = \{x \in \Omega : |u^{\epsilon m}(t)| < 1\}$ and $\Omega_2 = \{x \in \Omega : |u^{\epsilon m}(t)| \geq 1\}$. By (5.9) and Sobolev inequality

$$
\begin{aligned}
|J_2| &\leq \int_{\Omega_1} ||u^{\epsilon m}(t)|^{r-2} u^{\epsilon m} \ln |u^{\epsilon m}(t)| \Delta u_t^{\epsilon m}(t)| dx \\
&\quad + \int_{\Omega_2} ||u^{\epsilon m}(t)|^{r-2} u^{\epsilon m} \ln |u^{\epsilon m}(t)| \Delta u_t^{\epsilon m}(t)| dx \\
&\leq (e(r-1))^{-1} \int_\Omega |\Delta u_t^{\epsilon m}(t)| dx + (e(r-1))^{-1} \int_\Omega |u^{\epsilon m}(t)|^{r-1} |\Delta u_t^{\epsilon m}(t)| dx \\
&\leq 2(e(r-1))^{-2} + \frac{1}{8}|\Delta u_t^{\epsilon m}(t)|^2 + 2(e(r-1))^{-2} |u^{\epsilon m}(t)|_{2(r-1)}^{2(r-1)} + \frac{1}{8}|\Delta u_t^{\epsilon m}(t)|^2 \\
&\leq 2(e(r-1))^{-2} + \frac{1}{4}|\Delta u_t^{\epsilon m}(t)|^2 + 2C(e(r-1))^{-2} |\nabla u^{\epsilon m}(t)|^{2(r-1)} \\
&\leq C + \frac{1}{4}|\Delta u_t^{\epsilon m}(t)|^2
\end{aligned} \tag{5.24}
$$

where we have used $|x^{r-1} \ln x| \leq (e(r-1))^{-1}$ for $0 < x < 1$ and
$\ln x \leq (e(r-1))^{-1} x^{r-1}$, if $x \geq 1$.

Remark 5.1. *We note from the Cauchy-Schwarz inequality and Fubini's Theorem follows*

$$
\|g \diamond \nabla u\|_{L^2(Q)} \leq \|g\|_{L^1(0,\infty)} \|\nabla u\|_{L^2(Q)}
$$

Again from estimate (5.9) and remark 5.1

$$
\begin{aligned}
|J_3| &\leq \left(\int_0^t g(t-s) |\Delta u^{\epsilon m}(t)| ds \right) |\Delta u_t^{\epsilon m}(t)| \\
&\leq C \|g\|_{L^1(\mathbb{R}^+)} |\Delta u_t^{\epsilon m}(t)| \\
&\leq C + \frac{1}{4}|\Delta u_t^{\epsilon m}(t)|^2.
\end{aligned} \tag{5.25}
$$

Follows from (5.22)-(5.25) that

$$
\begin{aligned}
\frac{d}{dt}\left[\frac{1}{2}|\nabla u_t^{\epsilon m}(t)|^2 + \frac{1}{2}|\nabla \Delta u^{\epsilon m}(t)|^2 + \langle \Delta_p u^{\epsilon m}(t), \Delta u^{\epsilon m}(t)\rangle\right]& \\
+\frac{1}{2}|\Delta u_t^{\epsilon m}(t)|^2 + \frac{1}{\epsilon}(\beta(u_t^{\epsilon m}(t)), -\Delta u_t^{\epsilon m}(t))& \\
\leq C + C|\nabla \Delta u^{\epsilon m}(t)|^2.&
\end{aligned} \tag{5.26}
$$

Now, observe that

$$
\begin{aligned}
|\langle \Delta_p u^{\epsilon m}(t), \Delta u^{\epsilon m}(t)\rangle| &\leq \int_\Omega |\nabla u^{\epsilon m}(t)|^{p-1} |\nabla \Delta u^{\epsilon m}(t)| dx \\
&\leq |\Delta u^{\epsilon m}(t)|_{2(p-1)}^{p-1} |\nabla \Delta u^{\epsilon m}(t)| \\
&\leq C + |\nabla \Delta u^{\epsilon m}(t)|^2,
\end{aligned} \tag{5.27}
$$

and then

$$C + |\nabla\Delta u^{\epsilon m}(t)|^2 + \langle \Delta_p u^{\epsilon m}(t), \Delta u^{\epsilon m}(t)\rangle \geq 0.$$

Therefore, there exists $C_0 > 0$ such that

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}|\nabla u_t^{\epsilon m}(t)|^2 + \frac{1}{2}|\nabla\Delta u^{\epsilon m}(t)|^2 + \langle \Delta_p u^{\epsilon m}(t), \Delta u^{\epsilon m}(t)\rangle\right] \\ +\frac{1}{2}|\Delta u_t^{\epsilon m}(t)|^2 + \frac{1}{\epsilon}(\beta(u_t^{\epsilon m}(t)), -\Delta u_t^{\epsilon m}(t)) \\ \leq C_0 + C_0|\nabla\Delta u^{\epsilon m}(t)|^2 + \langle \Delta_p u^{\epsilon m}(t), \Delta u^{\epsilon m}(t)\rangle. \end{aligned} \tag{5.28}$$

Taking into account that $(\beta(u_t^{\epsilon m}(t), -\Delta u_t^{\epsilon m}(t)) \geq 0$, (5.21), integrating from 0 to t and applying Gronwall inequality, we obtain

$$|\nabla u_t^{\epsilon m}(t)|^2 + |\nabla\Delta u^{\epsilon m}(t)|^2 + \int_0^t |\Delta u_t^{\epsilon m}(t)|^2 \leq C, \tag{5.29}$$

then

$$u^{\epsilon m} \rightharpoonup u^{\epsilon} \text{ in } L^\infty(0,T;H^3_\Gamma(\Omega)), \quad \text{weakly star.} \tag{5.30}$$

$$u_t^{\epsilon m} \rightharpoonup u_t^{\epsilon} \text{ in } L^2(0,T;H^1_0(\Omega)\cap H^2(\Omega)), \quad \text{weakly} \tag{5.31}$$

$$\Delta u^{\epsilon m} \rightharpoonup \Delta u\epsilon \text{ in } L^\infty(0,T;H^1_0(\Omega)), \quad \text{weakly star.} \tag{5.32}$$

5.4. **Third estimate.** Let P_m be the ortogonal projection $P_m : L^2(\Omega) \to V_m$, that is

$$P_m\phi = \sum_{n=1}^{m}(\phi, w_j)w_j, \quad \phi \in L^2(\Omega).$$

Remark 5.2. *By remark 5.1, we observe that if*
$\psi \in L^2(0,T;H^1_0(\Omega))$ then $\int_0^t g(t-s)\psi(s)ds \in L^2(0,T;H^{-1}(\Omega))$
and by (5.12) $-\Delta_p u^{\epsilon m} \in L^2(0,T;(H^{-1}(\Omega))$.

We obtain using the notation and ideas of Lions [16], pages 75-76, remark 5.2 and estimates above that

$$u_{tt}^{\epsilon m} \rightharpoonup u_{tt}^{\epsilon} \text{ in } L^2(0,T;(H^{-1}(\Omega)), \quad \text{weakly.} \tag{5.33}$$

(5.31), (5.33) and Aubin-Lions compactness Theorem imply that there exists a subsequence from $(u_t^{\epsilon m})$, still denoted by $(u_t^{\epsilon m})$, such that

$$u_t^{\epsilon m} \to u_t^{\epsilon} \text{ strongly in } L^2(0,T;H^1_0(\Omega)) \text{ and a.e. in } Q. \tag{5.34}$$

Now, we are in position to prove Theorem 4.1.

5.5. **Strong solution.** Let $v \in L^2(0,T;H_0^1(\Omega))$ be $v(t) \in K$ a. e. for $t \in (0,T)$. From (4.6)$_1$ follows that

$$\begin{aligned}
&\int_0^T (u_{tt}^\epsilon, v-u_t^\epsilon)dt + \int_0^T (\Delta^2 u^\epsilon, v-u_t^\epsilon)dt + \int_0^T (-\Delta_p u^\epsilon, v-u_t^\epsilon)dt \\
&+\int_0^T \left(\int_0^t g(t-s)\Delta u^\epsilon(s)ds, v-u_t^\epsilon\right)dt + \int_0^T (-\Delta u_t^\epsilon, v-u_t^\epsilon)dt \\
&-\int_0^T (|u^\epsilon|^{r-2}u^\epsilon \ln|u^\epsilon|, v-u_t^\epsilon)dt = \frac{1}{\epsilon}\int_0^T (\beta(u_t^\epsilon), u_t^\epsilon - v)\,dt \\
&= \frac{1}{\epsilon}\int_0^T (\beta(u_t^\epsilon)-\beta v, u_t^\epsilon - v)\,dt \geq 0,
\end{aligned} \tag{5.35}$$

because $v \in K$ ($\beta(v)=0$) and β is monotone.

From (5.11), (5.12), (5.15), (5.20), (5.30), (5.31), (5.33), (5.34) and the Bannach-Steinhauss Theorem, it follows that there exists a subsequence $(u^\epsilon)_{0<\epsilon<1}$, such that it converge to u as $\epsilon \to 0$, that is

$$u_\epsilon \rightharpoonup u \text{ in } L^\infty(\mathbb{R}^+;H_0^1(\Omega)\cap H^2(\Omega)), \tag{5.36}$$

$$-\Delta_p u_\epsilon \rightharpoonup -\Delta_p u \text{ in } L^2(0,T;H^{-1}(\Omega)), \tag{5.37}$$

$$u_\epsilon \to u \text{ in } L^2(0,T;H_0^1(\Omega))\text{and a.e. in } Q, \tag{5.38}$$

$$u_\epsilon \rightharpoonup u \text{ in } L^\infty(0,T;H_\Gamma^3(\Omega)), \tag{5.39}$$

$$u_t^\epsilon \rightharpoonup u_t \text{ in } L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \tag{5.40}$$

$$u_{tt}^\epsilon \rightharpoonup u_{tt} \text{ in } L^2(0,T;H^{-1}(\Omega)), \tag{5.41}$$

$$|u^\epsilon|^{r-2}u^\epsilon \ln|u^\epsilon| \rightharpoonup |u|^{r-2}u\ln|u| \text{ in } L^2(0,T;L^2(\Omega)), \tag{5.42}$$

$$u_t^\epsilon \to u_t \text{ in } L^2(0,T;H_0^1(\Omega)) \text{ and a.e. in } Q. \tag{5.43}$$

The convergences above are sufficient to pass to the limit in (5.35) with $\epsilon > 0$ to conclude that (4.5) is valid. To complete the proof of Theorem 4.1, it remains to show that $u_t(t) \in K$ a.e.

In the position, we observe that using convergences (5.10)-(5.16) and (5.30)-(5.32), making $m \to \infty$ in (5.1), we can find u^ϵ such that

$$\begin{aligned}
&u_{tt}^\epsilon + \Delta^2 u^\epsilon - \Delta_p u^\epsilon + \int_0^t g(t-s)\Delta u^\epsilon(s)ds \\
&\quad - \Delta u_t^\epsilon - |u^\epsilon|^{r-2}u^\epsilon \ln|u^\epsilon| + \frac{1}{\epsilon}\beta(u_t^\epsilon) = 0 \text{ in } L^2(0,T;H^{-1}(\Omega)).
\end{aligned} \tag{5.44}$$

Then,

$$\beta(u_t^\epsilon) = \epsilon[-u_{tt}^\epsilon - \Delta^2 u^\epsilon + \Delta_p u^\epsilon - \int_0^t g(t-s)\Delta u^\epsilon(s)ds + \Delta u_t^\epsilon + |u^\epsilon|^{r-2}u^\epsilon \ln|u^\epsilon|]. \tag{5.45}$$

So,

$$\beta(u_t^\epsilon) \to 0 \text{ in } \mathcal{D}'(0,T;H^{-1}\Omega).$$

From (5.45) it follows that

$$\beta(u_t^\epsilon) \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)),$$

therefore

$$\beta(u_t^\epsilon) \rightharpoonup 0 \text{ weak in } L^2(0,T;H^{-1}\Omega). \tag{5.46}$$

On the other hand we deduce from (5.45) that

$$0 \leq \int_0^T (\beta(u_t^\epsilon), u_t^\epsilon)\, dt \leq \epsilon\, C. \tag{5.47}$$

Thus

$$\int_0^T (\beta(u_t^\epsilon), u_t^\epsilon) dt \longrightarrow 0. \tag{5.48}$$

We have that

$$\int_0^T (\beta(u_t^\epsilon) - \beta(\varphi), u_t^\epsilon - \varphi)\, dt \geq 0, \quad \forall \varphi \text{ in } L^2(0,T;H_0^1(\Omega)),$$

because β is a monotonous operator. Thus,

$$\int_0^T (\beta(u_t^\epsilon), u_t^\epsilon)\, dt - \int_0^T (\beta(u_t^\epsilon), \varphi)\, dt - \int_0^T (\beta(\varphi), u_t^\epsilon - \varphi)\, dt \geq 0. \tag{5.49}$$

From (5.40), (5.46) and (5.48) we obtain

$$\int_0^T (\beta(\varphi), u_t(t) - \varphi)\, dt \leq 0. \tag{5.50}$$

Taking $\varphi = u_t - \lambda v$, with $v \in L^2(0,T;H_0^1(\Omega))$ and $\lambda > 0$, we deduce using the hemicontinuity of β that

$$\beta(u_t(t)) = 0, \tag{5.51}$$

and this implies that $u_t(t) \in K$ a. e.

6. Uniqueness

Let u^1, u^2 two solutions of (4.5) , $w = u^2 - u^1$ and $t \in (0,T)$. Because $u_t \in L^2(0,T;H_0^1(\Omega)$, we can talking u_t^1 (resp. u_t^2) in the inequality (4.5) relative to v^2 (resp. v^1) and adding up the results we obtain

$$\begin{aligned}
&-\int_0^t (w_{tt}, w_t) ds - \int_0^t (\Delta^2 w, w_t) ds + \int_0^t (\Delta_p u^1, w_t) ds - \int_0^t (\Delta_p u^2, w_t) ds \\
&+ \int_0^t \left(\int_0^t g(t-s)\Delta w(s) ds, w_t \right) ds + \int_0^t (\Delta w_t, w_t) ds \\
&- \int_0^t (|u^1|^{r-2} u^1 \ln |u^1|, w_t) ds + \int_0^t (|u^2|^{r-2} u^2 \ln |u^2|, w_t) ds \geq 0,
\end{aligned}$$

thus, we have

$$\frac{1}{2}\int_0^t \frac{d}{dt}\left(|w_t(t)|^2+|\Delta w(t)|^2\right)ds+\int_0^t |\nabla w_t(t)|^2 ds$$
$$\leq \int_0^t \langle \Delta_p u^1(t)-\Delta_p u^2(t), w_t(t)\rangle ds+\int_0^t\int_0^t g(t-s)(\nabla w(s),\nabla w_t(t))dsd\sigma$$
$$\int_0^t \left(|u^1(t)|^{r-2}u^1(t)\ln|u^1(t)|-|u^2(t)|^{r-2}u^2(t)\ln|u^2(t)|, w_t(t)\right)ds.$$

By Lemma 2.2, we derive

$$\begin{aligned}
&\frac{1}{2}\int_0^t \frac{d}{dt}\left\{|w_t(t)|^2+|\Delta w(t)|^2-\left(\int_0^t g(s)ds\right)|\nabla w(t)|^2+(g\diamond\nabla w)(t)\right\}ds\\
&+\int_0^t |\nabla w_t(t)|^2 ds\leq \int_0^t |\langle \Delta_p u^1(t)-\Delta_p u^2(t), w_t(t)\rangle| ds\\
&+\int_0^t\int_\Omega \left(|u^1(t)|^{r-2}u^1(t)\ln|u^1(t)|-|u^2(t)|^{r-2}u^2(t)\ln|u^2(t)|, w_t(t)\right)dxds.
\end{aligned}\tag{6.1}$$

From Mean Value Theorem,

$$\begin{aligned}
&|\langle \Delta_p u^1(t)-\Delta_p u^2(t), w_t(t)\rangle|\\
&\leq C\left(|\nabla u^1(t)|_{2(p-1)}^{p-2}+|\nabla u_2(t)|_{2(p-1)}^{p-2}\right)|\nabla w(t)|_{2(p-1)}|\nabla w_t(t)|\\
&\leq C|\Delta w(t)|^2+\frac{1}{4}|\nabla w_t(t)|^2,
\end{aligned}\tag{6.2}$$

for some constant $C>0$, and

$$\begin{aligned}
&\int_0^t\int_\Omega \left(|u^1(t)|^{r-2}u^1(t)\ln|u^1(t)|-|u^2(t)|^{r-2}u^2(t)\ln|u^2(t)|, w_t(t)\right)dxds\\
&\leq \int_0^t\int_\Omega |\theta u^1(t)+(1-\theta)u^2(t))|^{r-2}|w(t)||w_t(t)|dxds\\
&+(r-1)\int_0^t\int_\Omega |\theta u^1(t)+(1-\theta)u^2|^{r-2}\ln|\theta u^1(t)\\
&+(1-\theta)u^2(t)||w(t)|w_t(t)|dxds=I_1+I_2, 0<\theta<1.
\end{aligned}\tag{6.3}$$

Hence, from the Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
&\int_\Omega |\theta u^1(t)+(1-\theta)u^2(t))|^{r-2}|w(t)||w_t(t)|dx\\
&\leq |\theta u^1(t)+(1-\theta)u^2(t)|_{n(r-2)}^{r-2}|w(t)|_{\frac{2n}{n-2}}|w_t(t)|\\
&\leq C_1^{r-2}C_2C_3|\Delta w(t)||\nabla w_t(t)|\leq C|\Delta w(t)|^2+\frac{1}{4}|\nabla w_t(t)|^2,
\end{aligned}\tag{6.4}$$

where C_1 , C_2 and C_3 are constants satisfying

$$|\theta u^1(t)+(1-\theta)u^2(t)|_{n(r-2)}^{r-2}|\leq C_1|\theta u^1(t)+(1-\theta)u^2(t)|,$$

$$|w(t)|_{\frac{2n}{n-2}}\leq C|w(t)|\leq C_2|\Delta w(t)| \text{ and } |w(t)|\leq C_3|\nabla w(t)|.$$

Also we used the condition $n(p-2) < \dfrac{2n}{n-2}$.

Now, using the calculation similar to (5.24), it follows that

$$\begin{aligned}&\int_\Omega |\theta u^1(t)+(1-\theta)u^2|^{r-2}\ln|\theta u^1(t)+(1-\theta)u^2(t)|^n dx\\ &\le (e(r-2)^{-n})|\Omega| + (e(r-2))^{-n}|\theta u^1(t)+(1-\theta)u^2(t)|_{n(r-2)}^{n(r-2)}\\ &\le (e(r-2)^{-n})|\Omega| + (e(r-2))^{-1}C_4|\theta u^1(t)+(1-\theta)u^2(t)|^{n(r-2)} \le C.\end{aligned} \tag{6.5}$$

Inserting (6.5) into I_2, we have

$$\begin{aligned}&I_2 = (r-1)\int_0^t\int_\Omega |\theta u^1(t)+(1-\theta)u^2|^{r-2}\ln|\theta u^1(t)\\ &+(1-\theta)u^2(t)||w(t)|w_t(t)|dxds\\ &\le (r-1)\int_0^t\left(\int_\Omega ||\theta u^1(t)+(1-\theta)u^2|^{r-2}\ln|\theta u^1(t)+\theta u^2(t)||^n dx\right)^{\frac{1}{n}}\\ &\times|w_t(t)||w(t)|_{\frac{2n}{n-2}}ds \le C|\Delta w(t)|^2+\frac{1}{4}|\nabla w_t(t)|^2.\end{aligned} \tag{6.6}$$

By (6.1), (6.2), (6.4) and (6.6) we get

$$\begin{aligned}&\int_0^t \frac{d}{dt}\left\{|w_t(t)|^2+|\Delta w(t)|^2-\left(\int_0^t g(s)ds\right)|\nabla w(t)|^2+(g\diamond\nabla w)(t)\right\}ds\\ &\qquad\qquad+\int_0^t|\nabla w_t(t)|^2ds \le C\int_0^t(|\Delta w(t)|^2+|\nabla w_t(t)|^2)ds.\end{aligned} \tag{6.7}$$

Putting,

$$\Phi(t) = |w_t(t)|^2+|\Delta w(t)|^2-\left(\int_0^t g(s)ds\right)|\nabla w(t)|^2+(g\diamond\nabla w)(t)$$

and using (H_3), we have

$$|\Delta w(t)|^2-\left(\int_0^t g(s)ds\right)|\nabla w(t)|^2 \ge l|\Delta w(t)|^2 \ge 0.$$

As $(g\diamond\nabla w)(t)\ge 0$, we have from (6.7) that $\int_0^t \frac{d}{dt}\Phi(t) \le C\Phi(t)$ and because $\Phi(0)=0$, follows from the Gronwall lemma that

$$|w_t(t)|^2+l|\Delta w(t)|^2 \le \Phi(t) \le 0,$$

which proves that $w=0$ in $H_0^1(\Omega)\cap H^2(\Omega)$.

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