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On Two One-Parameter Variants of the Hardy-Hilbert Integral Inequality

Christophe Chesneau[1,*], Josip Pečarić[2]

[1] *Department of Mathematics, LMNO, University of Caen-Normandie, Caen, 14032, France.*
christophe.chesneau@gmail.com

[2] *Croatian Academy of Sciences and Arts, Trg Nikole Šubića Zrinskog 11, Zagreb, 10000, Croatia*
jopecaric@gmail.com

[*] *Correspondence: christophe.chesneau@gmail.com*

Abstract. This article establishes two new one-parameter variants of the Hardy-Hilbert integral inequality. These results are derived by modifying an existing framework that incorporates the absolute difference between unity and the product of two variables, raised to an adjustable power parameter. By employing two distinct approaches, we obtain different weighted integral norms for the primary functions. The proofs are presented in comprehensive detail and are entirely self-contained.

1. Introduction

We begin by presenting a fundamental result in the theory of mathematical inequalities: the classical Hardy-Hilbert integral inequality. Let $p > 1$, $q = p/(p-1)$ satisfying $1/p + 1/q = 1$, and $f, g : (0, +\infty) \to (0, +\infty)$ be two (measurable) functions such that

$$\int_0^{+\infty} f^p(x)dx < +\infty, \quad \int_0^{+\infty} g^q(y)dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}. \tag{1}$$

Beyond the well-established sharpness of the upper bound and its optimal constant factor [8, 9, 18], the study of Hardy-Hilbert-type integral inequalities continues to expand. In particular, two-dimensional variations have received considerable attention [1, 2, 4–7, 10, 11, 16, 17], and recent efforts have pushed these results into higher dimensions [3, 12, 14, 15, 19, 20].

For the purposes of this study, we present a specific variant of the Hardy-Hilbert integral inequality that incorporates the absolute difference between unity and the product of two

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variables, as established in [5, Theorem 2.1]. Let $\lambda \in (0,1)$, $p > 1$, $q = p/(p-1)$, and $f, g : (0,+\infty) \to (0,+\infty)$ be two functions such that

$$\int_0^{+\infty} f^p(x) x^{(1-\lambda/2)p-1} dx < +\infty, \quad \int_0^{+\infty} g^q(y) y^{(1-\lambda/2)q-1} dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{|1-xy|^{\lambda}} f(x)g(y)dxdy$$
$$\leq 2B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[\int_0^{+\infty} f^p(x)x^{(1-\lambda/2)p-1}dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)y^{(1-\lambda/2)q-1}dy\right]^{1/q}, \tag{2}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $a, b > 0$, is the beta function at a and b.

This inequality serves as an alternative formulation to the one presented in [13, Corollary 3]. It possesses a notably more complex architecture than the classical Hardy-Hilbert integral inequality. Specifically, the primary integrand incorporates the function $|1-xy|^{\lambda}$, the constant factor is determined by the beta function, and the norms of f and g are modulated by power-type weights of x and y.

In this article, we propose a modification of this framework by considering a special double integral, given by

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{|1-xy|^{\lambda}(1+xy)^{\lambda}} dxdy$$

with $\lambda > 0$. The primary novelty lies in the denominator term, which establishes a multiplicative relationship between $|1-xy|^{\lambda}$ and $(1+xy)^{\lambda}$. This relationship can be treated in several ways, depending on the research objective.

To be more specific, this study pursues two primary objectives:

(1) For $\lambda \in (0,1)$, applying the algebraic identity

$$|1-xy|^{\lambda}(1+xy)^{\lambda} = |1-x^2y^2|^{\lambda},$$

we demonstrate how the inequality established in Equation (2) can be utilized to derive a Hardy-Hilbert integral inequality for the double integral under consideration.

(2) For the specific range $\lambda \in (1/2, 1)$, another approach is proposed. We establish a Hardy-Hilbert-type integral inequality through a systematic decomposition of the integrand and rigorous integral calculus.

The two results operate under different assumptions, yielding distinct weighted integral norms for the primary functions.

The rest of the article is as follows: Section 2 is dedicated to presenting our results, which are subsequently followed by concluding remarks in Section 3.

2. Results

2.1. A result derived from Equation (2). We present a result derived from the inequality in Equation (2).

Theorem 2.1. *Let $\lambda \in (0,1)$, $p > 1$, $q = p/(p-1)$, and $f, g : (0,+\infty) \to (0,+\infty)$ be two functions such that*

$$\int_0^{+\infty} f^p(x)x^{(1-\lambda)p-1}dx < +\infty, \qquad \int_0^{+\infty} g^q(y)y^{(1-\lambda)q-1}dy < +\infty.$$

Then we have

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{|1-xy|^{\lambda}(1+xy)^{\lambda}}dudv\\ &\leq B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[\int_0^{+\infty} f^p(x)x^{(1-\lambda)p-1}dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)y^{(1-\lambda)q-1}dy\right]^{1/q}.\end{aligned}$$

Proof. Let us consider the double integral

$$I = \int_0^{+\infty}\int_0^{+\infty} \frac{[f(x^{1/2})/x^{1/2}][g(y^{1/2})/y^{1/2}]}{|1-xy|^{\lambda}}dxdy.$$

Then, performing the changes of variables $x = u^2$ and $y = v^2$, and applying the algebraic identity $|1-u^2v^2|^{\lambda} = |1-uv|^{\lambda}(1+uv)^{\lambda}$, we get

$$\begin{aligned}I &= 4\int_0^{+\infty}\int_0^{+\infty} \frac{f(u)g(v)}{|1-u^2v^2|^{\lambda}}dudv\\ &= 4\int_0^{+\infty}\int_0^{+\infty} \frac{f(u)g(v)}{|1-uv|^{\lambda}(1+uv)^{\lambda}}dudv.\end{aligned} \tag{3}$$

We recognize the double integral under consideration (multiplied by 4).

On the other hand, we can bound I directly. Applying Equation (2) to the functions $f(x^{1/2})/x^{1/2}$ and $g(y^{1/2})/y^{1/2}$, making the changes of variables $x = u^2$ and $y = v^2$, and using $1/p + 1/q = 1$, we have

$$\begin{aligned}I &\leq 2B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[\int_0^{+\infty}\left[\frac{f(x^{1/2})}{x^{1/2}}\right]^p x^{(1-\lambda/2)p-1}dx\right]^{1/p}\left[\int_0^{+\infty}\left[\frac{g(y^{1/2})}{y^{1/2}}\right]^q y^{(1-\lambda/2)q-1}dy\right]^{1/q}\\ &= 2B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[\int_0^{+\infty} f^p(x^{1/2})x^{(1-\lambda)p/2-1}dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y^{1/2})y^{(1-\lambda)q/2-1}dy\right]^{1/q}\\ &= 2B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[2\int_0^{+\infty} f^p(u)u^{(1-\lambda)p-1}du\right]^{1/p}\left[2\int_0^{+\infty} g^q(v)v^{(1-\lambda)q-1}dv\right]^{1/q}\\ &= 4B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[\int_0^{+\infty} f^p(u)u^{(1-\lambda)p-1}du\right]^{1/p}\left[\int_0^{+\infty} g^q(v)v^{(1-\lambda)q-1}dv\right]^{1/q}.\end{aligned} \tag{4}$$

It follows from Equations (3) and (4) that

$$\begin{aligned}&4\int_0^{+\infty}\int_0^{+\infty} \frac{f(u)g(v)}{|1-uv|^{\lambda}(1+uv)^{\lambda}}dudv\\ &\leq 4B\left(\frac{\lambda}{2}, 1-\lambda\right)\left[\int_0^{+\infty} f^p(u)u^{(1-\lambda)p-1}du\right]^{1/p}\left[\int_0^{+\infty} g^q(v)v^{(1-\lambda)q-1}dv\right]^{1/q},\end{aligned}$$

so that, after a simplification and a minor change in the notation,

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{|1-xy|^\lambda(1+xy)^\lambda}dudv$$
$$\leq B\left(\frac{\lambda}{2},1-\lambda\right)\left[\int_0^{+\infty}f^p(x)x^{(1-\lambda)p-1}dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)y^{(1-\lambda)q-1}dy\right]^{1/q}.$$

This completes the proof. □

Compared with the inequality in Equation (2), we note that the constant factors differ by a multiplicative factor, and the weighted integral norms for the primary functions have been adapted accordingly.

In the section below, we outline an alternative strategy. Conceptually, this approach relies on a pointwise proof and applies the algebraic identity $|1-x^2|^\lambda = |1-x|^\lambda(1+x)^\lambda$ to evaluate a new constant factor.

2.2. **Another result.** The theorem below establishes this result alongside a complete, self-contained proof. Compared with Theorem 2.1, it focuses on the parameter range $\lambda \in (1/2, 1)$ and incorporates distinct weighted integral norms for the primary functions.

Theorem 2.2. *Let $\lambda \in (1/2, 1)$, $p > 1$, $q = p/(p-1)$, and $f, g : (0, +\infty) \to (0, +\infty)$ be two functions such that*

$$\int_0^{+\infty} f^p(x)x^{(2\lambda-1)p-1}dx < +\infty, \quad \int_0^{+\infty} g^q(y)y^{(2\lambda-1)q-1}dy < +\infty.$$

Then we have

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{|1-xy|^\lambda(1+xy)^\lambda}dxdy$$
$$\leq C\left[\int_0^{+\infty}f^p(x)x^{(2\lambda-1)p-1}dx\right]^{1/p}\left[\int_0^{+\infty}g^q(y)y^{(2\lambda-1)q-1}dy\right]^{1/q},$$

where

$$C = \frac{1}{2}\left(B(1-\lambda, 1-\lambda) + B(2\lambda-1, 1-\lambda)\right). \tag{5}$$

Proof. Using a special integrand decomposition and $1/p + 1/q = 1$, we can write

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)}{|1-xy|^\lambda(1+xy)^\lambda}dxdy$$
$$= \int_0^{+\infty}\int_0^{+\infty}\frac{f(x)y^{(1-2\lambda)/p}x^{(2\lambda-1)/q}}{|1-xy|^{\lambda/p}(1+xy)^{\lambda/p}} \times \frac{g(y)x^{(1-2\lambda)/q}y^{(2\lambda-1)/p}}{|1-xy|^{\lambda/q}(1+xy)^{\lambda/q}}dxdy. \tag{6}$$

Based on this, it follows from the Hölder integral inequality that

$$\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)y^{(1-2\lambda)/p}x^{(2\lambda-1)/q}}{|1-xy|^{\lambda/p}(1+xy)^{\lambda/p}} \times \frac{g(y)x^{(1-2\lambda)/q}y^{(2\lambda-1)/p}}{|1-xy|^{\lambda/q}(1+xy)^{\lambda/q}}dxdy$$
$$\leq A^{1/p}B^{1/q}, \tag{7}$$

where

$$A = \int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x) y^{1-2\lambda} x^{(2\lambda-1)p/q}}{|1-xy|^\lambda (1+xy)^\lambda} dxdy$$

and

$$B = \int_0^{+\infty} \int_0^{+\infty} \frac{g^q(y) x^{1-2\lambda} y^{(2\lambda-1)q/p}}{|1-xy|^\lambda (1+xy)^\lambda} dxdy.$$

We first analyze the term A, before proceeding to the evaluation of the term B.

For the term A, using $p/q = p-1$ and arranging the integral in a suitable way, we obtain

$$\begin{aligned} A &= \int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x) y^{1-2\lambda} x^{(2\lambda-1)p/q}}{|1-xy|^\lambda (1+xy)^\lambda} dxdy \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{f^p(x) y^{1-2\lambda} x^{(2\lambda-1)(p-1)}}{|1-xy|^\lambda (1+xy)^\lambda} dxdy \\ &= \int_0^{+\infty} f^p(x) x^{(2\lambda-1)p-1} I_x dx \end{aligned}$$

where

$$I_x = \int_0^{+\infty} \frac{(xy)^{1-2\lambda}}{|1-xy|^\lambda (1+xy)^\lambda} x dy. \tag{8}$$

Let us now determine I_x. Making the change of variables $z = xy$, we get

$$I_x = \int_0^{+\infty} \frac{z^{1-2\lambda}}{|1-z|^\lambda (1+z)^\lambda} dz.$$

Based on this expression, the Chasles integral relation, the changes of variables $u = z^2$ and $v = 1/u$, and the definition of the beta function give

$$\begin{aligned} I_x &= \int_0^1 \frac{z^{1-2\lambda}}{|1-z|^\lambda (1+z)^\lambda} dz + \int_1^{+\infty} \frac{z^{1-2\lambda}}{|1-z|^\lambda (1+z)^\lambda} dz \\ &= \int_0^1 \frac{z^{1-2\lambda}}{(1-z^2)^\lambda} dz + \int_1^{+\infty} \frac{z^{1-2\lambda}}{(z^2-1)^\lambda} dz \\ &= \frac{1}{2} \int_0^1 \frac{u^{-\lambda}}{(1-u)^\lambda} du + \frac{1}{2} \int_1^{+\infty} \frac{u^{-\lambda}}{(u-1)^\lambda} du \\ &= \frac{1}{2} \int_0^1 \frac{u^{-\lambda}}{(1-u)^\lambda} du + \frac{1}{2} \int_0^1 \frac{v^{2\lambda-2}}{(1-v)^\lambda} dv \\ &= \frac{1}{2} \left(\int_0^1 u^{(1-\lambda)-1} (1-u)^{(1-\lambda)-1} du + \int_0^1 v^{(2\lambda-1)-1} (1-v)^{(1-\lambda)-1} dv \right) \\ &= \frac{1}{2} \left(B(1-\lambda, 1-\lambda) + B(2\lambda-1, 1-\lambda) \right) \\ &= C, \end{aligned} \tag{9}$$

where C is given in Equation (5). Combining the above equations, we get

$$A = C \int_0^{+\infty} f^p(x) x^{(2\lambda-1)p-1} dx. \tag{10}$$

For the term B, we proceed similarly, paying attention to the variables x and y under consideration. We have

$$\begin{aligned} B &= \int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)x^{1-2\lambda}y^{(2\lambda-1)q/p}}{|1-xy|^\lambda(1+xy)^\lambda}dxdy \\ &= \int_0^{+\infty}\int_0^{+\infty} \frac{g^q(y)x^{1-2\lambda}y^{(2\lambda-1)(q-1)}}{|1-xy|^\lambda(1+xy)^\lambda}dxdy \\ &= \int_0^{+\infty} g^p(y)y^{(2\lambda-1)q-1}J_y dy, \end{aligned}$$

where

$$J_y = \int_0^{+\infty} \frac{(xy)^{1-2\lambda}}{|1-xy|^\lambda(1+xy)^\lambda} y dx = I_y$$

and I_y is defined as in Equation (8) with $x = y$. Therefore, using Equation (9), we have

$$J_y = C.$$

Combining the above equations, we get

$$B = C\int_0^{+\infty} g^q(y)y^{(2\lambda-1)q-1}dy. \tag{11}$$

It follows from Equations (6), (7), (10) and (11) and $1/p + 1/q = 1$ that

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{|1-xy|^\lambda(1+xy)^\lambda}dxdy \\ &\leq \left[C\int_0^{+\infty} f^p(x)x^{(2\lambda-1)p-1}dx\right]^{1/p}\left[C\int_0^{+\infty} g^q(y)y^{(2\lambda-1)q-1}dy\right]^{1/q} \\ &= C\left[\int_0^{+\infty} f^p(x)x^{(2\lambda-1)p-1}dx\right]^{1/p}\left[\int_0^{+\infty} g^q(y)y^{(2\lambda-1)q-1}dy\right]^{1/q}. \end{aligned}$$

This concludes the proof. □

Theorem 2.2 complements Theorem 2.1, though the differing weighted integral norms complicate any direct comparison.

3. Conclusion

In conclusion, this article establishes two new one-parameter Hardy-Hilbert integral inequalities by employing precise integral decomposition. These results, characterized by their explicit constant factors involving the beta function, provide a sophisticated extension to the classical theory of integral inequalities. Natural perspectives of this research include the investigation of Hardy-Hilbert integral inequalities derived from

- the double integral

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{|1-xy|^\lambda(1+xy)^\tau}dxdy$$

with $\lambda, \tau > 0$, the introduction of the parameter τ adding a new dimension to the problem,

- the triple integral
$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{f(x)g(y)h(z)}{|1-xyz|^{\lambda}(1+xyz)^{\lambda}}dxdydz,$$
with $\lambda>0$, where $h:(0,+\infty)\to(0,+\infty)$ is an additional function.

Each of these directions promises to yield novel results and further enrich the field of integral inequalities.

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References

[1] V. Adiyasuren, T. Batbold, M. Krnić, Hilbert-type inequalities involving differential operators, the best constants and applications, Math. Inequal. Appl. 18 (2015), 111–124.

[2] T. Batbold, M. Krnić, J. Pečarić, P. Vuković, Further Development of Hilbert-type Inequalities / Selected Topics in Hilbert-type Inequalities, Monographs in Inequalities 13, Element, Zagreb, (2017).

[3] A. Bényi, C.T. Oh, Best constant for certain multilinear integral operator, J. Inequal. Appl. 2006 (2006), 127694.

[4] Q. Chen, B.C. Yang, A survey on the study of Hilbert-type inequalities, J. Inequal. Appl. 2015 (2015), 1–29.

[5] C. Chesneau, Two contributions to the absolute-value-Hardy-Hilbert-type integral inequalities, Asia Math. 9 (2025), 12–19.

[6] C. Chesneau, On a Hardy-Hilbert-type integral inequality involving an adjustable function, Pecaric J. Math. Ineq. 2 (2026), 22–28.

[7] H. Du, Y. Miao, Several new Hardy-Hilbert's inequalities, Filomat 25 (2011), 153–162.

[8] G.H. Hardy, Note on a theorem of Hilbert concerning series of positive terms, Proc. London Math. Soc. 23 (1925), 45–46.

[9] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, (1934).

[10] M. Krnić, J. Pečarić, I. Perić, P. Vuković, Recent Advances in Hilbert-type Inequalities / A Unified Treatment of Hilbert-type Inequalities, Monographs in Inequalities 3, Element, Zagreb, (2012).

[11] Y. Li, Y. Qian, B. He, On further analogs of Hilbert's inequality, Int. J. Math. Math. Sci. 2007 (2007), 76329.

[12] W.T. Sulaiman, On Hardy-Hilbert's integral inequality, J. Inequal. Pure Appl. Math. 5 (2004), 1–9.

[13] W.T. Sulaiman, Hardy-Hilbert's integral inequalities via homogeneous functions and some other generalizations, Acta Comment. Univ. Tartu Math. 11 (2007), 23–32.

[14] W.T. Sulaiman, New types of Hardy-Hilbert's integral inequality, Gen. Math. Notes 2 (2011), 111–118.

[15] B. Sun, A multiple Hilbert-type integral inequality with the best constant factor, J. Inequal. Appl. 2007 (2007), 71049.

[16] J.S. Xu, Hardy-Hilbert's inequalities with two parameters, Adv. Math. 36 (2007), 63–76.

[17] B.C. Yang, The Norm of Operator and Hilbert-Type Inequalities, Science Press, Beijing, (2009).

[18] B.C. Yang, Hilbert-Type Integral Inequalities, Bentham Science Publishers, Sharjah, (2009).

[19] B.C. Yang, M. Krnić, On the norm of a multi-dimensional Hilbert-type operator, Sarajevo J. Math. 7 (2011), 223–243.

[20] W.Y. Zhong, B.C. Yang, On a multiple Hilbert-type integral inequality with the symmetric kernel, J. Inequal. Appl. 2007 (2007), 27962.