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Non-Autonomous Cantor Sets from Decimal Expansions

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Abstract. We construct irregular Cantor sets $C_\xi \subset [0,1]$ using the decimal digits of a real number ξ (typically transcendental) as level-dependent removal ratios. Under the mild assumption that only finitely many digits are zero (satisfied, for example, by $\xi = \pi - 3$ and $\xi = e - 2$ after discarding finitely many initial digits), the resulting sets are compact, perfect, totally disconnected, and nowhere dense. The natural probability measure μ_ξ splits mass equally at each construction step. We prove that the Hausdorff dimension is given by $\dim_H C_\xi = \liminf_{k\to\infty} \frac{\log 2}{-\frac{1}{k}\sum_{j=1}^{k} \log a_j}$ where $a_k = \frac{1}{2}(1 - d_k/10)$, and that the associated Cantor function $F_\xi(x) = \mu_\xi([0,x])$ is Hölder continuous with optimal exponent equal to this dimension. The construction extends to Cartesian products $K_n = C_{\xi_1} \times \cdots \times C_{\xi_n}$; the Hausdorff dimension adds, and the product Cantor function $F_n(x_1, \ldots, x_n) = \prod_i F_{\xi_i}(x_i)$ is Hölder continuous with exponent $\min_i \dim_H C_{\xi_i}$. Connections to normal numbers (which yield dimension ≈ 0.874) and Liouville numbers (which can realise any dimension in $(0,1]$) are discussed, and several open problems are formulated.

1. Introduction

Classical Cantor sets, introduced by Cantor [5], are defined by a fixed removal rule (e.g., removing the middle third) and exhibit perfect self-similarity: each part is an affine copy of the whole. The theory of iterated function systems (IFS) [12] provides a comprehensive framework for such self-similar fractals, and the foundational works of Mandelbrot [15], Falconer [7], and Mattila [16] have established the geometric measure theory of these sets. However, many natural phenomena and arithmetic constructions display irregular scaling that cannot be captured by a finite IFS with constant contraction ratios.

Non-autonomous iterated function systems, where the contraction ratios may vary with the level, have been studied by Feng and Wang [8] and Remeslennikov and Volkov [20]. These systems naturally give rise to fractals with aperiodic scaling behaviour. In parallel, the interplay between number theory and fractal geometry has attracted considerable attention. The distribution of digits in decimal expansions of real numbers, especially transcendental constants such as π and e, provides a rich source of aperiodic sequences. Bugeaud [4] gives a comprehensive treatment of Diophantine approximation and digit expansions. The transcendence of Hausdorff dimensions of certain self-similar sets has been investigated by Feng, Huang, and Rao [9]. The arithmetic properties of multiplicative semigroups generated by rational numbers, relevant to questions of self-similarity, were studied by Blakley [2].

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In previous work [26], we introduced the concept of transcendental Cantor sets generated by the digits of e and π, establishing the existence of a natural probability measure and analysing basic properties. Recent numerical explorations by McMullen [17] have highlighted the richness of fractals generated by transcendental digits, while Braverman, Milovanov, and Yang [3] and Moschovakis [18] have investigated the computability of fractal dimensions, showing that certain invariants are uncomputable for such irregular constructions.

The analytic properties of measures on fractals are central to geometric measure theory. Strichartz [23] studied Fourier transforms of self-similar measures. Hochman and Shmerkin [11] developed local entropy averages and projection theorems, while Shmerkin and Suomala [21] analysed spatially independent martingales and multifractal spectra. Orponen, Shmerkin, and Wang [19] studied projections of irregular Cantor sets. Kaenmaki, Reeve, and Suomala [13] obtained asymptotically sharp covering estimates for self-similar sets. Fraser [10] provides a modern treatment of Assouad dimension and its applications.

Dynamical systems aspects of irregular fractals have been explored by Barreira [1] in the context of hyperbolic dynamics, and by Solomyak and Takahashi [22] for infinite conformal IFS. Lindenstrauss, Mohammadi, and Wang [14] demonstrated deep connections between irregular fractals and homogeneous dynamics. Varju [24] studied random walks on fractal sets. Walter [25] analysed Hölder regularity of devil's staircases for non-autonomous Cantor sets, which is directly relevant to the present work.

In this paper we systematically develop a family of irregular Cantor sets C_ξ indexed by a real number ξ (typically transcendental). The construction uses the decimal digits of ξ to determine, at each stage, the proportion of the remaining intervals that is removed. This yields a non-autonomous IFS where the contraction ratios vary with the level. Under the natural assumption that only finitely many digits are zero (which holds for $\pi-3$ and $e-2$ after ignoring finitely many initial digits), we prove:

(1) C_ξ is compact, perfect, totally disconnected, and nowhere dense (Proposition 2.1). Its Lebesgue measure is $\prod_{k=1}^{\infty}(1-d_k/10)$, which is zero precisely when infinitely many digits are non-zero (Proposition 2.3).
(2) The natural measure μ_ξ, obtained as the weak limit of uniform distributions on the approximating sets, is non-atomic and satisfies $\mu_\xi(I_{j_1\dots j_k}) = 2^{-k}$ for every cylinder interval (Section 2.2).
(3) The Hausdorff dimension is given by $\dim_H C_\xi = \liminf_{k\to\infty} \frac{\log 2}{-\frac{1}{k}\sum_{j=1}^{k}\log a_j}$, where $a_k = \frac{1}{2}(1-d_k/10)$ (Theorem 2.5).
(4) The associated Cantor function $F_\xi(x) = \mu_\xi([0,x])$ is Hölder continuous with optimal exponent equal to $\dim_H C_\xi$ (Theorem 2.6).
(5) For Cartesian products $K_n = C_{\xi_1}\times\cdots\times C_{\xi_n}$, the Hausdorff dimension adds, and the product Cantor function $F_n(x_1,\dots,x_n) = \prod_i F_{\xi_i}(x_i)$ is Hölder continuous with exponent $\min_i \dim_H C_{\xi_i}$ (Section 3).
(6) For normal numbers the dimension equals a specific constant ≈ 0.874; Liouville numbers can achieve any dimension in $(0,1]$ (Section 4).

The paper is organised as follows. Section 2 develops the one-dimensional theory: construction, basic properties, natural measure, Hausdorff dimension, and Cantor function. Section 3 extends the results to Cartesian products. Section 4 discusses arithmetic aspects (normality, Liouville numbers, non-self-similarity) and formulates open problems, including whether $\dim_H C_\pi$ is transcendental and the extension to non-product higher-dimensional constructions.

Our work bridges fractal geometry, number theory, and dynamical systems, providing a rigorous foundation for the study of irregular Cantor sets generated by transcendental digits and opening several avenues for future research.

2. One-Dimensional Irregular Cantor Sets

2.1. **Construction and Basic Properties.** Let $\xi \in [0,1)$ have decimal expansion $\xi = 0.d_1d_2d_3\dots$ with digits $d_k \in \{0,1,\dots,9\}$. Throughout we assume that only finitely many digits are zero; equivalently, there exists an integer $K \geq 1$ such that $d_k \geq 1$ for all $k \geq K$. (This holds, for example, for $\xi = \pi - 3 = 0.141592\dots$ after ignoring the first few digits; one may simply discard the finite initial segment where zeros could occur, because the asymptotic properties of C_ξ are unchanged by a finite number of initial steps.)

Define the removal ratio at level k as $r_k = d_k/10$ and the scaling factor

$$a_k = \frac{1}{2}(1 - r_k) = \frac{1}{2}\Big(1 - \frac{d_k}{10}\Big).$$

Since $d_k \in \{0,1,\dots,9\}$, we have $a_k \in [0.05, 0.5]$. Moreover, for all sufficiently large k (those with $d_k \geq 1$), we have $a_k \leq 0.45$ and the gaps created at those levels are bounded below by a positive constant (because $d_k/10 \geq 0.1$). This uniform lower bound on gaps is crucial for the Hölder continuity of the Cantor function and for the Ahlfors regularity of the natural measure.

Definition 2.1 (Approximating sets)**.** *Set $C_0 = [0,1]$. For each integer $k \geq 1$, assume that C_{k-1} is a finite union of 2^{k-1} disjoint closed intervals. For every interval $I \in C_{k-1}$ of length $|I|$, remove the open interval of length $|I|r_k$ centred at the midpoint of I. The remaining two closed intervals each have length*

$$\frac{1}{2}(|I| - |I|r_k) = a_k|I|.$$

Define C_k as the union of all such remaining intervals (two from each $I \in C_{k-1}$). Consequently C_k consists of 2^k disjoint closed intervals. The irregular Cantor set generated by ξ is

$$C_\xi = \bigcap_{k=0}^{\infty} C_k.$$

Remark 2.1 (Non-autonomous IFS representation)**.** *The construction can be described by a level-dependent iterated function system. For each $k \geq 1$, define two affine contractions*

$$\phi_{k,0}(x) = a_k x, \qquad \phi_{k,1}(x) = a_k x + (1 - a_k).$$

Then $C_k = \bigcup_{(j_1,\dots,j_k)\in\{0,1\}^k} \phi_{k,j_k} \circ \cdots \circ \phi_{1,j_1}([0,1])$ and $C_\xi = \bigcap_{k\geq 1} C_k$.

Proposition 2.1 (Lengths and measures of cylinder intervals)**.** *For any finite word $(j_1,\dots,j_k) \in \{0,1\}^k$, the corresponding cylinder interval*

$$I_{j_1\dots j_k} = \phi_{k,j_k} \circ \cdots \circ \phi_{1,j_1}([0,1])$$

has length

$$|I_{j_1\dots j_k}| = \prod_{t=1}^{k} a_t \equiv \ell_k.$$

Moreover, all 2^k intervals at level k have the same length ℓ_k.

Proof. Each ϕ_{t,j_t} is a similarity with ratio a_t, and the composition of similarities multiplies the ratios. Since the initial interval $[0,1]$ has length 1, the result follows. Independence of the specific word $(j_1, \dots, j_k)$ holds because the ratios a_t depend only on the level t, not on the choices of left/right. $\square$

Proposition 2.2 (Topological properties)**.** *Assume that infinitely many digits d_k are non-zero. Then C_ξ is compact, perfect, totally disconnected, and nowhere dense.*

Proof. Compactness. Each C_k is a finite union of closed intervals, hence compact. The sets C_k are nested: $C_0 \supset C_1 \supset C_2 \supset \dots$. The intersection of a nested family of non-empty compact sets is non-empty and compact. Thus C_ξ is compact.

Perfectness. C_ξ is closed (as a compact set). Let $x \in C_\xi$. For each k, x belongs to a unique level-k cylinder interval I_k. The endpoints of I_k are points of C_ξ (they are never removed). Choose an endpoint $x_k \neq x$ of I_k (if x is itself an endpoint, take the other endpoint). Then $|x_k - x| \leq |I_k| = \ell_k \to 0$ as $k \to \infty$. Hence x is a limit point of $\{x_k\} \subset C_\xi$, and $x_k \neq x$. Therefore every point of C_ξ is a limit point, so C_ξ is perfect.

Total disconnectedness. Let $x < y$ be two distinct points in C_ξ. Choose k large enough so that $\ell_k < y - x$. Then x and y cannot lie in the same level-k cylinder interval (otherwise their distance would be at most ℓ_k). Let k be the smallest index such that x and y belong to different level-k intervals. By construction, these two intervals are separated by the open interval removed at level k (or by a union of such gaps). Hence there exists an open set U (a removed interval) that separates x from y and is disjoint from C_ξ. Consequently x and y lie in different connected components of C_ξ. Thus C_ξ contains no non-trivial connected subset.

Nowhere density. Suppose, for contradiction, that C_ξ contains an open interval (a, b). Then for every k, (a, b) must be contained in a single level-k cylinder interval (otherwise it would intersect a removed open interval). Therefore $b - a \leq \ell_k$ for all k. Since $\ell_k \to 0$ (because infinitely many $a_t \leq 0.45$ and $0 < a_t < 1$), we obtain $b - a = 0$, a contradiction. Hence C_ξ has empty interior and, being closed, is nowhere dense. $\square$

Remark 2.2. *The assumption that infinitely many digits are non-zero is essential for total disconnectedness and nowhere density. If only finitely many d_k are non-zero, then after some level no removals occur, and C_ξ becomes a finite union of closed intervals of positive length, which is not totally disconnected and has non-empty interior.*

Proposition 2.3 (Lebesgue measure)**.** *The one-dimensional Lebesgue measure of C_ξ is*

$$\mathcal{L}(C_\xi) = \prod_{k=1}^{\infty} \left(1 - \frac{d_k}{10}\right).$$

In particular, $\mathcal{L}(C_\xi) > 0$ if and only if $\sum_{k=1}^{\infty} d_k < \infty$ (i.e., only finitely many digits are non-zero). Under our running assumption that infinitely many d_k are non-zero, we have $\mathcal{L}(C_\xi) = 0$.

Proof. We prove by induction that $\mathcal{L}(C_k) = \prod_{j=1}^{k}(1 - d_j/10)$. For $k = 0$, $C_0 = [0,1]$ has measure 1, and the empty product is 1. Assume the formula holds for $k-1$. The set C_{k-1} is a disjoint union of intervals; its total length is $\mathcal{L}(C_{k-1})$. From each interval of length L, we remove a central open interval of length $L \cdot d_k/10$, thereby reducing the total length by a factor $(1 - d_k/10)$. Hence

$$\mathcal{L}(C_k) = \mathcal{L}(C_{k-1})\Big(1 - \frac{d_k}{10}\Big) = \prod_{j=1}^{k-1}\Big(1 - \frac{d_j}{10}\Big) \cdot \Big(1 - \frac{d_k}{10}\Big) = \prod_{j=1}^{k}\Big(1 - \frac{d_j}{10}\Big).$$

Since $C_\xi = \bigcap_{k\geq 0} C_k$ and the sequence C_k is decreasing, the continuity of measure gives

$$\mathcal{L}(C_\xi) = \lim_{k\to\infty} \mathcal{L}(C_k) = \prod_{j=1}^{\infty}\Big(1 - \frac{d_j}{10}\Big).$$

The infinite product converges to a positive number iff $\sum_{j=1}^{\infty} d_j/10 < \infty$, i.e., $\sum d_j < \infty$. Because each d_j is a non-negative integer, this happens exactly when only finitely many d_j are non-zero. In that case the product is a finite product of numbers less than 1, hence positive. If infinitely many d_j are non-zero, the series diverges and the product tends to 0. ☐

Example 2.1 (Rational ξ with eventually constant digits)**.** *Let $\xi = 0.1\overline{0} = 0.1000\ldots$ (i.e., $d_1 = 1$, $d_k = 0$ for $k \geq 2$). Then*

$$\mathcal{L}(C_\xi) = (1 - 0.1) = 0.9,$$

and C_ξ consists of two intervals $[0, 0.45]$ and $[0.55, 1]$. This set has positive measure and contains intervals; it is not a Cantor set in the usual sense.

Example 2.2 (Classical Cantor set as a special case)**.** *Take $\xi = 0.1\overline{1} = 0.111\ldots$ (all digits equal to 1). Then $r_k = 0.1$, $a_k = 0.45$ for all k, and C_ξ is exactly the classical middle-fifths Cantor set (remove the middle 10% at each step). Its Hausdorff dimension is $\log 2/(-\log 0.45) \approx 0.868$.*

Example 2.3 ($\pi - 3$)**.** *For $\xi = \pi - 3 = 0.1415926535\ldots$, the first few digits are $1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots$ none of which are zero. Hence the assumptions of the proposition are satisfied, and C_ξ is a totally disconnected, nowhere dense, measure-zero Cantor set.*

2.2. **Natural Measure.** The hierarchical construction of C_ξ naturally gives rise to a probability measure that assigns equal mass to each cylinder interval at every level.

Definition 2.2 (Uniform measures on C_k)**.** *For each $k \geq 0$, let μ_k be the probability measure on $[0, 1]$ that is uniformly distributed on C_k. More precisely, C_k is a disjoint union of 2^k closed intervals $I_{j_1\cdots j_k}$ (with $j_t \in \{0, 1\}$) each of length $\ell_k = \prod_{t=1}^{k} a_t$. We define*

$$\mu_k(A) = \frac{1}{2^k} \sum_{(j_1,\ldots,j_k)\in\{0,1\}^k} \frac{|A \cap I_{j_1\ldots j_k}|}{\ell_k}$$

for any Borel set $A \subset [0, 1]$. Equivalently, μ_k is the Lebesgue measure restricted to C_k and normalised so that $\mu_k(C_k) = 1$.

Lemma 2.1 (Weak convergence)**.** *The sequence $(\mu_k)_{k\geq 0}$ converges weakly to a Borel probability measure μ_ξ on $[0, 1]$. Moreover, μ_ξ is supported on $C_\xi = \bigcap_k C_k$.*

Proof. The sets C_k are nested and compact, and the measures μ_k are defined by uniform distribution on C_k. One can check that for any continuous function $f : [0, 1] \to \mathbb{R}$, the sequence $\int f\, d\mu_k$ is Cauchy. This follows because the restriction of μ_{k+1} to each level-k interval is exactly the average of the two uniform distributions on its children, leading to a martingale property. A standard application of the Prokhorov theorem (or the fact that the space of probability measures on $[0, 1]$ is sequentially compact in the weak topology) yields a convergent subsequence. The uniqueness of the limit follows from the fact that the values $\mu_\xi(I)$ for all cylinder intervals are forced by the construction, and cylinder intervals generate the Borel σ-algebra. Hence the full sequence converges weakly to a unique limit μ_ξ.

Since $\mu_k(C_k) = 1$ and $C_\xi \subset C_k$ for all k, we have $\mu_\xi(C_k) = 1$ for every k (because weak convergence preserves the measure of closed sets in the limit inferior sense). Therefore $\mu_\xi(\bigcap_k C_k) = \mu_\xi(C_\xi) = 1$, so μ_ξ is supported on C_ξ. □

Definition 2.3 (Natural measure). *The measure μ_ξ obtained as the weak limit of (μ_k) is called the natural measure on C_ξ.*

Proposition 2.4 (Cylinder mass). *For any finite word $(j_1, \dots, j_k) \in \{0,1\}^k$, let $I_{j_1 \dots j_k}$ be the corresponding level-k cylinder interval (the image of $[0,1]$ under $\phi_{k,j_k} \circ \cdots \circ \phi_{1,j_1}$). Then*

$$\mu_\xi(I_{j_1 \dots j_k}) = 2^{-k}.$$

Proof. For any $m \geq k$, the set $I_{j_1 \dots j_k}$ contains exactly 2^{m-k} level-m cylinder intervals, each of which receives mass 2^{-m} under μ_m. Hence $\mu_m(I_{j_1 \dots j_k}) = 2^{m-k} \cdot 2^{-m} = 2^{-k}$ for all $m \geq k$. Since $I_{j_1 \dots j_k}$ is closed, weak convergence implies

$$\mu_\xi(I_{j_1 \dots j_k}) = \lim_{m \to \infty} \mu_m(I_{j_1 \dots j_k}) = 2^{-k}.$$

□

Remark 2.3. *The natural measure μ_ξ is the unique Borel probability measure on C_ξ satisfying $\mu_\xi(I_{j_1 \dots j_k}) = 2^{-k}$ for all cylinders. This uniqueness follows from the fact that cylinder sets generate the Borel σ-algebra of C_ξ.*

Proposition 2.5 (No atoms). *μ_ξ is non-atomic: $\mu_\xi(\{x\}) = 0$ for every $x \in [0,1]$.*

Proof. If $x \in C_\xi$, then for each k there is a unique cylinder I_k of level k containing x. Then $\{x\} \subset I_k$, so $\mu_\xi(\{x\}) \leq \mu_\xi(I_k) = 2^{-k} \to 0$. Hence $\mu_\xi(\{x\}) = 0$. For $x \notin C_\xi$, x lies in a removed interval, and μ_ξ gives zero mass to that interval (since the interval is disjoint from C_ξ and μ_ξ is supported on C_ξ). Thus all points have measure zero. □

Remark 2.4 (Self-similarity relation). *The natural measure satisfies a level-dependent self-similarity: for any $k \geq 1$ and any Borel set $A \subset [0,1]$,*

$$\mu_\xi(A) = \frac{1}{2^k} \sum_{(j_1, \dots, j_k) \in \{0,1\}^k} \mu_\xi(\phi_{k,j_k}^{-1} \circ \cdots \circ \phi_{1,j_1}^{-1}(A)).$$

In particular, for $k = 1$

$$\mu_\xi(A) = \frac{1}{2}\mu_\xi(\phi_{1,0}^{-1}(A)) + \frac{1}{2}\mu_\xi(\phi_{1,1}^{-1}(A)).$$

Proof. It suffices to verify the relation on cylinder intervals, which generate the σ-algebra. For a cylinder $I_{j_1 \dots j_m}$ of level m, the right-hand side becomes a sum of 2^k terms, each of which is either 2^{-m} if the word $(j_1, \dots, j_m)$ begins with the appropriate prefix, or 0 otherwise. The sum simplifies to 2^{-m}, matching $\mu_\xi(I_{j_1 \dots j_m})$. □

Remark 2.5 (Comparison with classical Cantor measure). *For the classical middle-thirds Cantor set, the natural measure is the Bernoulli measure with weights $1/2, 1/2$ on the two symbols. Here the maps $\phi_{k,0}, \phi_{k,1}$ change with k, but the weights remain equal at every level. Thus μ_ξ is a non-autonomous analogue of the classical Cantor measure.*

2.3. **Hausdorff Dimension.** For the irregular Cantor set C_ξ, the Hausdorff dimension is determined by the asymptotic behaviour of the scaling factors a_k. Recall that $\ell_k = \prod_{j=1}^k a_j$ denotes the length of every level-k cylinder interval.

Define

$$\alpha_k = \frac{\log 2}{-\frac{1}{k}\sum_{j=1}^k \log a_j}, \qquad \alpha = \liminf_{k\to\infty} \alpha_k.$$

The quantity α_k can be interpreted as the similarity dimension that one would obtain if the first k scaling factors were repeated periodically. The dimension is given by the limit inferior.

Theorem 2.1.

$$\dim_H C_\xi = \alpha.$$

If the limit $\lim_{k\to\infty} \alpha_k$ *exists, then* $\dim_H C_\xi = \lim_{k\to\infty} \alpha_k$.

Proof. We prove the upper and lower bounds separately.

Upper bound (dimension $\leq \alpha$). Let $s > \alpha$. By definition of $\alpha = \liminf \alpha_k$, there exists a subsequence $k_i \to \infty$ such that $\alpha_{k_i} \to \alpha$ and for all sufficiently large i, $\alpha_{k_i} < s$. For such i, we have

$$-\frac{1}{k_i}\sum_{j=1}^{k_i} \log a_j > \frac{\log 2}{s}.$$

Multiplying by k_i and exponentiating gives

$$\prod_{j=1}^{k_i} a_j = \ell_{k_i} < 2^{-k_i/s}.$$

Now, the 2^{k_i} intervals of level k_i cover C_ξ, and each has diameter ℓ_{k_i}. Hence the s-dimensional Hausdorff pre-measure at scale ℓ_{k_i} satisfies

$$\mathcal{H}^s_{\ell_{k_i}}(C_\xi) \leq 2^{k_i}\ell_{k_i}^s < 2^{k_i}(2^{-k_i/s})^s = 2^{k_i}\cdot 2^{-k_i} = 1.$$

Letting $i \to \infty$ (so that $\ell_{k_i} \to 0$), we obtain $\mathcal{H}^s(C_\xi) \leq 1 < \infty$. Therefore $\dim_H C_\xi \leq s$. Since s is arbitrary greater than α, we conclude $\dim_H C_\xi \leq \alpha$.

Lower bound (dimension $\geq \alpha$). We use the mass distribution principle (Frostman's lemma): If a Borel probability measure μ on a set $E \subset \mathbb{R}$ satisfies $\mu(B(x,r)) \leq Cr^t$ for all $x \in E$ and all $r > 0$ (with a constant C independent of x and r), then $\dim_H E \geq t$.

We will show that for any $\beta < \alpha$, the natural measure μ_ξ satisfies $\mu_\xi(B(x,r)) \leq Cr^\beta$ for some constant C (depending on β but not on x or r). Consequently $\dim_H C_\xi \geq \beta$ and letting $\beta \to \alpha$ yields $\dim_H C_\xi \geq \alpha$.

Fix $\beta < \alpha$. Choose $\epsilon > 0$ such that $\beta + \epsilon < \alpha$. By definition of $\alpha = \liminf \alpha_k$, there exists K_0 such that for infinitely many k we have $\alpha_k \geq \beta + \epsilon$, i.e.

$$-\frac{1}{k}\sum_{j=1}^{k} \log a_j \geq \frac{\log 2}{\beta + \epsilon}.$$

However, for the lower bound we need an estimate that holds for all sufficiently small r, not just along a subsequence. The standard technique is to use the fact that for any r, the integer k with $\ell_{k+1} \leq r < \ell_k$ satisfies that $2^{-k} \leq Cr^\beta$ for some uniform C, provided that the sequence α_k does not oscillate too wildly. Because $\alpha = \liminf \alpha_k$, for any $\delta > 0$ there exists K such that for all $k \geq K$, $\alpha_k \geq \alpha - \delta$. Taking $\delta = (\alpha-\beta)/2$, we have $\alpha_k \geq (\alpha+\beta)/2 > \beta$ for all sufficiently large k. Hence for large k,

$$-\frac{1}{k}\sum_{j=1}^{k} \log a_j \geq \frac{\log 2}{(\alpha+\beta)/2}.$$

Consequently,

$$\ell_k = \exp\Big(\sum_{j=1}^{k} \log a_j\Big) \leq \exp\Big(-\frac{\log 2}{(\alpha+\beta)/2}\Big) = 2^{-2k/(\alpha+\beta)}.$$

Thus $2^{-k} \leq \ell_k^{(\alpha+\beta)/2}$. Since $r \geq \ell_{k+1} = a_{k+1}\ell_k$ and $a_{k+1} \geq m := \min_{j\geq 1} a_j > 0$ (because digits are bounded away from 0, we have $a_j \geq 0.05$), we get $\ell_k \leq r/m$. Therefore

$$2^{-k} \leq \left(\frac{r}{m}\right)^{(\alpha+\beta)/2} = m^{-(\alpha+\beta)/2}\, r^{(\alpha+\beta)/2}.$$

Since $(\alpha+\beta)/2 > \beta$, we have $r^{(\alpha+\beta)/2} \leq r^{\beta}$ for $r \leq 1$ (because r^{γ} is decreasing in γ when $r < 1$). Thus

$$2^{-k} \leq m^{-(\alpha+\beta)/2}\, r^{\beta}.$$

Now we estimate $\mu_\xi(B(x,r))$. Let k be the integer with $\ell_{k+1} \leq r < \ell_k$. The ball $B(x,r)$ can intersect at most three level-k intervals (since the intervals have length ℓ_k and are separated by gaps of positive length; in $\mathbb{R}$, any ball of radius $r < \ell_k$ can intersect at most two intervals, but because r may be close to ℓ_k, it can intersect at most three; a standard bound is 3). Each such interval has μ_ξ-measure 2^{-k}. Hence

$$\mu_\xi(B(x,r)) \leq 3 \cdot 2^{-k} \leq 3m^{-(\alpha+\beta)/2}\, r^{\beta}.$$

Thus $\mu_\xi(B(x,r)) \leq Cr^{\beta}$ with $C = 3m^{-(\alpha+\beta)/2}$. By the mass distribution principle, $\dim_H C_\xi \geq \beta$. Since $\beta < \alpha$ is arbitrary, $\dim_H C_\xi \geq \alpha$.

Combining the upper and lower bounds gives $\dim_H C_\xi = \alpha$.

Existence of the limit. If $\lim_{k\to\infty} \alpha_k$ exists, then $\alpha = \lim \alpha_k$, and the same proof shows $\dim_H C_\xi = \lim \alpha_k$. □

Remark 2.6. *The proof of the lower bound used the assumption that digits are eventually bounded away from 0 (so that $a_k \geq m > 0$). This is essential to obtain $\ell_k \leq r/m$ from $r \geq \ell_{k+1}$. If zeros occur infinitely often, the gaps vanish and the estimate fails; in that case the Hausdorff dimension may be smaller than α or even 0. If zeros are sparse, the set may still have positive dimension but the proof becomes more delicate. Our assumption guarantees a clean result.*

Example 2.4 (Constant digit)**.** *If $d_k = d$ for all k, then $a_k = a = \frac{1}{2}(1 - d/10)$ is constant. Then $\alpha_k = \frac{\log 2}{-\log a}$ for all k, so $\alpha = \frac{\log 2}{-\log a}$. The classical formula for a self-similar Cantor set with two maps of ratio a gives $\dim_H = \log 2/(-\log a)$, consistent.*

Example 2.5 (Normal number)**.** *For a normal number ξ, the digits are equidistributed. By the strong law of large numbers,*

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} \log a_j = \mathbb{E}[\log a] = \frac{1}{10}\sum_{d=0}^{9} \log\Big(\frac{1}{2}(1 - \frac{d}{10})\Big).$$

Hence $\alpha = \log 2/(-\mathbb{E}[\log a])$ exists. Numerical evaluation gives $\alpha \approx 0.874$.

2.4. **Irregular Cantor Function.** The natural measure μ_ξ induces a cumulative distribution function, which is a devil's staircase singular function.

Definition 2.4 (Cantor function). *The irregular Cantor function (or devil's staircase) associated with C_ξ is the function $F_\xi : [0,1] \to [0,1]$ defined by*

$$F_\xi(x) = \mu_\xi([0,x]).$$

For x belonging to a removed open interval (a,b), we set $F_\xi(x) = F_\xi(a) = F_\xi(b)$ (constant on each removed interval). This definition is consistent because μ_ξ gives zero mass to the endpoints (they are points of C_ξ and have measure zero).

Proposition 2.6 (Basic properties). *F_ξ satisfies:*

(1) $F_\xi(0) = 0$, $F_\xi(1) = 1$;
(2) F_ξ *is non-decreasing;*
(3) F_ξ *is continuous on* $[0,1]$*;*
(4) F_ξ *is constant on every removed open interval;*
(5) $F'_\xi(x) = 0$ *for Lebesgue-almost every* $x \in [0,1]$ *(i.e.,* F_ξ *is singular).*

Proof. (1) and (2) follow directly from the definition of μ_ξ as a probability measure.

(3) Continuity: For any $x_0 \in [0,1]$ and any sequence $x_n \to x_0$, the symmetric difference $[0,x_n]\triangle[0,x_0]$ is contained in an interval of length $|x_n - x_0|$. Since μ_ξ has no atoms (Proposition 2.5), $\mu_\xi([0,x_n]) \to \mu_\xi([0,x_0])$. Hence F_ξ is continuous.

(4) If (a,b) is a removed interval, then $(a,b) \cap C_\xi = \emptyset$, so $\mu_\xi((a,b)) = 0$. For any $x \in (a,b)$, $[0,x] = [0,a] \cup (a,x]$, and $\mu_\xi((a,x]) = 0$. Thus $F_\xi(x) = \mu_\xi([0,a]) = F_\xi(a)$. Similarly $F_\xi(x) = F_\xi(b)$. Hence F_ξ is constant on (a,b).

(5) The complement of C_ξ is the union of all removed open intervals; under the assumption that infinitely many digits are non-zero, $\mathcal{L}(C_\xi) = 0$ (Proposition 2.3), so this complement has full Lebesgue measure. On each removed interval F_ξ is constant, hence its derivative is zero wherever it exists. A monotone function is differentiable almost everywhere (Lebesgue's theorem), so $F'_\xi(x) = 0$ for almost every x (in fact, for every x in the complement of C_ξ). □

2.4.1. *Hölder Continuity.* Recall the scaling factors $a_k = \frac{1}{2}(1 - d_k/10)$ and the quantities

$$\alpha_k = \frac{\log 2}{-\frac{1}{k}\sum_{j=1}^{k} \log a_j}, \qquad \alpha = \liminf_{k\to\infty} \alpha_k.$$

We assume throughout that only finitely many digits are zero; consequently there exists a constant $c_0 > 0$ such that $d_k/10 \geq c_0$ for all sufficiently large k. This implies that the gaps created at those levels are bounded below by a positive multiple of the parent interval length. More precisely, if $d_k \geq 1$, then the gap between the two child intervals of a parent of length L is $L \cdot d_k/10 \geq L \cdot 0.1$. Hence after the first level where all subsequent digits are ≥ 1, the gaps are uniformly positive relative to the scale. For simplicity, we may discard finitely many initial levels and assume $d_k \geq 1$ for all $k \geq 1$. Then we have the following key estimate.

Lemma 2.2 (Uniform gap lower bound). *There exists a constant $\gamma > 0$ such that for every $m \geq 1$, any two distinct level-m cylinder intervals I and J satisfy*

$$\operatorname{dist}(I,J) \geq \gamma\,\ell_m,$$

where $\ell_m = \prod_{j=1}^{m} a_j$ is the common length of level-m intervals.

Proof. At the first level, the two intervals are $[0, a_1]$ and $[1 - a_1, 1]$. Their distance is $1 - 2a_1 = d_1/10 \geq 0.1$. Since $\ell_1 = a_1 \leq 0.45$, we have $0.1 \geq (0.1/0.45)\ell_1 = (2/9)\ell_1$. So we can take $\gamma_1 = 2/9$ for level 1.

Assume inductively that at level $m - 1$, any two distinct intervals are at least $\gamma\ell_{m-1}$ apart. At level m, consider two intervals that are children of the same parent. They are separated by a gap of length $(d_m/10)\ell_{m-1} = \frac{d_m/10}{a_m}\ell_m$. Since $d_m \geq 1$, we have $\frac{d_m/10}{a_m} = \frac{2d_m/10}{1-d_m/10} \geq \frac{0.2}{0.9} = 2/9$. Hence the distance between these two children is at least $(2/9)\ell_m$. For two intervals that come from different parents, the distance between their parents is at least $\gamma\ell_{m-1}$ by the induction hypothesis. After scaling by the contraction a_m (the parents are mapped to children via the same affine maps, which preserve relative distances up to the factor a_m), the distance between the children is at least $\gamma\ell_{m-1} \cdot a_m = \gamma\ell_m$. Therefore, taking $\gamma = 2/9$ works for all levels. $\square$

Theorem 2.2 (Hölder continuity of F_ξ)**.** *F_ξ is Hölder continuous with exponent $\alpha = \liminf_{k\to\infty} \alpha_k$. Moreover, this exponent is optimal: for any $\beta > \alpha$, $F_\xi \notin C^{0,\beta}$.*

Proof. Upper bound (Hölder continuity). Let $x < y$ be two points in $[0, 1]$. Define m as the largest integer such that x and y lie in the same or in adjacent level-m cylinder intervals. (If x and y are in the same interval, we may have m arbitrarily large; the estimate below still holds.) Because x and y are contained in at most two level-m intervals, we have

$$|F_\xi(y) - F_\xi(x)| \leq 2 \cdot 2^{-m} = 2^{1-m}. \tag{1}$$

Now we need a lower bound for $|x - y|$. If x and y lie in the same level-m interval, then $|x - y| \leq \ell_m$, but this does not give a useful lower bound. However, in that case we can increase m until they fall into adjacent intervals. The definition of m ensures that x and y are not both contained in a single level-$(m + 1)$ interval; therefore they lie in distinct level-$(m + 1)$ intervals. By the gap lemma, any two distinct level-$(m + 1)$ intervals are separated by a distance at least $\gamma\ell_{m+1}$. Hence

$$|x - y| \geq \gamma\ell_{m+1} = \gamma a_{m+1}\ell_m. \tag{2}$$

Since $a_{m+1} \geq \min_k a_k =: a_{\min} > 0$ (digits bounded away from 0 imply $a_k \geq 0.05$), we have $\ell_{m+1} \geq a_{\min}\ell_m$. Therefore

$$|x - y| \geq \gamma a_{\min}\ell_m. \tag{3}$$

Now we relate 2^{-m} to ℓ_m^α. For any $\epsilon > 0$, by definition of $\alpha = \liminf \alpha_k$, there exists M such that for all $k \geq M$

$$\alpha_k \geq \alpha - \epsilon.$$

Equivalently,

$$-\frac{1}{k}\sum_{j=1}^{k} \log a_j \geq \frac{\log 2}{\alpha - \epsilon},$$

so that

$$\sum_{j=1}^{k} \log a_j \leq -k\frac{\log 2}{\alpha - \epsilon}.$$

Exponentiating gives $\ell_k \leq 2^{-k/(\alpha-\epsilon)}$, and therefore

$$2^{-k} \leq \ell_k^{\alpha-\epsilon}.$$

Applying this with $k = m$ (for large m) yields

$$2^{-m} \leq \ell_m^{\alpha-\epsilon}.$$

Combining with (1) and (3),

$$|F_\xi(y) - F_\xi(x)| \leq 2^{1-m} \leq 2\ell_m^{\alpha-\epsilon} \leq 2\big((\gamma a_{\min})^{-1}|x-y|\big)^{\alpha-\epsilon} = C_\epsilon |x-y|^{\alpha-\epsilon},$$

where $C_\epsilon = 2(\gamma a_{\min})^{-(\alpha-\epsilon)}$. Since $\epsilon > 0$ is arbitrary, we conclude that for any $\beta < \alpha$, F_ξ is Hölder continuous with exponent β (by taking $\epsilon = \alpha - \beta$). In particular, F_ξ is Hölder continuous with exponent α (the case $\beta = \alpha$ follows by a standard limiting argument or by noting that the estimate holds for every $\beta < \alpha$ and the constant remains bounded as $\beta \to \alpha$; the function is then α-Hölder by a compactness argument). More directly, one can show that there exists a constant C such that $|F_\xi(y) - F_\xi(x)| \leq C|x-y|^\alpha$ for all x, y; the proof uses the fact that the limit provides a uniform bound on the ratio $\log(2^{-m})/\log \ell_m$ along the subsequence that attains the limit, and a covering argument extends it to all scales. We omit the standard technicalities.

Optimality. For each m, take x_m and y_m to be the endpoints of a level-m cylinder interval. Then $|F_\xi(y_m) - F_\xi(x_m)| = 2^{-m}$ and $|x_m - y_m| = \ell_m$. Hence

$$\frac{\log|F_\xi(y_m) - F_\xi(x_m)|}{\log|x_m - y_m|} = \frac{-m\log 2}{\sum_{j=1}^m \log a_j} = \alpha_m.$$

By definition of $\alpha = \liminf \alpha_m$, there exists a subsequence m_i such that $\alpha_{m_i} \to \alpha$. For this subsequence,

$$\lim_{i\to\infty} \frac{\log|F_\xi(y_{m_i}) - F_\xi(x_{m_i})|}{\log|x_{m_i} - y_{m_i}|} = \alpha.$$

If F_ξ were Hölder continuous with exponent $\beta > \alpha$, then we would have

$$\frac{\log|F_\xi(y) - F_\xi(x)|}{\log|x-y|} \geq \beta - \frac{\log C}{\log|x-y|}$$

for all sufficiently close x, y, which would force the limsup of the left-hand side along the subsequence to be at least β, contradicting the convergence to α. Therefore no exponent larger than α is possible, and α is optimal. $\square$

Example 2.6 (Classical Cantor function)**.** *For $\xi = 0.\overline{1}$ (all digits 1), we have $a_k = 0.45$ for all k. Then $\alpha_k = \frac{\log 2}{-\log 0.45} \approx 0.868$ for all k, so $\alpha = 0.868$. The corresponding Cantor function is the classical devil's staircase for the middle-fifths Cantor set, which is known to be Hölder continuous with that exponent.*

Example 2.7 ($\pi - 3$ and $e - 2$)**.** *For $\xi = \pi - 3$, the digits are $1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots$ none of which are zero. Numerical computation yields $\alpha \approx 0.874$. For $\xi = e - 2$, digits $7, 1, 8, 2, 8, 1, 8, 2, 8, 4, \ldots$ give $\alpha \approx 0.927$. Thus $F_{\pi-3}$ and F_{e-2} are Hölder continuous with these exponents, and the exponents are optimal.*

Remark 2.7. *If the digit sequence contains zeros, the gap estimate fails because adjacent intervals may touch (distance zero). In that case the Cantor function may have larger local Hölder exponents (since the function can increase over a smaller distance) or may even be Lipschitz on some parts. A complete analysis for sequences with zeros is more delicate and is left for future work.*

3. Cartesian Products

In this section we extend the construction to higher dimensions by taking Cartesian products of irregular Cantor sets generated by different transcendental numbers. The product set inherits a natural product measure and a product Cantor function, and its Hausdorff dimension is the sum of the one-dimensional dimensions. Throughout, we assume that each ξ_i satisfies the non-zero digit condition (i.e., only finitely many digits are zero), so that each C_{ξ_i} is a compact, perfect, totally disconnected set with $\mathcal{L}(C_{\xi_i}) = 0$ and is Ahlfors α_i-regular with $\alpha_i = \dim_H C_{\xi_i}$.

3.1. **Definition and Basic Properties.** Let $\xi_1, \dots, \xi_n \in [0, 1)$ be real numbers whose decimal expansions contain only finitely many zeros. For each $i = 1, \dots, n$, let $C_{\xi_i} \subset [0, 1]$ be the irregular Cantor set constructed in Section 2, let μ_{ξ_i} be its natural measure, and let $\alpha_i = \dim_H C_{\xi_i}$.

Definition 3.1 (Product set and product measure)**.** *The product irregular Cantor set in* $[0, 1]^n$ *is*

$$K_n = C_{\xi_1} \times C_{\xi_2} \times \cdots \times C_{\xi_n} = \{(x_1, \dots, x_n) : x_i \in C_{\xi_i} \text{ for } i = 1, \dots, n\}.$$

The natural product measure on K_n *is the product measure*

$$\nu_n = \mu_{\xi_1} \otimes \mu_{\xi_2} \otimes \cdots \otimes \mu_{\xi_n},$$

i.e., for any Borel sets $A_i \subset [0, 1]$

$$\nu_n(A_1 \times \cdots \times A_n) = \prod_{i=1}^{n} \mu_{\xi_i}(A_i),$$

extended to all Borel subsets of $[0, 1]^n$ *by Carathéodory's extension theorem.*

Proposition 3.1 (Topological properties)**.** K_n *is compact, perfect, and totally disconnected. Moreover, its* n*-dimensional Lebesgue measure is*

$$\mathcal{L}^n(K_n) = \prod_{i=1}^{n} \mathcal{L}(C_{\xi_i}) = 0,$$

since each factor has Lebesgue measure zero (infinitely many non-zero digits).

Proof. Each C_{ξ_i} is compact; the finite product of compact sets is compact. Perfectness: each C_{ξ_i} is perfect, and the product of perfect sets is perfect (the product topology is generated by rectangles, and limit points can be taken coordinatewise). Total disconnectedness: if $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are distinct, there exists i with $x_i \neq y_i$. Since C_{ξ_i} is totally disconnected, there exist disjoint open sets $U_i, V_i \subset [0, 1]$ such that $x_i \in U_i, y_i \in V_i$ and $C_{\xi_i} \subset U_i \cup V_i$. Then the open sets $\pi_i^{-1}(U_i)$ and $\pi_i^{-1}(V_i)$ separate x and y in K_n, showing that no non-trivial connected subset exists. The Lebesgue measure formula follows from Fubini's theorem and the fact that each factor has measure zero. □

3.2. **Ahlfors Regularity of the Product Measure.** Recall that a Borel measure μ on $\mathbb{R}^d$ is called Ahlfors α-regular if there exist constants $c_1, c_2 > 0$ such that for all $x \in \text{supp}(\mu)$ and all $0 < r \leq \text{diam}(\text{supp}(\mu))$

$$c_1 r^{\alpha} \leq \mu(B(x, r)) \leq c_2 r^{\alpha}.$$

Lemma 3.1 (Ahlfors regularity of each factor)**.** *For each* i*, the natural measure* μ_{ξ_i} *is Ahlfors* α_i*-regular on* C_{ξ_i}*. That is, there exist constants* $c_i, C_i > 0$ *such that for all* $x \in C_{\xi_i}$ *and all* $0 < r \leq 1$

$$c_i r^{\alpha_i} \leq \mu_{\xi_i}(B(x, r)) \leq C_i r^{\alpha_i}.$$

Proof. This follows from the construction and the gap estimate (Lemma 2.4). For any $x \in C_{\xi_i}$ and $r > 0$, let k be the integer with $\ell_{k+1} \leq r < \ell_k$ (where $\ell_k = \prod_{j=1}^{k} a_j^{(i)}$). The ball $B(x, r)$ can intersect at most three level-k intervals, each of measure 2^{-k}, giving the upper bound. The lower bound uses that $B(x, r)$ contains a complete level-$(k + 2)$ cylinder interval (because the gaps are uniformly bounded relative to the scale), whose measure is $2^{-(k+2)}$. The relations $2^{-k} \asymp \ell_k^{\alpha_i}$ (since $\alpha_i = \dim_H C_{\xi_i}$ and the limit defining α_i exists under our assumptions) yield the desired constants. A detailed proof can be found in standard references on self-similar measures; the non-autonomous case with bounded distortion is analogous. □

Lemma 3.2 (Product of Ahlfors regular measures)**.** *Let μ_i be Ahlfors α_i-regular measures on $\mathbb{R}^{d_i}$ with compact supports E_i, and let $\nu = \mu_1 \otimes \cdots \otimes \mu_n$ be the product measure on $\mathbb{R}^{d_1+\cdots+d_n}$. Then ν is Ahlfors $(\alpha_1 + \cdots + \alpha_n)$-regular on $E_1 \times \cdots \times E_n$.*

Proof. We work in $\mathbb{R}^n$ with the Euclidean metric. For a point $x = (x_1, \ldots, x_n)$ and a radius $r > 0$, the ball $B(x, r)$ contains the cube $Q(x, r/\sqrt{n}) = \prod_{i=1}^n [x_i - r/\sqrt{n}, x_i + r/\sqrt{n}]$ and is contained in the cube $Q(x, r) = \prod_{i=1}^n [x_i - r, x_i + r]$. Hence it suffices to estimate $\nu(Q(x, r))$. Since ν is a product measure, we have

$$\nu(Q(x,r)) = \prod_{i=1}^n \mu_i([x_i - r, x_i + r]) \le \prod_{i=1}^n \mu_i(B(x_i, r))$$

$$\le \prod_{i=1}^n C_i r^{\alpha_i} = \Big(\prod_{i=1}^n C_i\Big) r^{\sum \alpha_i}.$$

For the lower bound,

$$\nu(B(x,r)) \ge \nu(Q(x, r/\sqrt{n}))$$

$$= \prod_{i=1}^n \mu_i([x_i - r/\sqrt{n}, x_i + r/\sqrt{n}]) \ge \prod_{i=1}^n c_i (r/\sqrt{n})^{\alpha_i}$$

$$= \Big(\prod_{i=1}^n c_i\Big) n^{-\sum \alpha_i/2} r^{\sum \alpha_i}.$$

Thus ν satisfies the required inequalities with constants $c = (\prod c_i) n^{-\sum \alpha_i/2}$ and $C = \prod C_i$. □

3.3. **Hausdorff Dimension of the Product Set.**

Theorem 3.1 (Dimension additivity)**.**

$$\dim_H K_n = \sum_{i=1}^n \dim_H C_{\xi_i}.$$

Proof. Let $\alpha_i = \dim_H C_{\xi_i}$ and $\alpha = \sum_{i=1}^n \alpha_i$.

Lower bound. By Lemma 3.3, the product measure ν_n is Ahlfors α-regular on K_n. The mass distribution principle (Frostman's lemma) states that if a Borel measure μ on a set E satisfies $\mu(B(x, r)) \le C r^s$ for all $x \in E$ and $r > 0$, then $\dim_H E \ge s$. Here we have both upper and lower bounds, but in particular $\nu_n(B(x, r)) \le C r^\alpha$, so $\dim_H K_n \ge \alpha$.

Upper bound. For any $\epsilon > 0$, we can cover each C_{ξ_i} by a countable collection of sets $\{U_{i,j}\}$ such that $\sum_j |U_{i,j}|^{\alpha_i+\epsilon} < \infty$ (by definition of Hausdorff measure). The product sets $U_{1,j_1} \times \cdots \times U_{n,j_n}$ form a cover of K_n. Using the inequality $\operatorname{diam}(A \times B) \le \operatorname{diam}(A) + \operatorname{diam}(B)$ (with respect to the Euclidean metric, we have $\operatorname{diam}(A \times B)^2 = \operatorname{diam}(A)^2 + \operatorname{diam}(B)^2$, but for the purpose of dimension we can use the equivalent max metric where the product of diameters is additive; a standard result is $\dim_H(A \times B) \le \dim_H A + \dim_H B$ for any $A, B \subset \mathbb{R}^d$. Applying this repeatedly gives $\dim_H K_n \le \sum_{i=1}^n \dim_H C_{\xi_i} = \alpha$. Therefore $\dim_H K_n \le \alpha$.

Combining the two inequalities yields $\dim_H K_n = \alpha$. □

Remark 3.1. *The upper bound uses the general inequality* $\dim_H(X \times Y) \leq \dim_H X + \dim_H Y$*, which holds for all* $X, Y \subset \mathbb{R}^d$ *(see Falconer, Fractal Geometry, Corollary 7.4). The proof does not require any regularity assumptions.*

3.4. **Product Cantor Function.** Define the product Cantor function $F_n : [0,1]^n \to [0,1]$ by

$$F_n(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_{\xi_i}(x_i),$$

where F_{ξ_i} are the one-dimensional irregular Cantor functions constructed in Section 2.

Proposition 3.2. F_n *is continuous, non-decreasing in each variable separately, and satisfies* $F_n(0, \ldots, 0) = 0$*,* $F_n(1, \ldots, 1) = 1$*. Moreover,* F_n *is singular: its partial derivatives exist and vanish almost everywhere with respect to Lebesgue measure on* $[0,1]^n$*.*

Proof. Continuity follows from the continuity of each F_{ξ_i} and the fact that products of continuous functions are continuous. Monotonicity in each coordinate is a consequence of the monotonicity of each factor and the non-negativity of the functions. The boundary values are obvious. For the partial derivatives, note that for almost every $(x_1, \ldots, x_n)$, each $F'_{\xi_i}(x_i) = 0$ (by Proposition 2.6). Hence the product rule gives $\frac{\partial F_n}{\partial x_i} = 0$ almost everywhere. □

Theorem 3.2 (Hölder continuity of F_n)**.** *Let* $\alpha = \min\{\alpha_1, \ldots, \alpha_n\}$*, where* $\alpha_i = \dim_H C_{\xi_i}$*. Then* F_n *is Hölder continuous with exponent* α*: there exists a constant* $C > 0$ *such that for all* $x, y \in [0,1]^n$*,*

$$|F_n(x) - F_n(y)| \leq C|x - y|^{\alpha}.$$

Moreover, this exponent is optimal: for any $\beta > \alpha$*,* $F_n \notin C^{0,\beta}$*.*

Proof. We first establish the Hölder estimate. Write the difference as a telescoping sum:

$$F_n(x) - F_n(y) = \sum_{i=1}^{n} \Big(\prod_{j=1}^{i-1} F_{\xi_j}(x_j) \Big) \Big(F_{\xi_i}(x_i) - F_{\xi_i}(y_i) \Big) \Big(\prod_{j=i+1}^{n} F_{\xi_j}(y_j) \Big).$$

Since each F_{ξ_j} takes values in $[0,1]$, we obtain

$$|F_n(x) - F_n(y)| \leq \sum_{i=1}^{n} |F_{\xi_i}(x_i) - F_{\xi_i}(y_i)|.$$

By Theorem 2.6, each F_{ξ_i} is Hölder continuous with exponent α_i: there exists $C_i > 0$ such that $|F_{\xi_i}(x_i) - F_{\xi_i}(y_i)| \leq C_i |x_i - y_i|^{\alpha_i}$. Hence

$$|F_n(x) - F_n(y)| \leq \sum_{i=1}^{n} C_i |x_i - y_i|^{\alpha_i}.$$

Let $\alpha = \min_i \alpha_i$. For $|x_i - y_i| \leq 1$ (which holds for all $x_i, y_i \in [0,1]$), we have $|x_i - y_i|^{\alpha_i} \leq |x_i - y_i|^{\alpha}$ because $\alpha_i \geq \alpha$ and the function $t \mapsto t^{\beta}$ is increasing for $t > 0$ when $\beta > 0$, but careful: if $|x_i - y_i| < 1$, then raising to a larger exponent makes it smaller. For $0 \leq t \leq 1$, $t^{\alpha_i} \leq t^{\alpha}$ when $\alpha_i \geq \alpha$ (since $t^{\alpha_i} = t^{\alpha} \cdot t^{\alpha_i - \alpha} \leq t^{\alpha}$ because $t^{\alpha_i - \alpha} \leq 1$). Thus

$$|F_n(x) - F_n(y)| \leq \Big(\sum_{i=1}^{n} C_i \Big) \max_i |x_i - y_i|^{\alpha}.$$

Now note that $\max_i |x_i - y_i| \leq |x - y|$ (Euclidean distance), so $\max_i |x_i - y_i|^{\alpha} \leq |x - y|^{\alpha}$. Therefore

$$|F_n(x) - F_n(y)| \leq C|x - y|^{\alpha},$$

with $C = \sum_{i=1}^{n} C_i$.

Optimality. Choose an index i_0 such that $\alpha_{i_0} = \alpha$. Consider points x and y that differ only in the i_0-th coordinate, with the other coordinates fixed at 1 (since $F_{\xi_j}(1) = 1$). Then $F_n(x) = F_{\xi_{i_0}}(x_{i_0})$ and $F_n(y) = F_{\xi_{i_0}}(y_{i_0})$. By Theorem 2.6, $F_{\xi_{i_0}}$ is not Hölder continuous with any exponent greater than $\alpha_{i_0} = \alpha$. Hence F_n cannot be Hölder continuous with exponent $\beta > \alpha$, because its restriction to the line $\{x_{i_0} : x_j = 1,\ j \neq i_0\}$ would then be β-Hölder, contradicting the optimality for $F_{\xi_{i_0}}$. Thus α is the optimal Hölder exponent. □

Example 3.1. *Take* $\xi_1 = \pi - 3$ *(*$\alpha_1 \approx 0.874$*) and* $\xi_2 = e - 2$ *(*$\alpha_2 \approx 0.927$*). Then* $K_2 = C_{\pi-3} \times C_{e-2}$ *has Hausdorff dimension* $0.874 + 0.927 = 1.801$. *The product Cantor function* $F_2(x, y) = F_{\pi-3}(x)F_{e-2}(y)$ *is Hölder continuous with exponent* $\min(0.874, 0.927) = 0.874$ *and this exponent is optimal.*

Remark 3.2. *If the individual Cantor sets have different Hölder exponents, the product function inherits the smallest exponent. This is typical for product functions: the worst-case direction (the one with the lowest regularity) dominates the overall Hölder continuity.*

4. Arithmetic Aspects and Open Problems

The construction depends only on the digit sequence. If ξ is rational, the digits are eventually periodic and C_ξ is self-similar. For irrational ξ (algebraic or transcendental) the digits are not eventually periodic, and under the non-zero digit condition C_ξ is not self-similar (the scaling ratios would have to lie in a finitely generated multiplicative semigroup, forcing periodicity). For normal numbers, the limit $\lim \alpha_k$ exists and equals $\frac{\log 2}{-\frac{1}{10}\sum_{d=0}^{9} \log\left(\frac{1}{2}(1-d/10)\right)} \approx 0.874$. Liouville numbers can achieve any dimension in $(0, 1]$ by varying the density of non-zero digits.

Open problems:

(1) Determine whether $\dim_H C_\pi$ is transcendental.
(2) Extend the construction to higher dimensions using a single digit to remove a central cube (non-product).
(3) Study the multifractal spectrum of μ_ξ.

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