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$*$-K-Operator Frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$

Mohamed Rossafi[1,*], Roumaissae El Jazzar[2] and Ali Kacha[2]

[1] *LaSMA Laboratory Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, B. P. 1796 Fes Atlas, Morocco*
mohamed.rossafi@usmba.ac.ma

[2] *Laboratory of Partial Differential Equations, Spectral Algebra and Geometry Department of Mathematics, Faculty of Sciences, University Ibn Tofail, Kenitra, Morocco*
roumaissae.eljazzar@uit.ac.ma, ali.kacha@yahoo.fr

[*] *Correspondence: rossafimohamed@gmail.com*

Abstract. In this work, we introduce the concept of $*$-K-operator frames in Hilbert pro-C^*-modules, which is a generalization of K-operator frame. We present the analysis operator, the synthesis operator and the frame operator. We also give some properties and we study the tensor product of $*$-K-operator frame for Hilbert pro-C^*-modules.

1. Introduction

Duffin and Schaeffer introduced the notion of frame in nonharmonic Fourier analysis in 1952 [3]. In 1986 the work of Duffin and Schaeffer were reintroduced and developed by Grossman and Meyer [7]. The concept of frame on Hilbert space has already been successfully extended to pro-C^*-algebras and Hilbert modules. Many properties of frames in Hilbert C^*-modules are valid for frames of multipliers in Hilbert modules over pro-C^*-algebras [9].

Operator frames for $B(\mathcal{H})$ is a new notion of frames that Li and Cio introduced in [11] and generalized by Rossafi in [16]. In this work we introduce the notion of $*$-K-operator frame for the space $Hom^*_{\mathcal{A}}(\mathcal{X})$ of all adjointable operators on a Hilbert pro-C^*-module for $\mathcal{X}$.

This paper is divided into three sections. In section 2 we recall some fundamental definitions and notations of Hilbert pro-C^*-modules. In section 3 we introduce the $*$-K-operator Frame and we give some of its properties. Lastly we investigate tensor product of Hilbert pro-C^*-modules, we show that tensor product of $*$-K-operator frames for Hilbert pro-C^*-modules $\mathcal{X}$ and $\mathcal{Y}$, present an $*$-K-operator frames for $\mathcal{X} \otimes \mathcal{Y}$, and tensor product of their frame operators is the frame operator of their tensor product of $*$-K-operator frames.

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2. Preliminaries

The basic information about pro-C^*-algebras can be found in the works [4–6, 8, 12, 14, 15].

C^*-algebra whose topology is induced by a family of continuous C^*-seminorms instead of a C^*-norm is called pro-C^*-algebra. Hilbert pro-C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a pro-C^*-algebra rather than in the field of complex numbers.

Pro-C^*-algebra is defined as a complete Hausdorff complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms in the sens that a net $\{a_\alpha\}$ converges to 0 if and only if $p(a_\alpha)$ converges to 0 for all continuous C^*-seminorm p on $\mathcal{A}$ [8, 10, 15], and we have:

1) $p(ab) \leq p(a)p(b)$
2) $p(a^*a) = p(a)^2$

for all $a, b \in \mathcal{A}$

If the topology of pro-C^*-algebra is determined by only countably many C^*-seminorms, then it is called a σ-C^*-algebra.

We denote by $sp(a)$ the spectrum of a such that: $sp(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a$ is not invertible $\}$ for all $a \in \mathcal{A}$. Where $\mathcal{A}$ is unital pro-C^*-algebra with unite $1_{\mathcal{A}}$.

The set of all continuous C^*-seminorms on $\mathcal{A}$ is denoted by $S(\mathcal{A})$. If $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$, then $\mathcal{A}^+$ is a closed convex C^*-seminorms on $\mathcal{A}$.

Example 2.1. Every C^*-algebra is a pro-C^*-algebra.

Proposition 2.2. *[8] Let $\mathcal{A}$ be a unital pro-C^*-algebra with an identity $1_{\mathcal{A}}$. Then for any $p \in S(\mathcal{A})$, we have:*

(1) *$p(a) = p(a^*)$ for all $a \in A$*
(2) *$p(1_{\mathcal{A}}) = 1$*
(3) *If $a, b \in \mathcal{A}^+$ and $a \leq b$, then $p(a) \leq p(b)$*
(4) *If $1_{\mathcal{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathcal{A}}$*
(5) *If $a, b \in \mathcal{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$*
(6) *If $a, b, c \in \mathcal{A}$ and $a \leq b$ then $c^*ac \leq c^*bc$*
(7) *If $a, b \in \mathcal{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$*

Definition 2.3. [15] A pre-Hilbert module over pro-C^*-algebra $\mathcal{A}$, is a complex vector space E which is also a left $\mathcal{A}$-module compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle$ $E \times E \to \mathcal{A}$ which is $\mathbb{C}$-and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:

1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$
2) $\langle \xi, \xi \rangle \geq 0$ for every $\xi \in E$

3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$

for every $\xi, \eta \in E$. We say E is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C^*-module over $\mathcal{A}$). If E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)} \quad \xi \in E, p \in S(\mathcal{A})$$

Let $\mathcal{A}$ be a pro-C^*-algebra and let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and assume that I and J be countable index sets. A bounded $\mathcal{A}$-module map from $\mathcal{X}$ to $\mathcal{Y}$ is called an operators from $\mathcal{X}$ to $\mathcal{Y}$. We denote the set of all operator from $\mathcal{X}$ to $\mathcal{Y}$ by $Hom_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$.

Definition 2.4. [1] An $\mathcal{A}$-module map $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is adjointable if there is a map $T^* : \mathcal{Y} \longrightarrow \mathcal{X}$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi \in \mathcal{X}, \eta \in \mathcal{Y}$, and is called bounded if for all $p \in S(\mathcal{A})$, there is $M_p > 0$ such that $\bar{p}_{\mathcal{Y}}(T\xi) \leq M_p \bar{p}_{\mathcal{X}}(\xi)$ for all $\xi \in \mathcal{X}$.
We denote by $Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$, the set of all adjointable operator from $\mathcal{X}$ to $\mathcal{Y}$ and $Hom^*_{\mathcal{A}}(\mathcal{X}) = Hom^*_{\mathcal{A}}(\mathcal{X}, \mathcal{X})$

Definition 2.5. [1] Let $\mathcal{A}$ be a pro-C^*-algebra and $\mathcal{X}, \mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules. The operator $T : \mathcal{X} \to \mathcal{Y}$ is called uniformly bounded below, if there exists $C > 0$ such that for each $p \in S(\mathcal{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \leqslant C\bar{p}_{\mathcal{X}}(\xi), \qquad \text{for all } \xi \in \mathcal{X}$$

and is called uniformly bounded above if there exists $C' > 0$ such that for each $p \in S(\mathcal{A})$,

$$\bar{p}_{\mathcal{Y}}(T\xi) \geqslant C'\bar{p}_{\mathcal{X}}(\xi), \qquad \text{for all } \xi \in \mathcal{X}$$

$$\|T\|_\infty = \inf\{M : M \text{ is an upper bound for } T\}$$

$$\hat{p}_{\mathcal{Y}}(T) = \sup\{\bar{p}_{\mathcal{Y}}(T(x)) : \xi \in \mathcal{X}, \quad \bar{p}_{\mathcal{X}}(\xi) \leqslant 1\}$$

It's clear to see that, $\hat{p}(T) \leqslant \|T\|_\infty$ for all $p \in S(\mathcal{A})$.

Proposition 2.6. *[2]. Let $\mathcal{X}$ be a Hilbert module over pro-C^*-algebra $\mathcal{A}$ and T be an invertible element in $Hom^*_{\mathcal{A}}(\mathcal{X})$ such that both are uniformly bounded. Then for each $\xi \in \mathcal{X}$,*

$$\left\|T^{-1}\right\|_\infty^{-2} \langle \xi, \xi \rangle \leq \langle T\xi, T\xi \rangle \leq \|T\|_\infty^2 \langle \xi, \xi \rangle.$$

3. $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$

We begin this section with the definition of a K-operator frame.

Definition 3.1. Let $\{T_i\}_{i\in I}$ be a family of adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ over a unital pro-C^*-algebra, and let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$. $\{T_i\}_{i\in I}$ is called a K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$, if there exist two positive constants $A, B > 0$ such that

$$A\langle K^*\xi, K^*\xi \rangle \leq \sum_{i\in I} \langle T_i\xi, T_i\xi \rangle \leq B\langle \xi, \xi \rangle, \forall \xi \in \mathcal{X}. \tag{3.1}$$

The numbers A and B are called lower and upper bound of the K-operator frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle = \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle,$$

the K-operator frame is an A-tight. If $A = 1$, it is called a normalized tight K-operator frame or a Parseval K-operator frame.

We will now move to define the $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$.

Definition 3.2. Let $\{T_i\}_{i\in I}$ be a family of adjointable operators on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ over a unital pro-C^*-algebra, and let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$. $\{T_i\}_{i\in I}$ is called a $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{H})$, if there exists two nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle K^*\xi, K^*\xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}. \tag{3.2}$$

The elements A and B are called lower and upper bounds of the $*$-K-operator frame, respectively. If

$$A\langle K^*\xi, K^*\xi\rangle^* = \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle,$$

the $*$-K-operator frame is an A-tight. If $A = 1$, it is called a normalized tight $*$-K-operator frame or a Parseval $*$-K-operator frame.

Example 3.3. Let l^∞ be the set of all bounded complex-valued sequences. For any $u = \{u_j\}_{j\in\mathrm{N}}, v = \{v_j\}_{j\in\mathrm{N}} \in l^\infty$, we define

$$uv = \{u_j v_j\}_{j\in\mathrm{N}}, u^* = \{\bar{u}_j\}_{j\in\mathrm{N}}, \|u\| = \sup_{j\in\mathrm{N}}|u_j|.$$

Then $\mathcal{A} = \{l^\infty, \|.\|\}$ is a $\mathbb{C}^*$-algebra. Then $\mathcal{A}$ is pro-$\mathbb{C}^*$-algebra.

Let $\mathcal{X} = C_0$ be the set of all null sequences. For any $u, v \in \mathcal{X}$ we define

$$\langle u, v\rangle = uv^* = \{u_j\bar{u}_j\}_{j\in\mathrm{N}}.$$

Therefore $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module.

Define $f_j = \{f_i^j\}_{i\in\mathrm{N}^*}$ by $f_i^j = \frac{1}{2} + \frac{1}{i}$ if $i = j$ and $f_i^j = 0$ if $i \neq j$ $\forall j \in \mathbf{N}^*$.

Now define the adjointable operator $T_j : \mathcal{X} \to \mathcal{X}$, $T_j\{(\xi_i)_i\} = (\xi_i f_i^j)_i$.

Then for every $x \in \mathcal{X}$ we have

$$\sum_{j\in\mathrm{N}}\langle T_j\xi, T_j\xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathrm{N}^*}\langle\xi, \xi\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathrm{N}^*}.$$

So $\{T_j\}_j$ is a $\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathrm{N}^*}$-tight $*$-operator frame.

Let $K : \mathcal{H} \to \mathcal{H}$ defined by $K\xi = \{\frac{\xi_i}{i}\}_{i\in\mathrm{N}^*}$.

Then for every $\xi \in \mathcal{X}$ we have

$$\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in\mathrm{N}}\langle T_j\xi, T_j\xi\rangle = \{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathrm{N}^*}\langle\xi, \xi\rangle\{\frac{1}{2} + \frac{1}{i}\}_{i\in\mathrm{N}^*}.$$

This shows that $\{T_j\}_{j\in\mathbb{N}}$ is an $*$-K-operator frame with bounds $1, \{\frac{1}{2}+\frac{1}{i}\}_{i\in\mathbb{N}^*}$.

Remark 3.4. (1) Every $*$-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ is an $*$-K-operator frame, for any $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$: $K \neq 0$.

(2) If $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ is a surjective operator, then every $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$ is an $*$-operator frame.

Example 3.5. Let $\mathcal{X}$ be a finitely or countably generated Hilbert $\mathcal{A}$-module. $Hom^*_{A}(\mathcal{X})$. Let $K \in Hom^*_{\mathcal{A}}(\mathcal{X})$ an invertible element such that both are uniformly bounded and $K \neq 0$. Let $\{T_i\}_{i\in I}$ be an $*$-operator frame for $\mathcal{X}$ with bounds A and B, respectively. We have

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

Or

$$\langle K^*\xi, K^*\xi\rangle \leq \|K\|^2_{\infty}\langle \xi, \xi\rangle, \forall \xi \in \mathcal{X}.$$

Then

$$\|K\|^{-1}_{\infty}A\langle K^*\xi, K^*\xi\rangle(\|K\|^{-1}_{\infty}A)^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

So $\{T_i\}_{i\in I}$ is $*$-K-operator frame for $\mathcal{X}$ with bounds $\|K\|^{-1}_{\infty}A$ and B, respectively.

In what follows, we introduce the analysis, the synthesis and the frame operator. We also establish some properties.

Let $\{T_i\}_{i\in I}$ be an $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$. Define an operator $R : \mathcal{X} \to l^2(\mathcal{X})$ by $R\xi = \{T_i\xi\}_{i\in I}, \forall \xi \in \mathcal{X}$, then R is called the analysis operator. The adjoint of the analysis operator R, $R^* : l^2(\mathcal{X}) \to \mathcal{X}$ is given by $R^*(\{\xi_i\}_i) = \sum_{i\in I} T_i^*\xi_i, \forall \{\xi_i\}_i \in l^2(\mathcal{X})$. The operator R^* is called the synthesis operator. By composing R and R^*, the frame operator $S : \mathcal{X} \to \mathcal{X}$ is given by $S\xi = R^*R\xi = \sum_{i\in I} T_i^*T_i\xi$.

Note that S need not be invertible in general. But under some condition S will be invertible.

Theorem 3.6. *Let K be a surjective operators in $Hom^*_{\mathcal{A}}(\mathcal{X})$. If $\{T_i\}_{i\in I}$ is an $*$-K-operator frame for $Hom^*_{\mathcal{A}}(\mathcal{X})$, then the frame operator S is positive, invertible and adjointable. In addition we have the reconstruction formula, $\xi = \sum_{i\in I} T_i^*T_iS^{-1}\xi$, $\forall \xi \in \mathcal{X}$.*

Proof. We start by showing that, S is a self-adjoint operator. By definition we have $\forall \xi, \eta \in \mathcal{H}$

$$\begin{aligned}\langle S\xi, \eta\rangle &= \left\langle \sum_{i\in I} T_i^*T_i\xi, \eta \right\rangle \\ &= \sum_{i\in I}\langle T_i^*T_i\xi, \eta\rangle \\ &= \sum_{i\in I}\langle \xi, T_i^*T_i\eta\rangle\end{aligned}$$

$$= \left\langle \xi, \sum_{i\in I} T_i^* T_i \eta \right\rangle$$
$$= \langle \xi, S\eta \rangle.$$

Then S is a selfadjoint.

The operator S is clearly positive.

By (2) in Remark 3.4 $\{T_i\}_{i\in I}$ is an $*$-operator frame for $Hom_{\mathcal{A}}^*(\mathcal{X})$.

The definition of an $*$-operator gives

$$A_1\langle \xi, \xi\rangle A_1^* \leq \sum_{i\in I} \langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*.$$

Thus by the definition of norm in $l^2(\mathcal{X})$

$$\bar{p}_{\mathcal{X}}(R\xi)^2 = \bar{p}_{\mathcal{X}}(\sum_{i\in I}\langle T_i\xi, T_i\xi\rangle) \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle \xi, \xi\rangle), \forall \xi \in \mathcal{X}. \tag{3.3}$$

Therefore R is well defined and $\bar{p}_{\mathcal{X}}(R) \leq \bar{p}_{\mathcal{X}}(B)$. It's clear that R is a linear $\mathcal{A}$-module map. We will then show that the range of R is closed. Let $\{R\xi_n\}_{n\in\mathbb{N}}$ be a sequence in the range of R such that $\lim_{n\to\infty} R\xi_n = \eta$. For $n, m \in \mathbb{N}$, we have

$$p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \leq p(\langle R(\xi_n - \xi_m), R(\xi_n - \xi_m)\rangle) = \bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2.$$

Seeing that $\{R\xi_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{X}$, then

$p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*) \to 0$, as $n, m \to \infty$.

Note that for $n, m \in \mathbb{N}$,

$$\begin{aligned} p(\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle) &= p(A^{-1}A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*(A^*)^{-1}) \\ &\leq p(A^{-1})^2 p(A\langle \xi_n - \xi_m, \xi_n - \xi_m\rangle A^*). \end{aligned}$$

Thus the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is Cauchy and hence there exists $\xi \in \mathcal{X}$ such that $\xi_n \to \xi$ as $n \to \infty$. Again by (3.3), we have

$$\bar{p}_{\mathcal{X}}(R(\xi_n - \xi_m))^2 \leq \bar{p}_{\mathcal{X}}(B)^2 p(\langle \xi_n - \xi, \xi_n - \xi\rangle).$$

Thus $p(R\xi_n - R\xi) \to 0$ as $n \to \infty$ implies that $R\xi = \eta$. It is therefore concluded that the range of R is closed. We now show that R is injective. Let $\xi \in \mathcal{X}$ and $R\xi = 0$. Note that $A\langle \xi, \xi\rangle A^* \leq \langle R\xi, R\xi\rangle$ then $\langle \xi, \xi\rangle = 0$ so $\xi = 0$ i.e. R is injective.

For $\xi \in \mathcal{X}$ and $\{\xi_i\}_{i\in I} \in l^2(\mathcal{X})$ we have

$$\langle R\xi, \{\xi_i\}_{i\in I}\rangle = \langle \{T_i\xi\}_{i\in I}, \{\xi_i\}_{i\in I}\rangle = \sum_{i\in I}\langle T_i\xi, \xi_i\rangle = \sum_{i\in I}\langle \xi, T_i^*\xi_i\rangle = \langle \xi, \sum_{i\in I} T_i^*\xi_i\rangle.$$

Then $R^*(\{\xi_i\}_{i\in I}) = \sum_{i\in I} T_i^*\xi_i$. Since R is injective, then the operator R^* has closed range and $\mathcal{X} = range(R^*)$, therefore $S = R^*R$ is invertible

□

Let $K \in Hom^{*}_{\mathcal{A}}(\mathcal{X})$, in the following theorem we constructed an $*$-K-operator frame by using an $*$-operator frame.

Theorem 3.7. *Let $\{T_i\}_{i\in I}$ be an $*$-K-operator frame in $\mathcal{X}$ with bounds A, B and $K \in Hom^{*}_{\mathcal{A}}(\mathcal{X})$ be an invertible element such that both are uniformly bounded. Then $\{T_iK\}_{i\in I}$ is an $*$-K^*-operator frame in $\mathcal{X}$ with bounds A, $\|K\|_{\infty}B$. The frame operator of $\{T_iK\}_{i\in I}$ is $S^{'} = K^*SK$, where S is the frame operator of $\{T_i\}_{i\in I}$.*

Proof. From

$$A\langle \xi, \xi\rangle A^* \leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle \leq B\langle \xi, \xi\rangle B^*, \forall \xi \in \mathcal{X}.$$

We get for all $\xi \in \mathcal{X}$,

$$A\langle K\xi, K\xi\rangle A^* \leq \sum_{i\in I}\langle T_iK\xi, T_iK\xi\rangle \leq B\langle K\xi, K\xi\rangle B^* \leq \|K\|_{\infty}B\langle \xi, \xi\rangle(\|K\|_{\infty}B)^*.$$

Then $\{T_iK\}_{i\in I}$ is an $*$-K^*-operator frame in $\mathcal{X}$ with bounds A, $\|K\|_{\infty}B$.

By definition of S,we have $SK\xi = \sum_{i\in I} T_i^*T_iK\xi$. Then

$$K^*SK = K^*\sum_{i\in I} T_i^*T_iK\xi = \sum_{i\in I} K^*T_i^*T_iK\xi.$$

Hence $S^{'} = K^*SK$. □

Corollary 3.8. *Let $K \in Hom^{*}_{\mathcal{A}}(\mathcal{X})$ and $\{T_i\}_{i\in I}$ be an $*$-operator frame. Then $\{T_iS^{-1}K\}_{i\in I}$ is an $*$-K^*-operator frame, where S is the frame operator of $\{T_i\}_{i\in I}$.*

Proof. Result of the Theorem 3.7 for the $*$-operator frame $\{T_iS^{-1}\}_{i\in I}$. □

4. Tensor Product

We denote by $\mathcal{A}\otimes\mathcal{B}$, the minimal or injective tensor product of the pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, it is the completion of the algebraic tensor product $\mathcal{A}\otimes_{\text{alg}} \mathcal{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathcal{X}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{B}$. The algebraic tensor product $\mathcal{X}\otimes_{\text{alg}} \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathcal{A}\otimes\mathcal{B}$-module with the action of $\mathcal{A}\otimes\mathcal{B}$ on $\mathcal{X}\otimes_{\text{alg}} \mathcal{Y}$ defined by

$$(\xi\otimes\eta)(a\otimes b) = \xi a\otimes\eta b \text{ for all } \xi\in\mathcal{X}, \eta\in\mathcal{Y}, a\in\mathcal{A} \text{ and } b\in\mathcal{B}$$

and the inner product

$$\langle\cdot,\cdot\rangle : \left(\mathcal{X}\otimes_{\text{alg}} \mathcal{Y}\right)\times\left(\mathcal{X}\otimes_{\text{alg}} \mathcal{Y}\right) \to \mathcal{A}\otimes_{\text{alg}} \mathcal{B}. \text{ defined by}$$

$$\langle \xi_1\otimes\eta_1, \xi_2\otimes\eta_2\rangle = \langle \xi_1, \xi_2\rangle\otimes\langle \eta_1, \eta_2\rangle$$

And we know that for $z = \sum_{i=1}^{n}\xi_i\otimes\eta_i$ in $\mathcal{X}\otimes_{alg}\mathcal{Y}$ we have $\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j}\langle \xi_i, \xi_j\rangle_{\mathcal{A}}\otimes\langle \eta_i, \eta_j\rangle_{\mathcal{B}} \geq 0$ and $\langle z, z\rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z = 0$.

The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X} \otimes \mathcal{Y}$ over $\mathcal{A} \otimes \mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$.

If $P \in M(\mathcal{X})$ and $Q \in M(\mathcal{Y})$ then there is a unique adjointable module morphism $P \otimes Q : \mathcal{A} \otimes \mathcal{B} \to \mathcal{X} \otimes \mathcal{Y}$ such that $(P \otimes Q)(a \otimes b) = P(a) \otimes Q(b)$ and $(P \otimes Q)^*(a \otimes b) = P^*(a) \otimes Q^*(b)$ for all $a \in A$ and for all $b \in B$ (see, for example, cite The minimal or injective tensor product of the pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \otimes \mathcal{B}$, is the completion of the algebraic tensor product $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$ with respect to the topology determined by a family of C^*-seminorms. Suppose that $\mathcal{X}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{A}$ and $\mathcal{Y}$ is a Hilbert module over a pro-C^*-algebra $\mathcal{B}$. The algebraic tensor product $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is a pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module with the action of $\mathcal{A} \otimes \mathcal{B}$ on $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$ defined by

$$(\xi \otimes \eta)(a \otimes b) = \xi a \otimes \eta b \text{ for all } \xi \in \mathcal{X}, \eta \in \mathcal{Y}, a \in \mathcal{A} \text{ and } b \in \mathcal{B}$$

and the inner product

$$\langle \cdot, \cdot \rangle : \left(\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}\right) \times \left(\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}\right) \to \mathcal{A} \otimes_{\text{alg}} \mathcal{B}. \text{ defined by}$$

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \otimes \langle \eta_1, \eta_2 \rangle$$

We also know that for $z = \sum_{i=1}^{n} \xi_i \otimes \eta_i$ in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ we have $\langle z, z \rangle_{\mathcal{A}\otimes\mathcal{B}} = \sum_{i,j} \langle \xi_i, \xi_j \rangle_{\mathcal{A}} \otimes \langle \eta_i, \eta_j \rangle_{\mathcal{B}} \geq 0$ and $\langle z, z \rangle_{\mathcal{A}\otimes\mathcal{B}} = 0$ iff $z = 0$.

The external tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the Hilbert module $\mathcal{X} \otimes \mathcal{Y}$ over $\mathcal{A} \otimes \mathcal{B}$ obtained by the completion of the pre-Hilbert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes_{\text{alg}} \mathcal{Y}$.

If $P \in M(\mathcal{X})$ and $Q \in M(\mathcal{Y})$ then there is a unique adjointable module morphism $P \otimes Q : \mathcal{A} \otimes \mathcal{B} \to \mathcal{X} \otimes \mathcal{Y}$ such that $(P \otimes Q)(a \otimes b) = P(a) \otimes Q(b)$ and $(P \otimes Q)^*(a \otimes b) = P^*(a) \otimes Q^*(b)$ for all $a \in A$ and for all $b \in B$ (see, for example, [9])

Let I and J be countable index sets.

Theorem 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert pro-C^*-modules over unitary pro-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\{T_i\}_{i\in I} \subset Hom_{\mathcal{A}}^*(\mathcal{X})$ be an $*$-K-operator frame for $\mathcal{X}$ with bounds A and B and frame operators S_T and $\{P_j\}_{j\in J} \subset Hom_{\mathcal{B}}^*(\mathcal{Y})$ be an $*$-L-operator frame for $\mathcal{K}$ with bounds C and D and frame operators S_L. Then $\{T_i \otimes L_j\}_{i\in I, j\in J}$ is an $*$-K$\otimes$L-operator frame for Hibert $\mathcal{A} \otimes \mathcal{B}$-module $\mathcal{X} \otimes \mathcal{Y}$ with frame operator $S_T \otimes S_P$ and bounds $A \otimes C$ and $B \otimes D$.*

Proof. The defintion of $*$-K-operator frame $\{T_i\}_{i\in I}$ and $*$-L-operator frame $\{P_j\}_{j\in J}$ gives

$$A\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}} A^* \leq \sum_{i\in I} \langle T_i\xi, T_i\xi\rangle_{\mathcal{A}} \leq B\langle \xi, \xi\rangle_{\mathcal{A}} B^*, \forall \xi \in \mathcal{X}.$$

$$C\langle L^*\eta, L^*\eta\rangle_{\mathcal{B}} C^* \leq \sum_{j\in J} \langle P_j\eta, P_j\eta\rangle_{\mathcal{B}} \leq D\langle \eta, \eta\rangle_{\mathcal{B}} D^*, \forall \eta \in \mathcal{Y}.$$

Therefore

$$\begin{aligned}&(A\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}}A^*)\otimes(C\langle L^*\eta, L^*\eta\rangle_{\mathcal{B}}C^*)\\ &\leq \sum_{i\in I}\langle T_i\xi, T_i\xi\rangle_{\mathcal{A}}\otimes\sum_{j\in J}\langle P_j\eta, P_j\eta\rangle_{\mathcal{B}}\\ &\leq (B\langle\xi,\xi\rangle_{\mathcal{A}}B^*)\otimes(D\langle\eta,\eta\rangle_{\mathcal{B}}D^*), \forall\xi\in\mathcal{X}, \forall\eta\in\mathcal{Y}.\end{aligned}$$

Then

$$\begin{aligned}&(A\otimes C)(\langle K^*\xi, K^*\xi\rangle_{\mathcal{A}}\otimes\langle L^*\eta, L^*\eta\rangle_{\mathcal{B}})(A^*\otimes C^*)\\ &\leq \sum_{i\in I, j\in J}\langle T_i\xi, T_i\xi\rangle_{\mathcal{A}}\otimes\langle P_j\eta, P_j\eta\rangle_{\mathcal{B}}\\ &\leq (B\otimes D)(\langle\xi,\xi\rangle_{\mathcal{A}}\otimes\langle\eta,\eta\rangle_{\mathcal{B}})(B^*\otimes D^*), \forall\xi\in\mathcal{X}, \forall\eta\in\mathcal{Y}.\end{aligned}$$

Consequently we have

$$\begin{aligned}&(A\otimes C)\langle K^*\xi\otimes L^*\eta, K^*\xi\otimes L^*\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^*\\ &\leq \sum_{i\in I, j\in J}\langle T_i\xi\otimes P_j\eta, T_i\xi\otimes P_j\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\leq (B\otimes D)\langle\xi\otimes\eta, \xi\otimes\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*, \forall\xi\in\mathcal{X}, \forall\eta\in\mathcal{Y}.\end{aligned}$$

Then for all $\xi\otimes\eta$ in $\mathcal{X}\otimes\mathcal{Y}$ we have

$$\begin{aligned}&(A\otimes C)\langle(K\otimes L)^*(\xi\otimes\eta), (K\otimes L)^*(\xi\otimes\eta)\rangle_{\mathcal{A}\otimes\mathcal{B}}(A\otimes C)^*\\ &\leq \sum_{i\in I, j\in J}\langle(T_i\otimes P_j)(\xi\otimes\eta), (T_i\otimes P_j)(\xi\otimes\eta)\rangle_{\mathcal{A}\otimes\mathcal{B}}\\ &\leq (B\otimes D)\langle\xi\otimes\eta, \xi\otimes\eta\rangle_{\mathcal{A}\otimes\mathcal{B}}(B\otimes D)^*.\end{aligned}$$

The last inequality is true for every finite sum of elements in $\mathcal{X}\otimes_{alg}\mathcal{Y}$ and then it's true for all $z\in\mathcal{X}\otimes\mathcal{K}$. It shows that $\{T_i\otimes P_j\}_{i\in I, j\in J}$ is an $*$-$K\otimes L$-operator frame for Hilbert $\mathcal{A}\otimes\mathcal{B}$-module $\mathcal{X}\otimes\mathcal{Y}$ with lower and upper bounds $A\otimes C$ and $B\otimes D$, respectively.
By the definition of frame operator S_T and S_P we have

$$S_T\xi = \sum_{i\in I}T_i^*T_i\xi, \forall\xi\in\mathcal{X}.$$

$$S_P\eta = \sum_{j\in J}P_j^*P_j\eta, \forall\eta\in\mathcal{Y}.$$

Therefore

$$\begin{aligned}(S_T\otimes S_P)(\xi\otimes\eta) &= S_T\xi\otimes S_P\eta\\ &= \sum_{i\in I}T_i^*T_i\xi\otimes\sum_{j\in J}P_j^*P_j\eta\\ &= \sum_{i\in I, j\in J}T_i^*T_i\xi\otimes P_j^*P_j\eta\\ &= \sum_{i\in I, j\in J}(T_i^*\otimes P_j^*)(T_i\xi\otimes P_j\eta)\end{aligned}$$

$$= \sum_{i\in I, j\in J} (T_i^* \otimes P_j^*)(T_i \otimes P_j)(\xi \otimes \eta)$$
$$= \sum_{i\in I, j\in J} (T_i \otimes P_j)^*(T_i \otimes P_j)(\xi \otimes \eta).$$

Then by the uniqueness of frame operator, the last expression is equal to $S_{T\otimes P}(\xi\otimes\eta)$. Consequently we have $(S_T \otimes S_P)(\xi\otimes\eta) = S_{T\otimes P}(\xi\otimes\eta)$. The last equality is true for every finite sum of elements in $\mathcal{X} \otimes_{alg} \mathcal{Y}$ and then it's true for all $z \in \mathcal{X} \otimes \mathcal{Y}$. It follows that $(S_T \otimes S_P)(z) = S_{T\otimes P}(z)$. Thus $S_{T\otimes P} = S_T \otimes S_P$. □

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