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Generalized Trapezium-Type Inequalities via Modified (h,m)-Convex Functions of the Second Type

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Abstract. In this article, we establish a generalized framework for integral inequalities by introducing an integral family of trapezoidal fractional estimates. Using a generalized weight function $\varpi'(\tau)$, we prove a new integral identity that establishes clear relationships between general fractional operators and differentiable functions. Using this result together with the classical Hölder, Power Mean, and Young inequalities, we derive several precise upper bounds for the trapezoidal remainder by assuming that the absolute value of the first derivative belongs to the class of modified convex (h,m)-functions of the second type. Throughout the work, we show that several seminal results reported in the literature are special cases of our own. Finally, we provide concrete applications to precise error estimates for special means.

1. Introduction

The theory of integral inequalities has undergone rapid development in recent decades, establishing itself as one of the most dynamic research areas in applied mathematics due to its intrinsic relationship with optimization problems, numerical modeling, and operator theory. Within this field, the classic Hermite-Hadamard inequality stands as a fundamental result for bounding the integral average of convex functions, attracting the attention of numerous researchers and leading to the convergence of two highly fruitful conceptual branches: the use of new notions of convexity and the application of new integral operators.

Despite outstanding results, which constituted isolated solutions for each specific operator, the present work is precisely in order to overcome these limitations, providing a generalized

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framework, with weighted integrals, which not only unifies the known scattered criteria, but also refines the error bounds in functional classes of modified convexity, transforming complex theoretical results into directly applicable numerical tools.

Below we present the basic definitions of our work.

Definition 1. *([24])*
A set $\mathcal{I} \subset \mathbb{R}$ is said to be set convex set if

$$\tau\xi + (1-\tau)\varsigma \in \mathcal{I}$$

for each $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0,1]$.

Definition 2. *([24])*
The function $\psi : \mathcal{I} \to \mathbb{R}$, is a convex function if the following inequality holds:

$$\psi\left(\tau\xi + (1-\tau)\varsigma\right) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$$

for all $\xi, \varsigma \in \mathcal{I}$ and $\tau \in [0,1]$.

Readers interested in the ramifications and extensions of the classical notion of convexity can consult [22] where a fairly broad overview of this development is presented.

The Hermite-Hadamard Inequality, which we mentioned at the beginning, has the following form:

Suppose that ψ is a convex map on the closed real interval $[\xi, \varsigma] \in \mathbb{R}$ where $\xi \neq \varsigma$. Therefore (see [1, 10, 11, 19, 21])

$$\psi\left(\frac{\xi+\varsigma}{2}\right) \leq \frac{1}{\varsigma-\xi}\int_{\xi}^{\varsigma}\psi(\varkappa)d\varkappa \leq \frac{\psi(\xi)+\psi(\varsigma)}{2}.$$

For more recent developments related to the Hermite-Hadamard inequality, the reader is referred to [3, 6, 8, 9, 17, 18, 23] and references therein.

The class of convex functions that we will use in our work are the following (cf. [4, 5]):

Definition 3. *Let $\psi : \mathcal{I} \subseteq [0,\infty) \to [0,\infty)$ and $h : [0,1] \to [0,1]$, where $\mathcal{I}$ is an interval. If the inequality*

$$\psi\left(\xi\tau + m\varsigma(1-\tau)\right) \leq \psi(\xi)h^s(\tau) + m\psi(\varsigma)(1-h^s(\tau)) \tag{1}$$

holds for all $\varsigma, \xi \in \mathcal{I}$ and $\tau \in [0,1]$, where $s \in [0,1]$ and $m \in [0,1]$, then the function ψ is called a modified (h,m)–convex function of the first type on $\mathcal{I}$.

Definition 4. *Let $h : [0,1] \to [0,1]$ and $\psi : \mathcal{I} \subseteq [0,\infty) \to [0,\infty)$, where $\mathcal{I}$ is an interval. If the inequality*

$$\psi\left(\xi\tau + m\varsigma(1-\tau)\right) \leq \psi(\xi)h^s(\tau) + m\psi(\varsigma)(1-h(\tau))^s \tag{2}$$

holds for all $\varsigma, \xi \in \mathcal{I}$ and $\tau \in [0,1]$, where $s \in [0,1]$ and $m \in [0,1]$, then the function ψ is called a modified (h,m)–convex function of the second type on $\mathcal{I}$.

Remark 5. *This class of convex functions, as previously defined, contains many well-known definitions of convexity; however, we wish to establish a new, previously overlooked relationship. p-convex functions can be viewed as a particular case of the family of modified (h,m)-convex functions of the second type, under a suitable choice of parameters and a transformation of variables.*

We know that a function $f : I \subset (0,\infty) \to \mathbb{R}$ is defined as p-convex (with $p \neq 0$) if for all $x, y \in I$ and $\tau \in [0,1]$ the following holds:

$$f\left(\left[\tau x^p + (1-\tau)y^p\right]^{1/p}\right) \leq \tau f(x) + (1-\tau) f(y)$$

Obviously, if $p = 1$, we revert to ordinary convexity.

To establish that p-convexity is a special case of modified (h,m)-convex functions of the second type, a composition or change of variable is performed in the domain combined with the parameterization of the weights:

The parameters are set $m = 1$ and $s = 1$.

The weight function is taken as the identity function, that is, $h(\tau) = \tau$.

For the transformation of the domain (p-convexity), we define the function $\psi = f \circ g$, where $g(t) = t^{1/p}$. The linear inequality in the argument of ψ translates directly into the mean of order p in the argument of f.

Therefore, the p-convexity property of a function f is algebraically equivalent to saying that the transformed function $\psi(t) = f(t^{1/p})$ is an ordinary convex function, which is fully contained as the base case $(m = 1, s = 1, h(\tau) = \tau)$ of the modified (h,m)-convex functions of the second type.

The generalized integral operators with weight, which underpin our analysis, are presented next ([4, 5]).

Definition 6. *Let for some $\upsilon_1, \upsilon_2 \in \mathcal{I}$, $\psi \in L[\upsilon_1, \upsilon_2]$ and let $\varpi : [0,1] \to [0,+\infty)$ together with its second derivatives be an integrable function on $\mathcal{I}$. The generalized integral operators with weight, defined in both right–hand and left–hand forms, are given by:*

$$J^{\varpi}_{\upsilon_1+}\psi(r) = \int_{\upsilon_1}^{r} \varpi'\left(\frac{r-\sigma}{\upsilon_2-\upsilon_1}\right)\psi(\sigma)\, d\sigma, \quad r > \upsilon_1$$

and

$$J^{\varpi}_{\upsilon_2-}\psi(r) = \int_{r}^{\upsilon_2} \varpi'\left(\frac{\sigma-r}{\upsilon_2-\upsilon_1}\right)\psi(\sigma)\, d\sigma, \quad r < \upsilon_2.$$

Remark 7. *To better understand the scope of Definition 6, let us explore several specific cases of the kernel ϖ':*

1. *When $\varpi'(t) \equiv 1$, we recover the classical Riemann integral.*
2. *If $\varpi'(t) = \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}$, this yields the classical Riemann–Liouville fractional integral. The left–sided version can be defined analogously.*

3. *By choosing appropriate kernels ϖ', we can also recover several known operators: the k–Riemann–Liouville fractional integrals (see [20]), the right-sided fractional integrals of a function ψ with respect to another function h on $[v_1, v_2]$ (see [2]), the generalized right and left fractional integral operators of [12] and [26], as well as the integral operators discussed in [13] and [14]. All of these can be derived from Definition 6 by imposing suitable conditions on ϖ'.*

Naturally, several other well–known integral operators–both fractional and classical–can be obtained as special cases of the one previously discussed. However, we leave these explorations to interested readers (see, for example, [18, 27]).

The main objective of this work is to introduce and develop a generalized framework for the study of trapezoidal integral inequalities, overcoming the limitations imposed by the specific operators in the current literature. Specifically, we propose to prove a new integral identity using an abstract weighted kernel $\varpi'(\tau)$ that encompasses various cases found in the literature. From this lemma, we derive upper bounds for the trapezoidal residue, employing the classical inequalities of Hölder, Power Mean, and Young, under the condition that the modulus of the first derivative belongs to the class of functions (h, m)-convex modified of the second type. Later, we present applications to special half-forms to validate its practical applicability.

2. Main results

Lemma 8. *Let $v_1, v_2 \in \mathcal{I}$ with $v_1 < v_2$, and $\psi \in L[v_1, v_2]$. If weighted integral operators exist, then the following identity holds on the interval $[0, 1]$:*

$$J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1) = (v_2 - v_1)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau)\right]\psi(\tau v_1 + (1-\tau)v_2)d\tau. \tag{3}$$

Proof. Let's consider the left-hand operator evaluated at the upper bound, $r = v_2$, from the definition of 6:

$$J^w_{v_1+}\psi(v_2) = \int_{v_1}^{v_2} \varpi'\left(\frac{v_2-\sigma}{v_2-v_1}\right)\psi(\sigma)d\sigma$$

Making the change of variable $\sigma = \tau v_1 + (1-\tau)v_2$. Calculating the differential of the transformation, we obtain $d\sigma = (v_1 - v_2)d\tau = -(v_2 - v_1)d\tau$.

Let's evaluate the limits of integration for the new variable τ:

If $\sigma = v_1 \implies v_1 = \tau v_1 + (1-\tau)v_2 \implies (1-\tau)(v_1 - v_2) = 0 \implies \tau = 1$.

If $\sigma = v_2 \implies v_2 = \tau v_1 + (1-\tau)v_2 \implies \tau(v_1 - v_2) = 0 \implies \tau = 0$.

Formally substituting the expressions into the left-hand operator:

$$J^w_{v_1+}\psi(v_2) = \int_1^0 \varpi'\left(\frac{v_2 - [\tau v_1 + (1-\tau)v_2]}{v_2 - v_1}\right)\psi(\tau v_1 + (1-\tau)v_2)\left[-(v_2 - v_1)d\tau\right]$$

After some simple algebraic work we have:

$$J_{v_1+}^{w}\psi(v_2) = (v_2 - v_1)\int_0^1 \varpi'(\tau)\psi(\tau v_1 + (1-\tau)v_2)d\tau$$

Performing a similar procedure for the right operator evaluated at the lower end $r = v_1$:

$$J_{v_2-}^{w}\psi(v_1) = \int_{v_1}^{v_2} \varpi'\left(\frac{\sigma - v_1}{v_2 - v_1}\right)\psi(\sigma)d\sigma$$

After the change of variable $\sigma = \tau v_1 + (1-\tau)v_2$, and proceeding as before, we arrive at:

$$J_{v_2-}^{w}\psi(v_1) = (v_2 - v_1)\int_0^1 \varpi'(1-\tau)\psi(\tau v_1 + (1-\tau)v_2)d\tau$$

Adding term by term both integral expressions obtained over $[0,1]$ and factoring out the common linear term $(v_2 - v_1)$, we finally obtain:

$$J_{v_1+}^{w}\psi(v_2) + J_{v_2-}^{w}\psi(v_1) = (v_2 - v_1)\int_0^1 \left[\varpi'(\tau) + \varpi'(1-\tau)\right]\psi(\tau v_1 + (1-\tau)v_2)d\tau$$

The identity is proven. $\square$

Theorem 9. *(Bounding for Kernels with Monotonic Properties in Contractive Regime)*

Let $\psi : \mathcal{I} \to [0,\infty)$ be a modified (h,m)-convex function of the second type with $s, m \in (0,1]$. Let $v_1, v_2 \in \mathcal{I}$ be such that $v_1 < mv_2$. If ϖ' is monotonically non-decreasing and bounded on $[0,1]$, then the generalized double inequality holds:

$$\begin{aligned}
&\frac{\psi\left(\frac{v_1+mv_2}{2}\right)}{h^s\left(\frac{1}{2}\right)\left[1 + m\left(\frac{1-h(1/2)}{h(1/2)}\right)^s\right]}\int_0^1 \varpi'(\tau)d\tau \le \frac{J_{v_1+}^{w}\psi(v_2) + J_{v_2-}^{w}\psi(v_1)}{v_2 - v_1}\\
&\le \psi(v_1)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau)\right]h^s(\tau)d\tau + m\psi(v_2)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau)\right](1-h(\tau))^s d\tau
\end{aligned} \tag{4}$$

Proof. By virtue of the convexity class of the second type we are using, and evaluating at the symmetric midpoint $\tau = \frac{1}{2}$, we have:

$$\psi\left(\frac{x+my}{2}\right) \le h^s\left(\frac{1}{2}\right)\psi(x) + m\left(1 - h\left(\frac{1}{2}\right)\right)^s\psi(y)$$

Let:

$$x = \tau v_1 + (1-\tau)v_2, \quad y = (1-\tau)v_1 + \tau v_2$$

Substituting these expressions into the argument of the left-hand side of the initial midpoint inequality:

$$\frac{x+my}{2} = \frac{[\tau v_1 + (1-\tau)v_2] + m[(1-\tau)v_1 + \tau v_2]}{2} = \tau\left(\frac{v_1 + mv_2}{2}\right) + (1-\tau)\left(\frac{v_2 + mv_1}{2}\right)$$

Multiplying by $\varpi'(\tau)$ and integrating with respect to τ over $[0,1]$:

$$\begin{aligned}&\int_0^1 \varpi'(\tau)\psi\left(\frac{[\tau v_1+(1-\tau)v_2]+m[(1-\tau)v_1+\tau v_2]}{2}\right)d\tau\\ \leq\;& h^s\left(\frac{1}{2}\right)\int_0^1 \varpi'(\tau)\psi(\tau v_1+(1-\tau)v_2)d\tau+m\left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 \varpi'(\tau)\psi((1-\tau)v_1+\tau v_2)d\tau\end{aligned} \tag{5}$$

Let's analyze the second integral of the right side $\int_0^1 \varpi'(\tau)\psi((1-\tau)v_1+\tau v_2)d\tau$. Changing variables by $u=1-\tau$ $(d\tau=-du)$, we obtain:

$$\int_0^1 \varpi'(\tau)\psi((1-\tau)v_1+\tau v_2)d\tau=\int_0^1 \varpi'(1-u)\psi(uv_1+(1-u)v_2)du$$

Substituting this equivalence into (5) and grouping like terms, the right-hand side becomes:

$$\int_0^1\left[h^s\left(\frac{1}{2}\right)\varpi'(\tau)+m\left(1-h\left(\frac{1}{2}\right)\right)^s\varpi'(1-\tau)\right]\psi(\tau v_1+(1-\tau)v_2)d\tau$$

By using Lemma 8, the left member is reduced to the expression:

$$\frac{\psi\left(\frac{v_1+mv_2}{2}\right)}{h^s(1/2)+m(1-h(1/2))^s}\int_0^1 \varpi'(\tau)d\tau\leq\frac{J^w_{v_1+}\psi(v_2)+J^w_{v_2-}\psi(v_1)}{v_2-v_1}$$

Thus we obtain the first inequality.

To prove the second inequality, we take the exact identity obtained in Lemma 8:

$$\frac{J^w_{v_1+}\psi(v_2)+J^w_{v_2-}\psi(v_1)}{v_2-v_1}=\int_0^1\left[\varpi'(1-\tau)+\varpi'(\tau)\right]\psi(\tau v_1+(1-\tau)v_2)d\tau$$

Since the function ψ belongs to the modified convexity class of the second type, we have:

$$\psi(\tau v_1+(1-\tau)v_2)\leq h^s(\tau)\psi(v_1)+m(1-h(\tau))^s\psi(v_2)$$

Since by definition the weighted kernel satisfies $\varpi'(t)\geq 0$ for all $t\in[0,1]$, the sum of the kernels is strictly non-negative $[\varpi'(1-\tau)+\varpi'(\tau)]\geq 0$. Therefore, the inequality sign is preserved by monotonically multiplying both sides:

$$\left[\varpi'(1-\tau)+\varpi'(\tau)\right]\psi(\tau v_1+(1-\tau)v_2)\leq\left[\varpi'(1-\tau)+\varpi'(\tau)\right]\left[h^s(\tau)\psi(v_1)+m(1-h(\tau))^s\psi(v_2)\right]$$

Integrating the inequality term by term over the compact interval $[0,1]$ with respect to the variable τ:

$$\begin{aligned}&\int_0^1\left[\varpi'(1-\tau)+\varpi'(\tau)\right]\psi(\tau v_1+(1-\tau)v_2)d\tau\\ \leq\;&\int_0^1\left[\varpi'(1-\tau)+\varpi'(\tau)\right]\left[h^s(\tau)\psi(v_1)+m(1-h(\tau))^s\psi(v_2)\right]d\tau.\end{aligned}$$

By properties of the integral and factoring the non-negative scalar values $\psi(v_1)$ and $m\psi(v_2)$ we have:

$$\leq\psi(v_1)\int_0^1\left[\varpi'(1-\tau)+\varpi'(\tau)\right]h^s(\tau)d\tau+m\psi(v_2)\int_0^1\left[\varpi'(1-\tau)+\varpi'(\tau)\right](1-h(\tau))^s d\tau$$

From the Lemma 8 we finally obtain:

$$\frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{v_2 - v_1}$$
$$\leq \psi(v_1)\int_0^1 \left[\varpi'(1-\tau)+\varpi'(\tau)\right] h^s(\tau)d\tau + m\psi(v_2)\int_0^1 \left[\varpi'(1-\tau)+\varpi'(\tau)\right](1-h(\tau))^s d\tau$$

This completes the demonstration of the Theorem. □

Remark 10. *It is highly instructive to emphasize that the fundamental structure established in Theorem 9 serves as a general analytical framework containing the classical Hermite-Hadamard inequality as a simple and straightforward special case. Let us examine this in more detail. Consider $m = 1$ and $s = 1$, and specify the weight function as the identity function, namely, $h(\tau) = \tau$. Under these configurations, the definition of the modified convex function (h, m) of the second kind becomes the classical convex function. Let us define the derivative of the generalized kernel function as $\varpi'(t) \equiv 1$ for all $t \in [0, 1]$. Under this specific choice, the fractional integral operators $J^w_{v_1+}$ and $J^w_{v_2-}$ become the classical Riemann integrals:*

$$J^w_{v_1+}\psi(v_2) = J^w_{v_2-}\psi(v_1) = \int_{v_1}^{v_2}\psi(\sigma)d\sigma$$

Evaluating the above in Theorem 11, we have on the left-hand side:

$$\psi\left(\frac{v_1+v_2}{2}\right) \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2}\psi(\sigma)d\sigma$$

On the other hand, in the inequality on the right, evaluating the integrals of the linear weights Combined with $\varpi'(\tau) \equiv 1$, we get:

$$\frac{\psi(v_1)+\psi(v_2)}{2}$$

Combining these two results, we obtain the Hermite-Hadamard Inequality in its classical form:

$$\psi\left(\frac{v_1+v_2}{2}\right) \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2}\psi(\sigma)d\sigma \leq \frac{\psi(v_1)+\psi(v_2)}{2}$$

This reduction confirms the mathematical consistency of Theorem 9, demonstrating that it represents a generalized result for Hermite-Hadamard type inequalities.

Theorem 11. *Under the same hypotheses as Theorem 9, if the kernel ϖ' exhibits an asymmetric or higher-order structure that requires the use of bounds at the endpoints of the domain, the following inequality holds:*

$$\frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{v_2 - v_1} \leq \left[\max_{t\in[0,1]}\varpi'(t)\right]\cdot\left[\psi(v_1)\int_0^1 h^s(\tau)d\tau + m\psi(v_2)\int_0^1(1-h(\tau))^s d\tau\right] + \mathcal{R}_\varpi(\psi) \tag{6}$$

where $\mathcal{R}_\varpi(\psi)$ represents the integral coupling residue.

Proof. Starting from Lemma 8:

$$\frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{v_2 - v_1} = \int_0^1 \varpi'(1-\tau)\psi(\tau v_1 + (1-\tau)v_2)d\tau + \int_0^1 \varpi'(\tau)\psi(\tau v_1 + (1-\tau)v_2)d\tau$$

Applying the modified convexity hypothesis of the second type to each integral separately:

$$\int_0^1 \varpi'(1-\tau)\psi(\tau v_1+(1-\tau)v_2)d\tau \le \psi(v_1)\int_0^1 \varpi'(1-\tau)h^s(\tau)d\tau + m\psi(v_2)\int_0^1 \varpi'(1-\tau)(1-h(\tau))^s d\tau$$

$$\int_0^1 \varpi'(\tau)\psi(\tau v_1 + (1-\tau)v_2)d\tau \le \psi(v_1)\int_0^1 \varpi'(\tau)h^s(\tau)d\tau + m\psi(v_2)\int_0^1 \varpi'(\tau)(1-h(\tau))^s d\tau$$

Adding both results gives us the bound:

$$\psi(v_1)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau)\right] h^s(\tau)d\tau + m\psi(v_2)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau)\right](1-h(\tau))^s d\tau$$

Since the generalized kernel $\varpi' : [0,1] \to [0,\infty)$ is an integrable and bounded function on the compact interval $[0,1]$, by the uniform boundedness theorem there exists a non-negative real supremum for the kernel and its reflection. We define:

$$\mathcal{M} = \sup_{t\in[0,1]} \varpi'(t)$$

Because if $t \in [0,1]$ then $(1-t) \in [0,1]$, it is verified identically that:

$$\varpi'(\tau) \le \mathcal{M} \quad \varpi'(1-\tau) \le \mathcal{M} \quad \forall \tau \in [0,1]$$

Therefore, by linearity we have:

$$\varpi'(1-\tau) + \varpi'(\tau) \le \mathcal{M} + \mathcal{M} = 2\mathcal{M}$$

With this, we can increase both integrals and finally obtain:

$$\frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{v_2 - v_1} \le 2\left[\sup_{t\in[0,1]} \varpi'(t)\right] \cdot \left[\psi(v_1)\int_0^1 h^s(\tau)d\tau + m\psi(v_2)\int_0^1 (1-h(\tau))^s d\tau\right]$$

Where $\mathcal{R}_\varpi(\psi)$ is explicitly defined as:

$$\mathcal{R}_\varpi(\psi) = \psi(v_1)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau) - 2\mathcal{M}\right] h^s(\tau)d\tau + m\psi(v_2)\int_0^1 \left[\varpi'(1-\tau) + \varpi'(\tau) - 2\mathcal{M}\right](1-h(\tau))^s d\tau$$

□

Remark 12. *Analytical Properties of $\mathcal{R}_\varpi(\psi)$*

Since $\mathcal{M}$ is the absolute supremum of $\varpi'(t)$ in $[0,1]$, it holds by definition that $\varpi'(\tau) \le \mathcal{M}$ and $\varpi'(1-\tau) \le \mathcal{M}$. Therefore:

$$\varpi'(1-\tau) + \varpi'(\tau) \le 2\mathcal{M} \implies \left[\varpi'(1-\tau) + \varpi'(\tau) - 2\mathcal{M}\right] \le 0$$

Since the weight functions are non-negative ($h^s(\tau) \geq 0$, $(1-h(\tau))^s \geq 0$) and the functional values are also non-negative ($\psi(v_1) \geq 0$, $m\psi(v_2) \geq 0$), integrating a non-positive function yields a non-positive result:

$$\mathcal{R}_{\varpi}(\psi) \leq 0$$

This demonstrates that $\mathcal{R}_{\varpi}(\psi)$ acts as a "buffer" or error-control term. If we omit it, the inequality still holds (since we would be adding a negative term to the right-hand side, making the upper bound a little looser), but keeping it explicitly in Theorem 11 gives it absolute precision.

Remark 13. *Let's see how the results of [16] are deduced as a direct particular case.*

Corollary 14. *If in Theorem 9 we fix the parameters at $m = 1$, $s = 1$, and select the identity weight function $h(\tau) = \tau$, further defining the weighted kernel under the fractional Riemann-Liouville standard $\varpi'(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, the generalized inequality becomes that of Theorem 3.1 of [16].*

Proof. By fixing $m = 1, s = 1, h(\tau) = \tau$, the modified convexity condition of the second type reduces to the classical convexity:

$$\psi(\tau v_1 + (1-\tau)v_2) \leq \tau\psi(v_1) + (1-\tau)\psi(v_2)$$

Evaluating the weight integrals of the Theorem 9 with $\varpi'(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$:

$$\mathcal{K}_1 = \int_0^1 \left[\frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\right]\tau d\tau = \frac{1}{\Gamma(\alpha)}\left[\int_0^1 \tau(1-\tau)^{\alpha-1}d\tau + \int_0^1 \tau^{\alpha}d\tau\right]$$

Using the definition of the Beta function from [16], $\int_0^1 \tau(1-\tau)^{\alpha-1}d\tau = B(2,\alpha) = \frac{\Gamma(2)\Gamma(\alpha)}{\Gamma(\alpha+2)} = \frac{1}{\alpha(\alpha+1)}$. For its part, the second elementary integral gives $\frac{1}{\alpha+1}$. Making the change of variables $v = x^p$, required to obtain the p-convexity of f (where $\psi(t) = f(t^{1/p})$), the differentials drag exactly the multiplicative hyperbolic algebraic terms $-\mu f(\mu)\gamma^{\frac{2}{p}-1} + \nu f(\mu)\gamma^{\frac{1}{p}}(1-\gamma)^{\frac{1}{p}-1}$.

Let us consider the right side of our generalized inequality (Theorem 9) after the substitution of the Riemann-Liouville kernels and the weight functions:

$$\mathbb{R} = \psi(\xi)\int_0^1 \frac{1}{\Gamma(\alpha)}\left[(1-\tau)^{\alpha-1} + \tau^{\alpha-1}\right]\tau\, d\tau + \psi(\varsigma)\int_0^1 \frac{1}{\Gamma(\alpha)}\left[(1-\tau)^{\alpha-1} + \tau^{\alpha-1}\right](1-\tau)\, d\tau$$

After performing the necessary calculations and substituting, we obtain

$$\mathbb{R} = \frac{1}{\Gamma(\alpha)}\left[f(\mu)\cdot\frac{1}{\alpha} + f(\nu)\cdot\frac{1}{\alpha}\right] = \frac{f(\mu)+f(\nu)}{\alpha\Gamma(\alpha)} = \frac{f(\mu)+f(\nu)}{\Gamma(\alpha+1)}$$

Similarly, the left member is reduced to

$$\frac{J^{\alpha}_{\mu^+,p}f(\nu) + J^{\alpha}_{\nu^-,p}f(\mu)}{(\nu^p-\mu^p)^{\alpha}} \leq \frac{f(\mu)+f(\nu)}{\Gamma(\alpha+1)}$$

From here

$$J^{\alpha}_{\mu^+,p}f(\nu) + J^{\alpha}_{\nu^-,p}f(\mu) \leq \frac{(\nu^p-\mu^p)^{\alpha}}{\Gamma(\alpha+1)}[f(\mu)+f(\nu)]$$

From which we obtain Theorem 3.1 of [16]. □

For convenience, we will denote the right-hand side of equation (3) as follows:

$$\mathcal{A}_{\varpi}(\psi; v_1, v_2) = \frac{\psi(v_1) + \psi(v_2)}{2} \int_0^1 \varpi'(\tau) d\tau - \frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{2(v_2 - v_1)}. \tag{7}$$

So, we have

Lemma 15. *Let $\varpi : [0,1] \to [0,\infty)$ be a differentiable monotonic function with integrable derivative ϖ' on $[0,1]$. Let $\psi : \mathcal{I} \to \mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ such that $\psi' \in L[v_1, v_2]$ for $v_1, v_2 \in \mathcal{I}$ with $v_1 < v_2$. The following identity is verified:*

$$\mathcal{A}_{\varpi}(\psi; v_1, v_2) = \frac{v_2 - v_1}{2} \int_0^1 \mathcal{K}_{\varpi}(\tau) \psi'(\tau v_1 + (1-\tau) v_2) d\tau \tag{8}$$

where the control core is attached $\mathcal{K}_{\varpi}(\tau) : [0,1] \to \mathbb{R}$ is explicitly defined by:

$$\mathcal{K}_{\varpi}(\tau) = \int_0^\tau \varpi'(t) dt - \int_\tau^1 \varpi'(t) dt$$

Proof. To prove the identity, we will start from the right side, integrating by parts we obtain:

$$\mathcal{I} = \int_0^1 \mathcal{K}_{\varpi}(\tau) \psi'(\tau v_1 + (1-\tau) v_2) d\tau$$

From here:

$$\mathcal{I} = \left[\mathcal{K}_{\varpi}(\tau) \left(-\frac{\psi(\tau v_1 + (1-\tau) v_2)}{v_2 - v_1} \right) \right]_0^1 - \int_0^1 \left(-\frac{\psi(\tau v_1 + (1-\tau) v_2)}{v_2 - v_1} \right) (2\varpi'(\tau) d\tau)$$

By properties:

$$\mathcal{I} = -\frac{1}{v_2 - v_1} \left[\mathcal{K}_{\varpi}(1)\psi(v_1) - \mathcal{K}_{\varpi}(0)\psi(v_2) \right] + \frac{2}{v_2 - v_1} \int_0^1 \varpi'(\tau) \psi(\tau v_1 + (1-\tau) v_2) d\tau$$

As

$$\left[\mathcal{K}_{\varpi}(1)\psi(v_1) - \mathcal{K}_{\varpi}(0)\psi(v_2) \right] = \left(\int_0^1 \varpi'(\tau) d\tau \right) \psi(v_1) - \left(-\int_0^1 \varpi'(\tau) d\tau \right) \psi(v_2)$$

Have

$$\mathcal{I} = -\frac{\psi(v_1) + \psi(v_2)}{v_2 - v_1} \int_0^1 \varpi'(\tau) d\tau + \frac{2}{v_2 - v_1} \int_0^1 \varpi'(\tau) \psi(\tau v_1 + (1-\tau) v_2) d\tau$$

From the definition it is easy to see that

$$\int_0^1 \varpi'(\tau) \psi(\tau v_1 + (1-\tau) v_2) d\tau = \frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{2(v_2 - v_1)}$$

Then

$$\mathcal{I} = -\frac{\psi(v_1) + \psi(v_2)}{v_2 - v_1} \int_0^1 \varpi'(\tau) d\tau + \frac{J^w_{v_1+}\psi(v_2) + J^w_{v_2-}\psi(v_1)}{(v_2 - v_1)^2}$$

After multiplying by the factor $\frac{v_2 - v_1}{2}$, rearranging and grouping conveniently, we have

$$\frac{\psi(v_1)+\psi(v_2)}{2}\int_0^1 \varpi'(\tau)d\tau - \frac{J^w_{v_1+}\psi(v_2)+J^w_{v_2-}\psi(v_1)}{2(v_2-v_1)} = \frac{v_2-v_1}{2}\int_0^1 \mathcal{K}_\varpi(\tau)\psi'(\tau v_1+(1-\tau)v_2)d\tau$$

Which is the desired identity. This completes the proof. □

Remark 16. *We want to point out some details to illustrate the scope of the above Lemma.*

If we consider convex functions and consider the weight $\varpi'(r)=r$ from here we derive Lemma 2.1 of [7] one of the fundamental results in the Theory of Integral Inequalities.

Conversely, if $\varpi'(r)=r^\alpha$ and we consider convex functions, from Lemma 8 we obtain without difficulty, Lemma 2 of [25].

Theorem 17. *Let $\psi:\mathcal{I}\to\mathbb{R}$ be a differentiable function on $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$). Let $v_1,v_2\in\mathcal{I}$ with $v_1<mv_2$ and $\psi'\in L[v_1,v_2]$. If $|\psi'|^q$ is a modified (h,m)-convex function of the second type on $\mathcal{I}$ for $s,m\in(0,1]$ and real conjugates $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| \le \left(\int_0^1 |\mathcal{K}_\varpi(\tau)|^p\,d\tau\right)^{\frac{1}{p}} \left(|\psi'(v_1)|^q\int_0^1 h^s(\tau)d\tau + m|\psi'(v_2)|^q\int_0^1(1-h(\tau))^s d\tau\right)^{\frac{1}{q}} \tag{9}$$

Proof. We take the absolute value in both members of the equation (8):

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| = \left|\int_0^1 \mathcal{K}_\varpi(\tau)\psi'(\tau v_1+(1-\tau)v_2)d\tau\right|$$

By properties we have:

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| \le \int_0^1 |\mathcal{K}_\varpi(\tau)|\cdot|\psi'(\tau v_1+(1-\tau)v_2)|\,d\tau$$

Let the non-negative functions be $f(\tau)=|\mathcal{K}_\varpi(\tau)|$ and $g(\tau)=|\psi'(\tau v_1+(1-\tau)v_2)|$. Applying Hölder's inequality for conjugate exponents $p,q>1$, we obtain:

$$\int_0^1 f(\tau)g(\tau)d\tau \le \left(\int_0^1 [f(\tau)]^p d\tau\right)^{\frac{1}{p}}\left(\int_0^1 [g(\tau)]^q d\tau\right)^{\frac{1}{q}}$$

Substituting our explicit functions:

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| \le \left(\int_0^1 |\mathcal{K}_\varpi(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\left(\int_0^1 |\psi'(\tau v_1+(1-\tau)v_2)|^q\,d\tau\right)^{\frac{1}{q}}$$

Since $|\psi'|^q$ is (h,m)-modified convex of the second type, we expand the argument of the second integral of the right side:

$$|\psi'(\tau v_1+(1-\tau)v_2)|^q \le h^s(\tau)|\psi'(v_1)|^q + m(1-h(\tau))^s|\psi'(v_2)|^q$$

Because the integral operator preserves the order of functional inequalities, we integrate the upper bound on the compact interval $[0,1]$ with respect to the variable τ:

$$\int_0^1 |\psi'(\tau v_1 + (1-\tau)v_2)|^q \, d\tau \leq \int_0^1 \left[h^s(\tau)|\psi'(v_1)|^q + m(1-h(\tau))^s|\psi'(v_2)|^q\right] d\tau$$

For linearity, we extract the non-negative real constants $|\psi'(v_1)|^q$ and $m|\psi'(v_2)|^q$ from the integrals:

$$\int_0^1 |\psi'(\tau v_1 + (1-\tau)v_2)|^q \, d\tau \leq |\psi'(v_1)|^q \int_0^1 h^s(\tau)d\tau + m|\psi'(v_2)|^q \int_0^1 (1-h(\tau))^s d\tau$$

Then:

$$|\mathcal{A}_\varpi(\psi; v_1, v_2)| \leq \left(\int_0^1 |\mathcal{K}_\varpi(\tau)|^p \, d\tau\right)^{\frac{1}{p}} \left(|\psi'(v_1)|^q \int_0^1 h^s(\tau)d\tau + m|\psi'(v_2)|^q \int_0^1 (1-h(\tau))^s d\tau\right)^{\frac{1}{q}}$$

The proof of the Theorem is complete. □

Theorem 18. *Under the same differentiability and integrability conditions of Theorem 17, if the modulus of the derivative raised to the power $q \geq 1$ ($|\psi'|^q$) is a modified (h, m) convex function of the second type on $\mathcal{I}$ for $s, m \in (0, 1]$, the following estimate holds:*

$$|\mathcal{A}_\varpi(\psi; v_1, v_2)| \leq \left(\int_0^1 |\mathcal{K}_\varpi(\tau)| \, d\tau\right)^{1-\frac{1}{q}} \times$$
$$\left(|\psi'(v_1)|^q \int_0^1 |\mathcal{K}_\varpi(\tau)| \, h^s(\tau)d\tau + m|\psi'(v_2)|^q \int_0^1 |\mathcal{K}_\varpi(\tau)| \, (1-h(\tau))^s d\tau\right)^{\frac{1}{q}}. \tag{10}$$

Proof. As in the previous case, we take the absolute value in the equation (8):

$$|\mathcal{A}_\varpi(\psi; v_1, v_2)| \leq \int_0^1 |\mathcal{K}_\varpi(\tau)| \cdot |\psi'(\tau v_1 + (1-\tau)v_2)| \, d\tau$$

The Power Mean inequality states that for any non-negative measure, the following holds:

$$\int_0^1 f(\tau)g(\tau)d\tau \leq \left(\int_0^1 f(\tau)d\tau\right)^{1-\frac{1}{q}} \left(\int_0^1 f(\tau)[g(\tau)]^q d\tau\right)^{\frac{1}{q}}$$

Assigning again $f(\tau) = |\mathcal{K}_\varpi(\tau)|$ and $g(\tau) = |\psi'(\tau v_1 + (1-\tau)v_2)|$, we rewrite the expression:

$$|\mathcal{A}_\varpi(\psi; v_1, v_2)| \leq \left(\int_0^1 |(1-\tau)v_2)|^q \, d\tau\right)^{\frac{1}{q}}$$

Using the modified (h, m)-convexity of the second type:

$$|\psi'(\tau v_1 + (1-\tau)v_2)|^q \leq h^s(\tau)|\psi'(v_1)|^q + m(1-h(\tau))^s|\psi'(v_2)|^q$$

Since by formal definition the modular core satisfies $|\mathcal{K}_\varpi(\tau)| \geq 0$, we multiply member by member preserving the inequality:

$$|\mathcal{K}_\varpi(\tau)| \cdot |\psi'(\tau v_1 + (1-\tau)v_2)|^q \leq |\mathcal{K}_\varpi(\tau)| \left[h^s(\tau)|\psi'(v_1)|^q + m(1-h(\tau))^s|\psi'(v_2)|^q\right]$$

We integrate the previous inequality with respect to τ over the interval $[0,1]$:

$$\int_0^1 |\mathcal{K}_\varpi(\tau)|\cdot|\psi'(\tau v_1+(1-\tau)v_2)|^q\,d\tau \le \int_0^1 |\mathcal{K}_\varpi(\tau)|\left[h^s(\tau)|\psi'(v_1)|^q + m(1-h(\tau))^s|\psi'(v_2)|^q\right]d\tau$$

Where do you have:

$$\int_0^1 |\mathcal{K}_\varpi(\tau)|\cdot|\psi'(\tau v_1+(1-\tau)v_2)|^q\,d\tau \le |\psi'(v_1)|^q\int_0^1 |\mathcal{K}_\varpi(\tau)|\,h^s(\tau)d\tau + m|\psi'(v_2)|^q\int_0^1 |\mathcal{K}_\varpi(\tau)|\,(1-h(\tau))^s d\tau$$

For all of the above we have:

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| \le \left(\int_0^1 |\mathcal{K}_\varpi(\tau)|\,d\tau\right)^{1-\frac{1}{q}}\left(|\psi'(v_1)|^q\int_0^1 |\mathcal{K}_\varpi(\tau)|\,h^s(\tau)d\tau + m|\psi'(v_2)|^q\int_0^1 |\mathcal{K}_\varpi(\tau)|\,(1-h(\tau))^s d\tau\right)^{\frac{1}{q}}$$

Which completes the proof of the Theorem. □

Theorem 19. *Under the previous differentiability and integrability assumptions, let $p,q>1$ be real conjugates such that $\frac{1}{p}+\frac{1}{q}=1$. If $|\psi'|^q$ is a modified (h,m)-convex function of the second type on $\mathcal{I}$ with parameters $s,m\in(0,1]$, then the following upper bound holds for any tunable scaling parameter $\epsilon>0$:*

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| \le \frac{\epsilon^p}{p}\int_0^1 |\mathcal{K}_\varpi(\tau)|^p\,d\tau + \frac{\epsilon^{-q}}{q}\left[|\psi'(v_1)|^q\int_0^1 h^s(\tau)d\tau + m|\psi'(v_2)|^q\int_0^1 (1-h(\tau))^s d\tau\right]. \tag{11}$$

Proof. As before, we start by taking the modulus in the equation (8):

$$|\mathcal{A}_\varpi(\psi;v_1,v_2)| \le \int_0^1 |\mathcal{K}_\varpi(\tau)|\cdot|\psi'(\tau v_1+(1-\tau)v_2)|\,d\tau$$

To obtain a more applicable, flexible form, we rewrite the integrand product by simultaneously multiplying and dividing by $\epsilon>0$:

$$|\mathcal{K}_\varpi(\tau)|\cdot|\psi'(\tau v_1+(1-\tau)v_2)| = (\epsilon\,|\mathcal{K}_\varpi(\tau)|)\cdot\left(\epsilon^{-1}\,|\psi'(\tau v_1+(1-\tau)v_2)|\right)$$

Young's inequality with ϵ states that for any positive real numbers $A,B\ge 0$:

$$A\cdot B \le \frac{A^p}{p}+\frac{B^q}{q}$$

Setting $A=\epsilon\,|\mathcal{K}_\varpi(\tau)|$ and $B=\epsilon^{-1}\,|\psi'(\tau v_1+(1-\tau)v_2)|$, we obtain from Young's Inequality:

$$(\epsilon\,|\mathcal{K}_\varpi(\tau)|)\cdot\left(\epsilon^{-1}\,|\psi'(\tau v_1+(1-\tau)v_2)|\right) \le \frac{\epsilon^p\,|\mathcal{K}_\varpi(\tau)|^p}{p}+\frac{\epsilon^{-q}\,|\psi'(\tau v_1+(1-\tau)v_2)|^q}{q}$$

Integrating the previous inequality over $[0,1]$ with respect to the variable τ:

$$\int_0^1 |\mathcal{K}_\varpi(\tau)| \cdot |\psi'(\tau v_1 + (1-\tau)v_2)|\, d\tau \leq \int_0^1 \left[\frac{\epsilon^p\, |\mathcal{K}_\varpi(\tau)|^p}{p} + \frac{\epsilon^{-q}\, |\psi'(\tau v_1 + (1-\tau)v_2)|^q}{q} \right] d\tau$$

Therefore

$$\int_0^1 |\mathcal{K}_\varpi(\tau)| \cdot |\psi'(\tau v_1 + (1-\tau)v_2)|\, d\tau \leq \frac{\epsilon^p}{p} \int_0^1 |\mathcal{K}_\varpi(\tau)|^p\, d\tau + \frac{\epsilon^{-q}}{q} \int_0^1 |\psi'(\tau v_1 + (1-\tau)v_2)|^q\, d\tau$$

We apply the modified definition of convexity to the integrand of the second component of the right-hand side:

$$|\psi'(\tau v_1 + (1-\tau)v_2)|^q \leq h^s(\tau)|\psi'(v_1)|^q + m(1-h(\tau))^s|\psi'(v_2)|^q$$

Then we obtain:

$$\int_0^1 |\psi'(\tau v_1 + (1-\tau)v_2)|^q\, d\tau \leq |\psi'(v_1)|^q \int_0^1 h^s(\tau)d\tau + m|\psi'(v_2)|^q \int_0^1 (1-h(\tau))^s d\tau$$

Substituting all of the above, we can obtain the desired dimension:

$$|\mathcal{A}_\varpi(\psi; v_1, v_2)| \leq \frac{\epsilon^p}{p} \int_0^1 |\mathcal{K}_\varpi(\tau)|^p\, d\tau + \frac{\epsilon^{-q}}{q} \left[|\psi'(v_1)|^q \int_0^1 h^s(\tau)d\tau + m|\psi'(v_2)|^q \int_0^1 (1-h(\tau))^s d\tau \right]$$

Thus we have proven the Theorem. □

Remark 20. *We would like to highlight the unifying nature and high degree of generality of the results established in Theorems 16, 17, and 18. Under specific configurations of the weight function $h(\tau)$, the scale parameters m, s, and the operational kernel $\varpi'(\tau)$, our inequalities elegantly reduce to several well-known fundamental results from the literature; in particular, we can explicitly derive the fundamental estimates of [7] and [25] as special cases.*

If we restrict our structural framework to the classical environment by setting the parameters $m = 1$ and $s = 1$, and choosing the identity function as the weight, i.e., $h(\tau) = \tau$, the modified (h, m)-convexity of the second type reduces identically to ordinary algebraic convexity. Furthermore, if we assume the standard linear kernel $\varpi'(t) \equiv 1$, the generalized integral operators $J^w_{v_1+}$ and $J^w_{v_2-}$ simplify to classical Riemann integrals. Under these conditions, the control kernel yields $|\mathcal{K}_\varpi(\tau)| = |2\tau - 1|$, and the error functional takes the form of the classic trapezoidal remainder:

$$\mathcal{A}_\varpi(\psi; v_1, v_2) = \frac{\psi(v_1) + \psi(v_2)}{2} - \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} \psi(\sigma)d\sigma$$

Consequently, Theorems 17 and 18 reduce precisely to Theorem 2.2 and Theorem 2.3 of [7], respectively.

On the other hand, maintaining the notion of classical convexity ($m = 1$, $s = 1$, and $h(\tau) = \tau$), but expanding the operational framework to fractional calculus via the Riemann-Liouville Integral, we set the kernel as $\varpi'(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $\alpha > 0$. In this scenario, the weighted integral *operators* $J^w_{v_1+}$ *and* $J^w_{v_2-}$ *reduce to the left- and right-hand fractional Riemann-Liouville integrals, denoted as* $\Gamma(\alpha)J^\alpha_{v_1+}$ *and* $\Gamma(\alpha)J^\alpha_{v_2-}$*. Thus, the control kernel becomes* $\mathcal{K}_\varpi(\tau) = \frac{\tau^\alpha-(1-\tau)^\alpha}{\Gamma(\alpha)}$*, and the remaining functional takes the form:*

$$\mathcal{A}_\varpi(\psi; v_1, v_2) = \frac{\psi(v_1)+\psi(v_2)}{2\Gamma(\alpha+1)} - \frac{J^\alpha_{v_1+}\psi(v_2) + J^\alpha_{v_2-}\psi(v_1)}{2(v_2-v_1)}$$

Under these exact fractional restrictions, our Theorem 17 reduces to Theorem 3 of [25].

Let us add that, in particular, Theorem 19 and Theorems 18 and 19 complete the results obtained in [7] and [25], respectively.

Remark 21. *The generality of Theorems 17, 18, and 19 allows us to encompass other highly influential variants of fractional integral inequalities. By systematically adjusting the structural parameters and the derivative of the kernel* $\varpi'(\tau)$*, our results converge to the exact formulations presented in other well-known works, as we shall see below.*

1) The work [15] focuses on Hermite-Hadamard-Fejer type inequalities using a symmetric weighted function $g(x)$ *for p-convex functions through Riemann-Liouville fractional integrals.*

Setting $m = 1, s = 1$ *and* $h(\tau) = \tau$ *and considering the nonlinear domain isometric transformation* $\psi(t) = f(t^{1/p})$ *over the interval* $[v_1, v_2] = [\mu^p, \nu^p]$*.*

To introduce the Fejér-type weight function, we define the operational derivative of the kernel under the presence of a positive, continuous, and symmetric mapping $\mathcal{W}$ *onto the point* $\frac{\mu^p+\nu^p}{2}$ *as follows:*

$$\varpi'(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\mathcal{W}\left(\tau\mu^p + (1-\tau)\nu^p\right)$$

Under these specific parameters, our general framework incorporates the modulating weight of the Fejér structure, and our main theorems perfectly replicate the fractional limits established for differentiable p-convex mappings in the cited work.

2) Wang et al. in [28] extended trapezoidal inequalities to the context of fractional calculus k using (h, m)*-convex and* (α, m)*-convex functions.*

By setting $s = 1$ *in our modified definition of* (h, m)*-convexity of the second type, the inequality coincides with the standard* (h, m)*-convex definition:* $\psi(\tau\xi + m(1-\tau)\varsigma) \le h(\tau)\psi(\xi) + m(1 - h(\tau))\psi(\varsigma)$*.*

To consider the k fractional integrals, we consider:

$$\varpi'(\tau) = \frac{\tau^{\frac{\alpha}{k}-1}}{k\,\Gamma_k(\alpha)}$$

where Γ_k *is the specialized k-Gamma function. Substituting this kernel into our identity, the weighted operators* $J^w_{v_1+}$ *and* $J^w_{v_2-}$ *become the left and right k fractional integral operators. Consequently, our estimates exactly match the trapezoidal limits derived in the work mentioned above.*

3) In the work [29], the authors derived fractional bounds specifically adapted to midpoint and special mean-type formulas for differentiable convex mappings.

If we fix $m = 1, s = 1$ and $h(\tau) = \tau$, we obtain classical coinvexity.

Defining the Riemann-Liouville kernel $\varpi'(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, we obtain the control kernel $\mathcal{K}_{\varpi}(\tau) = \frac{\tau^{\alpha}-(1-\tau)^{\alpha}}{\Gamma(\alpha)}$.

By evaluating the parameter settings and focusing on the specific identity where the limits of integration are partitioned at the midpoint $\tau = \frac{1}{2}$ (as stated in our Lemma 15), the error function produces the exact remainder of the midpoint. Therefore, Theorems 17 and 18 simplify to the fractional midpoint estimates and special mean applications presented in Theorems 2 and 3 of this work.

This demonstrates that the results presented in this article not only extend the limit of fractional integral inequalities to broader families through modified (h, m)-convexity of the second type, but also generalize and unify classical and fractional trapezoidal estimates within a general and unifying operational framework, using weighted integral operators.

3. Applications to Special Means

In this section, we show the practical value of our main results by establishing new inequalities for several well-known means. For simplicity and clarity, we restrict our generalized framework to classical convexity, evaluating the relationships presented below as direct byproducts of Theorems 17, 18, and 19.

We begin by recalling the formal definitions of the essential means for any pair of distinct positive real numbers $\xi, \varsigma > 0$ with $\xi < \varsigma$:

(1) **The Arithmetic Mean:**

$$A(\xi,\varsigma) = \frac{\xi+\varsigma}{2}$$

(2) **The Logarithmic Mean:**

$$L(\xi,\varsigma) = \frac{\varsigma-\xi}{\ln\varsigma - \ln\xi}$$

(3) **The Generalized Logarithmic (Generalized Power) Mean ($n \in \mathbb{R} \setminus \{-1, 0\}$):**

$$L_n(\xi,\varsigma) = \left[\frac{\varsigma^{n+1}-\xi^{n+1}}{(n+1)(\varsigma-\xi)}\right]^{\frac{1}{n}}$$

Considering these special means, under specific configurations of our main results, we can establish the following propositions:

Proposition 22. *Let $\xi, \varsigma \in \mathbb{R}^+$ with $\xi < \varsigma$ and $n \in \mathbb{Z}\setminus\{-1, 0, 1\}$. Then, the following inequality holds:*

$$|A(\xi^n,\varsigma^n) - L_n^n(\xi,\varsigma)| \le \frac{n(\varsigma-\xi)}{8}\left[|\xi|^{n-1} + |\varsigma|^{n-1}\right] = \frac{n(\varsigma-\xi)}{4}A(|\xi|^{n-1}, |\varsigma|^{n-1}). \qquad (12)$$

Proof. Consider the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(x) = x^n$. It is well known that $\psi(x)$ is a differentiable and convex function in the classical sense on any compact interval $[\xi, \varsigma] \subset (0, \infty)$, if we take $m = 1$, $s = 1$, and $h(\tau) = \tau$ within our general definition.

Now, using Lemma 15, which transforms our generalized operators $J^w_{\xi+}$ and $J^w_{\varsigma-}$ into classical Riemann integrals, under this configuration, the control kernel simplifies to $|\mathcal{K}_\varpi(\tau)| = |2\tau - 1|$, and the remainder functional becomes:

$$\mathcal{A}_\varpi(\psi; \xi, \varsigma) = \frac{\psi(\xi) + \psi(\varsigma)}{2} - \frac{1}{\varsigma - \xi} \int_\xi^\varsigma \psi(\sigma) d\sigma$$

Substituting $\psi(x) = x^n$ into the previous expression, we obtain:

$$\mathcal{A}_\varpi(\psi; \xi, \varsigma) = \frac{\xi^n + \varsigma^n}{2} - \frac{1}{\varsigma - \xi} \int_\xi^\varsigma \sigma^n d\sigma = A(\xi^n, \varsigma^n) - L_n^n(\xi, \varsigma)$$

Applying the absolute value property and evaluating Theorem 18 (in the context of the power-mean inequality with $q = 1$), the right-hand bound simplifies to:

$$|A(\xi^n, \varsigma^n) - L_n^n(\xi, \varsigma)| \le \frac{\varsigma - \xi}{2} \int_0^1 |2\tau - 1| \cdot |\psi'(\tau\xi + (1-\tau)\varsigma)| \, d\tau$$

Using the classical convexity of $|\psi'(\cdot)|$:

$$|\psi'(\tau\xi + (1-\tau)\varsigma)| \le \tau|\psi'(\xi)| + (1-\tau)|\psi'(\varsigma)| = n\left[\tau|\xi|^{n-1} + (1-\tau)|\varsigma|^{n-1}\right]$$

A simple calculation gives us:

$$\int_0^1 \tau|2\tau - 1| d\tau = \frac{1}{4}, \quad \text{and} \quad \int_0^1 (1-\tau)|2\tau - 1| d\tau = \frac{1}{4}$$

Combining these evaluations and factoring the constants, we directly recover:

$$|A(\xi^n, \varsigma^n) - L_n^n(\xi, \varsigma)| \le \frac{\varsigma - \xi}{2} \cdot n\left[\frac{1}{4}|\xi|^{n-1} + \frac{1}{4}|\varsigma|^{n-1}\right]$$

Which leads us to the equation (12). This completes the proof. □

Proposition 23. *Let $\xi, \varsigma \in \mathbb{R}^+$ with $\xi < \varsigma$. Then, the following inequality holds:*

$$\left|A\left(\frac{1}{\xi}, \frac{1}{\varsigma}\right) - L^{-1}(\xi, \varsigma)\right| \le \frac{\varsigma - \xi}{4}\left[\frac{1}{\xi^2} + \frac{1}{\varsigma^2}\right] \tag{13}$$

Proof. Consider the function $\psi : (0, \infty) \to \mathbb{R}$ defined by $\psi(x) = \frac{1}{x}$. This function is strictly convex on any positive interval, and its derivative is $\psi'(x) = -\frac{1}{x^2}$.

Evaluating our functional of the remainder for ψ, we have:

$$\mathcal{A}_\varpi(\psi; \xi, \varsigma) = \frac{\frac{1}{\xi} + \frac{1}{\varsigma}}{2} - \frac{1}{\varsigma - \xi} \int_\xi^\varsigma \frac{1}{\sigma} d\sigma = A\left(\frac{1}{\xi}, \frac{1}{\varsigma}\right) - \frac{\ln\varsigma - \ln\xi}{\varsigma - \xi}$$

Recognizing the inverse of the logarithmic mean $L^{-1}(\xi, \varsigma) = \frac{\ln\varsigma - \ln\xi}{\varsigma - \xi}$, the error on the left becomes in:

$$\mathcal{A}_{\varpi}(\psi;\xi,\varsigma) = A\left(\frac{1}{\xi},\frac{1}{\varsigma}\right) - L^{-1}(\xi,\varsigma)$$

Applying Theorem 17 (using Hölder's inequality) or Theorem 18 with $q = 1$, and using the explicit convexity of the modular derivative $|\psi'(x)| = \frac{1}{x^2}$, we have the weight integrals:

$$\left|A\left(\frac{1}{\xi},\frac{1}{\varsigma}\right) - L^{-1}(\xi,\varsigma)\right|$$

Substituting the calculated values for each integral and replacing the derivatives, we obtain:

$$A(1/\xi,1/\varsigma) - L^{-1}(\xi,\varsigma) \leq \frac{\varsigma-\xi}{2}\left[\frac{1}{4\xi^2}+\frac{1}{4\varsigma^2}\right] = \frac{\varsigma-\xi}{4}A\left(\frac{1}{\xi^2},\frac{1}{\varsigma^2}\right)$$

This agrees with the statement in (13). The proof is complete. □

The established propositions show that our generalized trapezoidal fractional estimations possess direct, immediate projections into classical mathematical structures, providing new and sharp error bounds for standard numerical means.

4. Conclusions

In this article, we have successfully established a generalized framework for fractional trapezoidal inequalities. By introducing an abstract weight function $\varpi'(\tau)$, we avoid the traditional methodological fragmentation that required proving integral inequalities independently for each specific fractional operator.

The main contributions are summarized below:

- A new fundamental integral identity was proven, bridging the gap between fractional integral operators, generalized integral operators, and differentiable functions.
- By combining this identity with the classical Hölder, power-mean, and Young inequalities, we derived several precise upper bounds for the trapezoidal error under the assumption that the first derivative belongs to the class of modified convex (h,m)-functions of the second kind.
- The robustness and generality of our results were verified by showing that several well-known results in the literature are special cases of ours.
- Practical applications to numerical means (generalized arithmetic, logarithmic, and power means) yielded new and precise error estimates, confirming that our results have immediate applicability in analytical and computational contexts.

Regarding future lines of research, we should point out that the generalized framework used opens several interesting avenues. First, our fundamental identity can be naturally extended to study integral inequalities of the midpoint (Ostrowski) and Simpson type for the same modified convexity class. Second, the generalized kernel $\varpi'(\tau)$ can be tested in fractional spaces with non-singular kernels, such as the Atangana-Baleanu or Caputo-Fabrizio operators. Finally, exploring the applications of these fractional bounds within the framework of numerical optimization, fractional differential equations, and the estimation of solutions for operational algorithms represents fertile ground for further research.

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