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Unconditional and Conditional Two-Power-Parameter Cyclic Inequalities

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Abstract. In this paper, we introduce unconditional and conditional cyclic inequalities involving two power parameters. By analyzing the non-negativity of difference functions, we generalize classical bounds for cyclic rational sums and establish new analytical results. We further determine the precise parameter ranges for which these global inequalities hold and discuss the conditions for the equality. Finally, several special cases of interest are highlighted.

1. Introduction

1.1. **Context.** An inequality involving a set of variables is termed cyclic if it remains invariant under a forward cyclic permutation. For three variables a, b, c, this is represented as $(a, b, c) \mapsto (b, c, a)$; that is, the variable a becomes b, the variable b becomes c, and the variable c wraps around to become a. In this three-variable system, the three valid cyclic permutations are given by

$$(a, b, c) \mapsto (b, c, a) \mapsto (c, a, b).$$

While every symmetric inequality is cyclic, the converse does not hold. This directional symmetry admits a broader class of algebraic relations in which the ordering of variables determines precise bounds.

Several classical results rely on cyclic structures to establish fundamental relationships among variables. Applications are found in optimization, real analysis, and contest mathematics. For three variables $a, b, c > 0$, representative examples include those below.

- **The Nesbitt inequality:** Serving as the canonical cyclic rational inequality, it asserts that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

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- **Cyclic Cauchy-Schwarz applications:** Derived from the Titu lemma, these yield sharp asymmetric inequalities, such as
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c.$$
- **Fourth-degree cyclic polynomial bounds:** Requiring targeted factorization or applications of the arithmetic mean-geometric mean inequality, a prominent fourth-degree bound states that
$$a^3b+b^3c+c^3a\geq abc(a+b+c).$$
- **Cyclic sum of cubes:** Demonstrating that a cyclic sum of mixed-degree terms can be bounded by a symmetric sum of pure powers, this classical result shows that
$$a^3+b^3+c^3\geq a^2b+b^2c+c^2a.$$

Historically, much of the literature on cyclic inequalities has been confined to specific discrete domains or isolated single-parameter special cases (see, e.g., [1,2,5–12]; see also [3,4] for recent developments).

1.2. **Motivation: unconditional and conditional cyclic inequalities.** This paper extends classical frameworks by introducing unconditional and conditional cyclic inequalities involving two power parameters $p,q\in\mathbb{R}$, and three variables $a,b,c>0$. The exposition is structured into two principal parts.

The first part addresses the unconditional case. To illustrate this regime, our first theorem establishes that, for any $p,q\in\mathbb{R}$, the following inequality holds:
$$\frac{2a^{2p+q}+a^q(b^{2p}+c^{2p})}{b^q+c^q}+\frac{2b^{2p+q}+b^q(c^{2p}+a^{2p})}{c^q+a^q}+\frac{2c^{2p+q}+c^q(a^{2p}+b^{2p})}{a^q+b^q}$$
$$\geq\frac{b^{2p+q}+c^{2p+q}}{b^q+c^q}+\frac{c^{2p+q}+a^{2p+q}}{c^q+a^q}+\frac{a^{2p+q}+b^{2p+q}}{a^q+b^q}+a^{2p}+b^{2p}+c^{2p}.$$
We emphasize that this bound is unconditional, meaning it imposes no restrictions on the signs or magnitudes of p and q. The necessity of employing two power parameters in this setting may be debatable. Indeed, note that, under the reparameterization $x=a^p$, $y=b^p$, $z=c^p$, and $r=q/p$, this inequality can be rewritten as follows:
$$\frac{2x^{r+2}+x^r(y^2+z^2)}{y^r+z^r}+\frac{2y^{r+2}+y^r(z^2+x^2)}{z^r+x^r}+\frac{2z^{r+2}+z^r(x^2+y^2)}{x^r+y^r}$$
$$\geq\frac{y^{r+2}+z^{r+2}}{y^r+z^r}+\frac{z^{r+2}+x^{r+2}}{z^r+x^r}+\frac{x^{r+2}+y^{r+2}}{x^r+y^r}+x^2+y^2+z^2.$$
It thus deals with only a single power parameter, r. However, this alternate formulation restricts the parameter space by implicitly requiring $p\neq 0$. In addition, it obscures the distinct roles played by p (which governs the base power scaling) and q (which acts as a weighting exponent in the denominator sums). To preserve full generality, we choose to retain the two independent power parameters p and q throughout our framework. The proofs are based on the analysis

of cyclic difference functions and the sign properties of power expressions. By establishing a variety of general unconditional bounds of this type, we provide a unified mechanism capable of recovering classical cyclic inequalities while also generating new ones.

The second part is devoted to the conditional case. To illustrate this regime, our first theorem presents the two inequalities below.

- For $pq > 0$ (i.e., p and q share the same sign):

$$\frac{a^p}{b^q+c^q}+\frac{b^p}{c^q+a^q}+\frac{c^p}{a^q+b^q}\geq\frac{1}{2}\left(\frac{b^p+c^p}{b^q+c^q}+\frac{c^p+a^p}{c^q+a^q}+\frac{a^p+b^p}{a^q+b^q}\right).$$

- For $pq < 0$ (i.e., p and q have opposite signs):

$$\frac{a^p}{b^q+c^q}+\frac{b^p}{c^q+a^q}+\frac{c^p}{a^q+b^q}\leq\frac{1}{2}\left(\frac{b^p+c^p}{b^q+c^q}+\frac{c^p+a^p}{c^q+a^q}+\frac{a^p+b^p}{a^q+b^q}\right).$$

We emphasize that this bound is conditional, meaning that the direction of the inequality is strictly governed by the sign of the product pq. Unlike the unconditional results, the parameter space in this regime is partitioned, requiring explicit conditions to dictate whether a lower or upper bound is attained. As in the unconditional case, the proofs rely on analyzing cyclic difference functions and evaluating the sign properties of power expressions. Once again, by establishing a variety of general conditional bounds of this type, we provide a unified mechanism capable of recovering classical cyclic inequalities while also generating new ones.

In a sense, this work bridges the gap between classical cyclic inequality techniques and generalized parameterized frameworks through a unified collection of theorems. Furthermore, the equality conditions for every result are discussed, typically holding for $a = b = c$, or reducing to identities for trivial values of p and q.

1.3. **Organization.** The remainder of this paper is organized as follows: Section 2 outlines the primary notation and conventions. Section 3 presents the main results concerning unconditional two-parameter inequalities, while Section 4 investigates the conditional cases. Finally, Section 5 offers concluding remarks and highlights potential directions for future research.

2. Notation and conventions

In this paper, the analytical inequality analysis relies on a framework of variables, operators, and difference functions to evaluate algebraic bounds.

Regarding the variables and parameters, $a, b, c > 0$ represent the main variables, $p, q \in \mathbb{R}$ denote the primary recurrent parameters, and $k, r \in \mathbb{R}$ serve as secondary parameters.

The cyclic summation may employ the operator $\sum_{\text{cyc}}$ to denote a forward cyclic permutation sum. For a given function $f(a, b, c)$ depending on the three variables, it is defined as

$$\sum_{\text{cyc}} f(a,b,c) = f(a,b,c) + f(b,c,a) + f(c,a,b).$$

For example, the Nesbitt inequality can be written compactly as

$$\sum_{\text{cyc}} \frac{a}{b+c} \geq \frac{3}{2}$$

and the fundamental inequality $a^2+b^2+c^2 \geq ab+bc+ca$ can be expressed as

$$\sum_{\text{cyc}} a^2 \geq \sum_{\text{cyc}} ab.$$

The operator $\sum_{\text{cyc}}$ is obviously linear. It also satisfies the following index-shift invariance:

$$\sum_{\text{cyc}} f(a,b,c) = \sum_{\text{cyc}} f(b,c,a) = \sum_{\text{cyc}} f(c,a,b).$$

To maintain conciseness, we adopt the cyclic sum notation $\sum_{\text{cyc}}$ throughout the proofs, whereas the theorem statements are written in explicitly expanded form.

Additionally, we introduce the difference function $\mathcal{D}_\theta(a,b,c)$, formed by subtracting one side of a target inequality from the other, up to an appropriate positive scaling factor. Here, θ denotes the parameter vector. This framework reduces each proof to establishing the sign of $\mathcal{D}_\theta(a,b,c)$.

In each theorem, the phrase "equality holds for $a=b=c$" indicates a sufficient condition for equality. For parametric boundary cases (e.g., $q=0$, $p=0$, $p=q$, or $p+q=0$), the cyclic difference function often satisfies $\mathcal{D}_\theta(a,b,c) \equiv 0$, transforming the inequality into an identity valid with no restrictions on a,b,c.

As a final convention, relations and inequalities satisfied for some variables are understood to hold for all cyclic permutations $(a,b,c) \mapsto (b,c,a) \mapsto (c,a,b)$ without explicit repetition.

3. Unconditional two-power-parameter cyclic inequalities

This section presents nine unconditional two-power-parameter cyclic inequalities of varying types. Each proof immediately follows its corresponding statement, and remarks are provided to highlight special cases of interest.

Theorem 3.1. *Let $a,b,c>0$ and $p,q \in \mathbb{R}$. The following inequality unconditionally holds:*

$$\frac{2a^{2p+q}+a^q(b^{2p}+c^{2p})}{b^q+c^q} + \frac{2b^{2p+q}+b^q(c^{2p}+a^{2p})}{c^q+a^q} + \frac{2c^{2p+q}+c^q(a^{2p}+b^{2p})}{a^q+b^q}$$
$$\geq \frac{b^{2p+q}+c^{2p+q}}{b^q+c^q} + \frac{c^{2p+q}+a^{2p+q}}{c^q+a^q} + \frac{a^{2p+q}+b^{2p+q}}{a^q+b^q} + a^{2p}+b^{2p}+c^{2p}.$$

The equality holds for $a=b=c$ (when $q=0$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^{2p+q}+a^q(b^{2p}+c^{2p})}{b^q+c^q} - \sum_{\text{cyc}} \frac{b^{2p+q}+c^{2p+q}}{b^q+c^q} - \sum_{\text{cyc}} a^{2p}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{2a^{2p+q}+a^q(b^{2p}+c^{2p})-(b^{2p+q}+c^{2p+q})-a^{2p}(b^q+c^q)}{b^q+c^q}.$$

Noting that

$$\begin{aligned}
&2a^{2p+q}+a^q(b^{2p}+c^{2p})-(b^{2p+q}+c^{2p+q})-a^{2p}(b^q+c^q)\\
&=2a^{2p+q}+a^qb^{2p}+a^qc^{2p}-b^{2p+q}-c^{2p+q}-a^{2p}b^q-a^{2p}c^q\\
&=(a^{2p+q}-a^{2p}b^q+a^qb^{2p}-b^{2p+q})+(a^{2p+q}-a^{2p}c^q+a^qc^{2p}-c^{2p+q})\\
&=\Big(a^{2p}(a^q-b^q)+b^{2p}(a^q-b^q)\Big)+\Big(a^{2p}(a^q-c^q)+c^{2p}(a^q-c^q)\Big)\\
&=(a^{2p}+b^{2p})(a^q-b^q)+(a^{2p}+c^{2p})(a^q-c^q),
\end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{(a^{2p}+b^{2p})(a^q-b^q)}{b^q+c^q}+\sum_{\text{cyc}}\frac{(a^{2p}+c^{2p})(a^q-c^q)}{b^q+c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{(a^{2p}+c^{2p})(a^q-c^q)}{b^q+c^q}\longrightarrow\frac{(b^{2p}+a^{2p})(b^q-a^q)}{c^q+a^q}=-\frac{(a^{2p}+b^{2p})(a^q-b^q)}{c^q+a^q}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_{p,q}(a,b,c)&=\sum_{\text{cyc}}\frac{(a^{2p}+b^{2p})(a^q-b^q)}{b^q+c^q}-\sum_{\text{cyc}}\frac{(a^{2p}+b^{2p})(a^q-b^q)}{c^q+a^q}\\
&=\sum_{\text{cyc}}(a^{2p}+b^{2p})(a^q-b^q)\left(\frac{1}{b^q+c^q}-\frac{1}{c^q+a^q}\right)\\
&=\sum_{\text{cyc}}(a^{2p}+b^{2p})(a^q-b^q)\frac{(c^q+a^q)-(b^q+c^q)}{(a^q+c^q)(b^q+c^q)}\\
&=\sum_{\text{cyc}}\frac{(a^{2p}+b^{2p})(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)}.
\end{aligned}$$

Because $a,b,c>0$, we have $a^{2p}+b^{2p}>0$ and $(a^q+c^q)(b^q+c^q)>0$ (and cyclically). Moreover, we have $(a^q-b^q)^2\geq 0$ for any $q\in\mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}}\frac{(a^{2p}+b^{2p})(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)}\geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c)\geq 0$, i.e.,

$$\begin{aligned}
&\frac{2a^{2p+q}+a^q(b^{2p}+c^{2p})}{b^q+c^q}+\frac{2b^{2p+q}+b^q(c^{2p}+a^{2p})}{c^q+a^q}+\frac{2c^{2p+q}+c^q(a^{2p}+b^{2p})}{a^q+b^q}\\
&\geq\frac{b^{2p+q}+c^{2p+q}}{b^q+c^q}+\frac{c^{2p+q}+a^{2p+q}}{c^q+a^q}+\frac{a^{2p+q}+b^{2p+q}}{a^q+b^q}+a^{2p}+b^{2p}+c^{2p}.
\end{aligned}$$

The equality holds for $a=b=c$, yielding a common value of $6a^{2p}$. □

Remark 3.1. *When $p = 0$, the inequality in Theorem 3.1 simplifies to*

$$\frac{a^q}{b^q + c^q} + \frac{b^q}{c^q + a^q} + \frac{c^q}{a^q + b^q} \geq \frac{3}{2},$$

which is exactly the Nesbitt inequality applied to the variables $x = a^q$, $y = b^q$, and $z = c^q$. This highlights that Theorem 3.1 serves as a two-parameter polynomial power generalization of classical cyclic rational inequalities.

Theorem 3.2. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The following inequality unconditionally holds:*

$$\begin{aligned}
&\frac{a^{2p+q}(b^p + c^p) + a^{p+q}(b^{2p} + c^{2p})}{b^q + c^q} + \frac{b^{2p+q}(c^p + a^p) + b^{p+q}(c^{2p} + a^{2p})}{c^q + a^q} \\
&+ \frac{c^{2p+q}(a^p + b^p) + c^{p+q}(a^{2p} + b^{2p})}{a^q + b^q} \\
&\geq \frac{a^{2p}(b^{p+q} + c^{p+q}) + a^p(b^{2p+q} + c^{2p+q})}{b^q + c^q} + \frac{b^{2p}(c^{p+q} + a^{p+q}) + b^p(c^{2p+q} + a^{2p+q})}{c^q + a^q} \\
&+ \frac{c^{2p}(a^{p+q} + b^{p+q}) + c^p(a^{2p+q} + b^{2p+q})}{a^q + b^q}.
\end{aligned}$$

The equality holds for $a = b = c$ (when $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{a^{2p+q}(b^p + c^p) + a^{p+q}(b^{2p} + c^{2p})}{b^q + c^q} - \sum_{\text{cyc}} \frac{a^{2p}(b^{p+q} + c^{p+q}) + a^p(b^{2p+q} + c^{2p+q})}{b^q + c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{a^{2p+q}(b^p + c^p) + a^{p+q}(b^{2p} + c^{2p}) - a^{2p}(b^{p+q} + c^{p+q}) - a^p(b^{2p+q} + c^{2p+q})}{b^q + c^q}.$$

Observing that

$$\begin{aligned}
&a^{2p+q}(b^p + c^p) + a^{p+q}(b^{2p} + c^{2p}) - a^{2p}(b^{p+q} + c^{p+q}) - a^p(b^{2p+q} + c^{2p+q}) \\
&= (a^{2p+q}b^p + a^{p+q}b^{2p} - a^{2p}b^{p+q} - a^p b^{2p+q}) \\
&+ (a^{2p+q}c^p + a^{p+q}c^{2p} - a^{2p}c^{p+q} - a^p c^{2p+q}) \\
&= a^p b^p(a^{p+q} + a^q b^p - a^p b^q - b^{p+q}) + a^p c^p(a^{p+q} + a^q c^p - a^p c^q - c^{p+q}) \\
&= a^p b^p\,(a^q(a^p + b^p) - b^q(a^p + b^p)) + a^p c^p\,(a^q(a^p + c^p) - c^q(a^p + c^p)) \\
&= a^p b^p(a^p + b^p)(a^q - b^q) + a^p c^p(a^p + c^p)(a^q - c^q),
\end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{a^p b^p(a^p + b^p)(a^q - b^q)}{b^q + c^q} + \sum_{\text{cyc}} \frac{a^p c^p(a^p + c^p)(a^q - c^q)}{b^q + c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^p c^p (a^p + c^p)(a^q - c^q)}{b^q + c^q} \longrightarrow \frac{b^p a^p (b^p + a^p)(b^q - a^q)}{c^q + a^q} = -\frac{a^p b^p (a^p + b^p)(a^q - b^q)}{c^q + a^q}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^p b^p (a^p + b^p)(a^q - b^q)}{b^q + c^q} - \sum_{\text{cyc}} \frac{a^p b^p (a^p + b^p)(a^q - b^q)}{c^q + a^q} \\
&= \sum_{\text{cyc}} a^p b^p (a^p + b^p)(a^q - b^q) \left(\frac{1}{b^q + c^q} - \frac{1}{c^q + a^q} \right) \\
&= \sum_{\text{cyc}} a^p b^p (a^p + b^p)(a^q - b^q) \frac{(c^q + a^q) - (b^q + c^q)}{(a^q + c^q)(b^q + c^q)} \\
&= \sum_{\text{cyc}} \frac{a^p b^p (a^p + b^p)(a^q - b^q)^2}{(a^q + c^q)(b^q + c^q)}.
\end{aligned}$$

Because $a, b, c > 0$, we have $a^p b^p > 0$, $a^p + b^p > 0$, and $(a^q + c^q)(b^q + c^q) > 0$ (and cyclically). Moreover, we have $(a^q - b^q)^2 \geq 0$ for any $q \in \mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^p b^p (a^p + b^p)(a^q - b^q)^2}{(a^q + c^q)(b^q + c^q)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\begin{aligned}
&\frac{a^{2p+q}(b^p + c^p) + a^{p+q}(b^{2p} + c^{2p})}{b^q + c^q} + \frac{b^{2p+q}(c^p + a^p) + b^{p+q}(c^{2p} + a^{2p})}{c^q + a^q} \\
&+ \frac{c^{2p+q}(a^p + b^p) + c^{p+q}(a^{2p} + b^{2p})}{a^q + b^q} \\
&\geq \frac{a^{2p}(b^{p+q} + c^{p+q}) + a^p(b^{2p+q} + c^{2p+q})}{b^q + c^q} + \frac{b^{2p}(c^{p+q} + a^{p+q}) + b^p(c^{2p+q} + a^{2p+q})}{c^q + a^q} \\
&+ \frac{c^{2p}(a^{p+q} + b^{p+q}) + c^p(a^{2p+q} + b^{2p+q})}{a^q + b^q}.
\end{aligned}$$

The equality holds for $a = b = c$, yielding a common value of $6a^{3p}$. □

Remark 3.2. *When $p = 1$ and $q = -1$, the inequality in Theorem 3.2 simplifies to*

$$\frac{b^2 + c^2}{a(b + c)} + \frac{c^2 + a^2}{b(c + a)} + \frac{a^2 + b^2}{c(a + b)} \geq 2 \left(\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \right),$$

which provides a new reciprocal extension of Nesbitt-type inequalities.

Theorem 3.3. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The following inequality unconditionally holds:*

$$\begin{aligned}
&\frac{2a^{2p+q} + a^{p+q}(b^p + c^p) + a^q(b^{2p} + c^{2p})}{b^q + c^q} + \frac{2b^{2p+q} + b^{p+q}(c^p + a^p) + b^q(c^{2p} + a^{2p})}{c^q + a^q} \\
&+ \frac{2c^{2p+q} + c^{p+q}(a^p + b^p) + c^q(a^{2p} + b^{2p})}{a^q + b^q} \\
&\geq \frac{b^{2p+q} + c^{2p+q} + a^p(b^{p+q} + c^{p+q})}{b^q + c^q} + \frac{c^{2p+q} + a^{2p+q} + b^p(c^{p+q} + a^{p+q})}{c^q + a^q} \\
&+ \frac{a^{2p+q} + b^{2p+q} + c^p(a^{p+q} + b^{p+q})}{a^q + b^q} + a^{2p} + b^{2p} + c^{2p}.
\end{aligned}$$

The equality holds for $a = b = c$ (when $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\begin{aligned}
\mathcal{D}_{p,q}(a, b, c) = &\sum_{\text{cyc}} \frac{2a^{2p+q} + a^{p+q}(b^p + c^p) + a^q(b^{2p} + c^{2p})}{b^q + c^q} \\
&- \sum_{\text{cyc}} \frac{b^{2p+q} + c^{2p+q} + a^p(b^{p+q} + c^{p+q})}{b^q + c^q} - \sum_{\text{cyc}} a^{2p}.
\end{aligned}$$

Combining the terms into the sums over a common denominator and splitting the numerator across two lines for compactness, we get

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{\begin{array}{c} 2a^{2p+q} + a^{p+q}(b^p + c^p) + a^q(b^{2p} + c^{2p}) \\ - \left(b^{2p+q} + c^{2p+q} + a^p(b^{p+q} + c^{p+q})\right) - a^{2p}(b^q + c^q) \end{array}}{b^q + c^q}.$$

Noting that

$$\begin{aligned}
&2a^{2p+q} + a^{p+q}(b^p + c^p) + a^q(b^{2p} + c^{2p}) - \left(b^{2p+q} + c^{2p+q} + a^p(b^{p+q} + c^{p+q})\right) \\
&- a^{2p}(b^q + c^q) \\
&= 2a^{2p+q} + a^{p+q}b^p + a^{p+q}c^p + a^q b^{2p} + a^q c^{2p} - b^{2p+q} - c^{2p+q} - a^p b^{p+q} \\
&- a^p c^{p+q} - a^{2p}b^q - a^{2p}c^q \\
&= (a^{2p+q} - a^{2p}b^q + a^{p+q}b^p - a^p b^{p+q} + a^q b^{2p} - b^{2p+q}) \\
&+ (a^{2p+q} - a^{2p}c^q + a^{p+q}c^p - a^p c^{p+q} + a^q c^{2p} - c^{2p+q}) \\
&= \left(a^{2p}(a^q - b^q) + a^p b^p(a^q - b^q) + b^{2p}(a^q - b^q)\right) \\
&+ \left(a^{2p}(a^q - c^q) + a^p c^p(a^q - c^q) + c^{2p}(a^q - c^q)\right) \\
&= (a^{2p} + a^p b^p + b^{2p})(a^q - b^q) + (a^{2p} + a^p c^p + c^{2p})(a^q - c^q),
\end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{(a^{2p} + a^p b^p + b^{2p})(a^q - b^q)}{b^q + c^q} + \sum_{\text{cyc}} \frac{(a^{2p} + a^p c^p + c^{2p})(a^q - c^q)}{b^q + c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{(a^{2p}+a^pc^p+c^{2p})(a^q-c^q)}{b^q+c^q} \longrightarrow \frac{(b^{2p}+b^pa^p+a^{2p})(b^q-a^q)}{c^q+a^q}$$
$$= -\frac{(a^{2p}+a^pb^p+b^{2p})(a^q-b^q)}{c^q+a^q}.$$

Therefore, we have

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{(a^{2p}+a^pb^p+b^{2p})(a^q-b^q)}{b^q+c^q} - \sum_{\text{cyc}} \frac{(a^{2p}+a^pb^p+b^{2p})(a^q-b^q)}{c^q+a^q}$$
$$= \sum_{\text{cyc}} (a^{2p}+a^pb^p+b^{2p})(a^q-b^q)\left(\frac{1}{b^q+c^q} - \frac{1}{c^q+a^q}\right)$$
$$= \sum_{\text{cyc}} (a^{2p}+a^pb^p+b^{2p})(a^q-b^q)\frac{(c^q+a^q)-(b^q+c^q)}{(a^q+c^q)(b^q+c^q)}$$
$$= \sum_{\text{cyc}} \frac{(a^{2p}+a^pb^p+b^{2p})(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)}.$$

Because $a,b,c>0$, we have $a^{2p}+a^pb^p+b^{2p}>0$ and $(a^q+c^q)(b^q+c^q)>0$ (and cyclically). Moreover, we have $(a^q-b^q)^2\geq 0$ for any $q\in\mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^{2p}+a^pb^p+b^{2p})(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c)\geq 0$, i.e.,

$$\frac{2a^{2p+q}+a^{p+q}(b^p+c^p)+a^q(b^{2p}+c^{2p})}{b^q+c^q} + \frac{2b^{2p+q}+b^{p+q}(c^p+a^p)+b^q(c^{2p}+a^{2p})}{c^q+a^q}$$
$$+ \frac{2c^{2p+q}+c^{p+q}(a^p+b^p)+c^q(a^{2p}+b^{2p})}{a^q+b^q}$$
$$\geq \frac{b^{2p+q}+c^{2p+q}+a^p(b^{p+q}+c^{p+q})}{b^q+c^q} + \frac{c^{2p+q}+a^{2p+q}+b^p(c^{p+q}+a^{p+q})}{c^q+a^q}$$
$$+ \frac{a^{2p+q}+b^{2p+q}+c^p(a^{p+q}+b^{p+q})}{a^q+b^q} + a^{2p}+b^{2p}+c^{2p}.$$

The equality holds for $a=b=c$, yielding a common value of $9a^{2p}$. □

Remark 3.3. *When $p=1$ and $q=-1$, the inequality in Theorem 3.3 simplifies to*

$$\frac{b^2+c^2}{a^2(b+c)} + \frac{c^2+a^2}{b^2(c+a)} + \frac{a^2+b^2}{c^2(a+b)} \geq \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}.$$

Theorem 3.4. *Let $a,b,c>0$ and $p,q\in\mathbb{R}$. The following inequality unconditionally holds:*

$$\frac{a^pb^p(a^{2q}+b^{2q})+2(ab)^{p+q}}{a^p+b^p} + \frac{b^pc^p(b^{2q}+c^{2q})+2(bc)^{p+q}}{b^p+c^p}$$
$$+ \frac{c^pa^p(c^{2q}+a^{2q})+2(ca)^{p+q}}{c^p+a^p} \leq 2(a^{p+2q}+b^{p+2q}+c^{p+2q}).$$

The equality holds for $a=b=c$ (when $p+q=0$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = 2(a^{p+2q}+b^{p+2q}+c^{p+2q}) - \sum_{\text{cyc}} \frac{a^p b^p (a^{2q}+b^{2q}) + 2(ab)^{p+q}}{a^p+b^p}.$$

Observing that $2(a^{p+2q}+b^{p+2q}+c^{p+2q}) = \sum_{\text{cyc}}(a^{p+2q}+b^{p+2q})$ and combining the terms into the sums over a common denominator, we derive

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}}(a^{p+2q}+b^{p+2q}) - \sum_{\text{cyc}} \frac{a^p b^p (a^{2q}+b^{2q}) + 2(ab)^{p+q}}{a^p+b^p} \\ &= \sum_{\text{cyc}} \frac{(a^{p+2q}+b^{p+2q})(a^p+b^p) - a^p b^p (a^{2q}+b^{2q}) - 2(ab)^{p+q}}{a^p+b^p}.\end{aligned}$$

Noting that

$$\begin{aligned}&(a^{p+2q}+b^{p+2q})(a^p+b^p) - a^p b^p (a^{2q}+b^{2q}) - 2(ab)^{p+q} \\ &= (a^{2p+2q} + a^{p+2q}b^p + a^p b^{p+2q} + b^{2p+2q}) - (a^{p+2q}b^p + a^p b^{p+2q}) - 2a^{p+q}b^{p+q} \\ &= a^{2p+2q} - 2a^{p+q}b^{p+q} + b^{2p+2q} \\ &= (a^{p+q}-b^{p+q})^2,\end{aligned}$$

we have

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{(a^{p+q}-b^{p+q})^2}{a^p+b^p}.$$

Because $a,b,c>0$, we have $a^p+b^p>0$ (and cyclically). Moreover, we have $(a^{p+q}-b^{p+q})^2 \geq 0$ for any $p,q \in \mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^{p+q}-b^{p+q})^2}{a^p+b^p} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\begin{aligned}&\frac{a^p b^p (a^{2q}+b^{2q}) + 2(ab)^{p+q}}{a^p+b^p} + \frac{b^p c^p (b^{2q}+c^{2q}) + 2(bc)^{p+q}}{b^p+c^p} \\ &+ \frac{c^p a^p (c^{2q}+a^{2q}) + 2(ca)^{p+q}}{c^p+a^p} \leq 2(a^{p+2q}+b^{p+2q}+c^{p+2q}).\end{aligned}$$

The equality holds for $a=b=c$, yielding a common value of $6a^{p+2q}$. □

Remark 3.4. *When $p=1$ and $q=0$, the inequality in Theorem 3.4 simplifies to*

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \leq \frac{a+b+c}{2},$$

which is the well-known harmonic mean-arithmetic mean cyclic inequality.

Furthermore, when $p=1$ and $q=1$, the inequality simplifies to

$$ab(a+b) + bc(b+c) + ca(c+a) \leq 2(a^3+b^3+c^3),$$

which is a classical inequality derived from $(a+b)(a-b)^2 \geq 0$.

Theorem 3.5. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The following inequality unconditionally holds:*

$$\frac{(a^{2p}+b^{2p})(a^{2q}+b^{2q})}{2(a^{p+q}+b^{p+q})} + \frac{(b^{2p}+c^{2p})(b^{2q}+c^{2q})}{2(b^{p+q}+c^{p+q})} + \frac{(c^{2p}+a^{2p})(c^{2q}+a^{2q})}{2(c^{p+q}+a^{p+q})} \geq a^{p+q}+b^{p+q}+c^{p+q}.$$

The equality holds for $a = b = c$ (when $p = q$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{(a^{2p}+b^{2p})(a^{2q}+b^{2q})}{2(a^{p+q}+b^{p+q})} - (a^{p+q}+b^{p+q}+c^{p+q}).$$

Observing that $a^{p+q}+b^{p+q}+c^{p+q} = \sum_{\text{cyc}}(a^{p+q}+b^{p+q})/2$ and combining the terms into the sums over a common denominator, we can write

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{(a^{2p}+b^{2p})(a^{2q}+b^{2q})}{2(a^{p+q}+b^{p+q})} - \sum_{\text{cyc}} \frac{a^{p+q}+b^{p+q}}{2} \\ &= \sum_{\text{cyc}} \frac{(a^{2p}+b^{2p})(a^{2q}+b^{2q}) - (a^{p+q}+b^{p+q})^2}{2(a^{p+q}+b^{p+q})}.\end{aligned}$$

Noting that

$$\begin{aligned}&(a^{2p}+b^{2p})(a^{2q}+b^{2q}) - (a^{p+q}+b^{p+q})^2 \\ &= (a^{2p+2q}+a^{2p}b^{2q}+b^{2p}a^{2q}+b^{2p+2q}) - (a^{2p+2q}+2a^{p+q}b^{p+q}+b^{2p+2q}) \\ &= a^{2p}b^{2q} - 2a^{p+q}b^{p+q} + b^{2p}a^{2q} \\ &= (a^p b^q - b^p a^q)^2,\end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{(a^p b^q - b^p a^q)^2}{2(a^{p+q}+b^{p+q})}.$$

Because $a, b, c > 0$, we have $2(a^{p+q}+b^{p+q}) > 0$ (and cyclically). Moreover, we have $(a^p b^q - b^p a^q)^2 \geq 0$ for any $p, q \in \mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p b^q - b^p a^q)^2}{2(a^{p+q}+b^{p+q})} \geq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \geq 0$, i.e.,

$$\frac{(a^{2p}+b^{2p})(a^{2q}+b^{2q})}{2(a^{p+q}+b^{p+q})} + \frac{(b^{2p}+c^{2p})(b^{2q}+c^{2q})}{2(b^{p+q}+c^{p+q})} + \frac{(c^{2p}+a^{2p})(c^{2q}+a^{2q})}{2(c^{p+q}+a^{p+q})} \geq a^{p+q}+b^{p+q}+c^{p+q}.$$

The equality holds for $a = b = c$, yielding a common value of $3a^{p+q}$. □

Remark 3.5. *When $p = 1$ and $q = 0$, the inequality in Theorem 3.5 reduces to*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq a+b+c,$$

which is a well-known consequence of the Titu lemma. Thus, Theorem 3.5 serves as a two-parameter family generalization of classical polynomial rational inequalities.

Remark 3.6. *Setting $q = -p$, Theorem 3.5 yields*

$$\sum_{cyc}\frac{(a^{2p}+b^{2p})(a^{-2p}+b^{-2p})}{4}\geq 3.$$

This can also be proved using the arithmetic mean-geometric mean inequality as $(x+y)(1/x+1/y)\geq(2\sqrt{xy})\left(2\sqrt{1/(xy)}\right)=4$, and applied to $x=a^{2p}$ and $y=b^{2p}$.

Theorem 3.6. *Let $a,b,c>0$ and $p,q\in\mathbb{R}$. The following inequality unconditionally holds:*

$$\begin{aligned}
&\frac{a^{p+q}}{b^q+c^q}\left(\frac{b^p}{a^p+b^p}+\frac{c^p}{a^p+c^p}\right)+\frac{b^{p+q}}{c^q+a^q}\left(\frac{c^p}{b^p+c^p}+\frac{a^p}{b^p+a^p}\right)\\
&+\frac{c^{p+q}}{a^q+b^q}\left(\frac{a^p}{c^p+a^p}+\frac{b^p}{c^p+b^p}\right)\\
&\geq\frac{a^p}{b^q+c^q}\left(\frac{b^{p+q}}{a^p+b^p}+\frac{c^{p+q}}{a^p+c^p}\right)+\frac{b^p}{c^q+a^q}\left(\frac{c^{p+q}}{b^p+c^p}+\frac{a^{p+q}}{b^p+a^p}\right)\\
&+\frac{c^p}{a^q+b^q}\left(\frac{a^{p+q}}{c^p+a^p}+\frac{b^{p+q}}{c^p+b^p}\right).
\end{aligned}$$

The equality holds for $a=b=c$ (when $q=0$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^{p+q}}{b^q+c^q}\left(\frac{b^p}{a^p+b^p}+\frac{c^p}{a^p+c^p}\right)-\sum_{\text{cyc}}\frac{a^p}{b^q+c^q}\left(\frac{b^{p+q}}{a^p+b^p}+\frac{c^{p+q}}{a^p+c^p}\right).$$

Factorizing and arranging the terms, we get

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{1}{b^q+c^q}\left(\frac{a^{p+q}b^p-a^pb^{p+q}}{a^p+b^p}+\frac{a^{p+q}c^p-a^pc^{p+q}}{a^p+c^p}\right).$$

Observing that

$$\frac{a^{p+q}b^p-a^pb^{p+q}}{a^p+b^p}+\frac{a^{p+q}c^p-a^pc^{p+q}}{a^p+c^p}=\frac{a^pb^p(a^q-b^q)}{a^p+b^p}+\frac{a^pc^p(a^q-c^q)}{a^p+c^p},$$

we can write

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^pb^p(a^q-b^q)}{(a^p+b^p)(b^q+c^q)}+\sum_{\text{cyc}}\frac{a^pc^p(a^q-c^q)}{(a^p+c^p)(b^q+c^q)}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^pc^p(a^q-c^q)}{(a^p+c^p)(b^q+c^q)}\longrightarrow\frac{b^pa^p(b^q-a^q)}{(b^p+a^p)(c^q+a^q)}=-\frac{a^pb^p(a^q-b^q)}{(a^p+b^p)(c^q+a^q)}.$$

Therefore, we have

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)}{(a^p + b^p)(b^q + c^q)} - \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)}{(a^p + b^p)(c^q + a^q)} \\ &= \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)}{a^p + b^p} \left(\frac{1}{b^q + c^q} - \frac{1}{c^q + a^q} \right) \\ &= \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)}{a^p + b^p} \frac{(c^q + a^q) - (b^q + c^q)}{(a^q + c^q)(b^q + c^q)} \\ &= \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)^2}{(a^p + b^p)(a^q + c^q)(b^q + c^q)}.\end{aligned}$$

Because $a, b, c > 0$, we have $a^p b^p > 0$ and $(a^p + b^p)(a^q + c^q)(b^q + c^q) > 0$ (and cyclically). Moreover, we have $(a^q - b^q)^2 \geq 0$ for any $q \in \mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)^2}{(a^p + b^p)(a^q + c^q)(b^q + c^q)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\begin{aligned}&\frac{a^{p+q}}{b^q + c^q} \left(\frac{b^p}{a^p + b^p} + \frac{c^p}{a^p + c^p} \right) + \frac{b^{p+q}}{c^q + a^q} \left(\frac{c^p}{b^p + c^p} + \frac{a^p}{b^p + a^p} \right) \\ &+ \frac{c^{p+q}}{a^q + b^q} \left(\frac{a^p}{c^p + a^p} + \frac{b^p}{c^p + b^p} \right) \\ &\geq \frac{a^p}{b^q + c^q} \left(\frac{b^{p+q}}{a^p + b^p} + \frac{c^{p+q}}{a^p + c^p} \right) + \frac{b^p}{c^q + a^q} \left(\frac{c^{p+q}}{b^p + c^p} + \frac{a^{p+q}}{b^p + a^p} \right) \\ &+ \frac{c^p}{a^q + b^q} \left(\frac{a^{p+q}}{c^p + a^p} + \frac{b^{p+q}}{c^p + b^p} \right).\end{aligned}$$

The equality holds for $a = b = c$, yielding a common value of $(3/2)a^p$. □

Remark 3.7. *When $p = 1$ and $q = -1$, the inequality in Theorem 3.6 simplifies to*

$$a^2 b^2 + b^2 c^2 + c^2 a^2 \geq abc(a + b + c),$$

which is a classical degree-four cyclic inequality.

Theorem 3.7. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The following inequality unconditionally holds:*

$$\begin{aligned}&\frac{a^{p+q}(b^p + c^p)}{(b^q + c^q)^2} + \frac{b^{p+q}(c^p + a^p)}{(c^q + a^q)^2} + \frac{c^{p+q}(a^p + b^p)}{(a^q + b^q)^2} \\ &\geq \frac{a^p(b^{p+q} + c^{p+q})}{(b^q + c^q)^2} + \frac{b^p(c^{p+q} + a^{p+q})}{(c^q + a^q)^2} + \frac{c^p(a^{p+q} + b^{p+q})}{(a^q + b^q)^2}.\end{aligned}$$

The equality holds for $a = b = c$ (when $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+q}(b^p+c^p)}{(b^q+c^q)^2} - \sum_{\text{cyc}} \frac{a^p(b^{p+q}+c^{p+q})}{(b^q+c^q)^2}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+q}(b^p+c^p) - a^p(b^{p+q}+c^{p+q})}{(b^q+c^q)^2}.$$

Noting that

$$\begin{aligned} &a^{p+q}(b^p+c^p) - a^p(b^{p+q}+c^{p+q}) = a^{p+q}b^p + a^{p+q}c^p - a^pb^{p+q} - a^pc^{p+q} \\ &= (a^{p+q}b^p - a^pb^{p+q}) + (a^{p+q}c^p - a^pc^{p+q}) = a^pb^p(a^q-b^q) + a^pc^p(a^q-c^q), \end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^pb^p(a^q-b^q)}{(b^q+c^q)^2} + \sum_{\text{cyc}} \frac{a^pc^p(a^q-c^q)}{(b^q+c^q)^2}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^pc^p(a^q-c^q)}{(b^q+c^q)^2} \longrightarrow \frac{b^pa^p(b^q-a^q)}{(c^q+a^q)^2} = -\frac{a^pb^p(a^q-b^q)}{(c^q+a^q)^2}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^pb^p(a^q-b^q)}{(b^q+c^q)^2} - \sum_{\text{cyc}} \frac{a^pb^p(a^q-b^q)}{(c^q+a^q)^2} \\ &= \sum_{\text{cyc}} a^pb^p(a^q-b^q)\left(\frac{1}{(b^q+c^q)^2} - \frac{1}{(c^q+a^q)^2}\right) \\ &= \sum_{\text{cyc}} a^pb^p(a^q-b^q)\frac{(c^q+a^q)^2-(b^q+c^q)^2}{(b^q+c^q)^2(c^q+a^q)^2} \\ &= \sum_{\text{cyc}} \frac{a^pb^p(a^q-b^q)^2(a^q+b^q+2c^q)}{(a^q+c^q)^2(b^q+c^q)^2}. \end{aligned}$$

Because $a,b,c>0$, we have $a^pb^p>0$, $a^q+b^q+2c^q>0$, and $(a^q+c^q)^2(b^q+c^q)^2>0$ (and cyclically). Moreover, we have $(a^q-b^q)^2 \geq 0$ for any $q \in \mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^pb^p(a^q-b^q)^2(a^q+b^q+2c^q)}{(a^q+c^q)^2(b^q+c^q)^2} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\begin{aligned} &\frac{a^{p+q}(b^p+c^p)}{(b^q+c^q)^2} + \frac{b^{p+q}(c^p+a^p)}{(c^q+a^q)^2} + \frac{c^{p+q}(a^p+b^p)}{(a^q+b^q)^2} \\ &\geq \frac{a^p(b^{p+q}+c^{p+q})}{(b^q+c^q)^2} + \frac{b^p(c^{p+q}+a^{p+q})}{(c^q+a^q)^2} + \frac{c^p(a^{p+q}+b^{p+q})}{(a^q+b^q)^2}. \end{aligned}$$

The equality holds for $a=b=c$, yielding a common value of $(3/2)a^{2p-q}$. □

Remark 3.8. *When $p=1$ and $q=-1$, the inequality in Theorem 3.7 simplifies to*

$$\frac{bc}{a(b+c)}+\frac{ca}{b(c+a)}+\frac{ab}{c(a+b)}\geq 2\left(\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}+\frac{ab}{(a+b)^2}\right).$$

Theorem 3.8. *Let $a,b,c>0$ and $p,q\in\mathbb{R}$. The following inequality unconditionally holds:*

$$\begin{aligned}&\frac{a^{p+q}(b^p+c^p)}{b^q+c^q}+\frac{b^{p+q}(c^p+a^p)}{c^q+a^q}+\frac{c^{p+q}(a^p+b^p)}{a^q+b^q}\\&\geq\frac{a^p(b^{p+q}+c^{p+q})}{b^q+c^q}+\frac{b^p(c^{p+q}+a^{p+q})}{c^q+a^q}+\frac{c^p(a^{p+q}+b^{p+q})}{a^q+b^q}.\end{aligned}$$

The equality holds for $a=b=c$ (when $q=0$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^{p+q}(b^p+c^p)}{b^q+c^q}-\sum_{\text{cyc}}\frac{a^p(b^{p+q}+c^{p+q})}{b^q+c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^{p+q}(b^p+c^p)-a^p(b^{p+q}+c^{p+q})}{b^q+c^q}.$$

Observing that

$$\begin{aligned}&a^{p+q}(b^p+c^p)-a^p(b^{p+q}+c^{p+q})=a^pa^qb^p+a^pa^qc^p-a^pb^pb^q-a^pc^pc^q\\&=(a^pa^qb^p-a^pb^pb^q)+(a^pa^qc^p-a^pc^pc^q)\\&=a^pb^p(a^q-b^q)+a^pc^p(a^q-c^q),\end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^pb^p(a^q-b^q)}{b^q+c^q}+\sum_{\text{cyc}}\frac{a^pc^p(a^q-c^q)}{b^q+c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^pc^p(a^q-c^q)}{b^q+c^q}\longrightarrow\frac{b^pa^p(b^q-a^q)}{c^q+a^q}=-\frac{a^pb^p(a^q-b^q)}{c^q+a^q}.$$

Therefore, we have

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c)&=\sum_{\text{cyc}}\frac{a^pb^p(a^q-b^q)}{b^q+c^q}-\sum_{\text{cyc}}\frac{a^pb^p(a^q-b^q)}{c^q+a^q}\\&=\sum_{\text{cyc}}a^pb^p(a^q-b^q)\left(\frac{1}{b^q+c^q}-\frac{1}{c^q+a^q}\right)\\&=\sum_{\text{cyc}}a^pb^p(a^q-b^q)\frac{(c^q+a^q)-(b^q+c^q)}{(a^q+c^q)(b^q+c^q)}\\&=\sum_{\text{cyc}}\frac{a^pb^p(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)}.\end{aligned}$$

Because $a, b, c > 0$, we have $a^p b^p > 0$ and $(a^q + c^q)(b^q + c^q) > 0$ (and cyclically). Moreover, we have $(a^q - b^q)^2 \geq 0$ for any $q \in \mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)^2}{(a^q + c^q)(b^q + c^q)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \geq 0$, i.e.,

$$\begin{aligned} &\frac{a^{p+q}(b^p + c^p)}{b^q + c^q} + \frac{b^{p+q}(c^p + a^p)}{c^q + a^q} + \frac{c^{p+q}(a^p + b^p)}{a^q + b^q} \\ &\geq \frac{a^p(b^{p+q} + c^{p+q})}{b^q + c^q} + \frac{b^p(c^{p+q} + a^{p+q})}{c^q + a^q} + \frac{c^p(a^{p+q} + b^{p+q})}{a^q + b^q}. \end{aligned}$$

The equality holds for $a = b = c$, yielding a common value of $3a^{2p}$. □

Remark 3.9. *When $p = 1$ and $q = 1$, the inequality in Theorem 3.8 reduces to*

$$a^2 + b^2 + c^2 \geq \frac{a(b^2 + c^2)}{b + c} + \frac{b(c^2 + a^2)}{c + a} + \frac{c(a^2 + b^2)}{a + b}.$$

Furthermore, when $p = 2$ and $q = -1$, the inequality simplifies to

$$\frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + b^2}{a + b} \geq a + b + c.$$

Theorem 3.9. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The following inequality unconditionally holds:*

$$\frac{a^q(b^p + c^p)}{b^q + c^q} + \frac{b^q(c^p + a^p)}{c^q + a^q} + \frac{c^q(a^p + b^p)}{a^q + b^q} \geq \frac{b^q c^p + c^q b^p}{b^q + c^q} + \frac{c^q a^p + a^q c^p}{c^q + a^q} + \frac{a^q b^p + b^q a^p}{a^q + b^q}.$$

The equality holds for $a = b = c$ (when $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{a^q(b^p + c^p)}{b^q + c^q} - \sum_{\text{cyc}} \frac{b^q c^p + c^q b^p}{b^q + c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{a^q(b^p + c^p) - (b^q c^p + c^q b^p)}{b^q + c^q}.$$

Noting that

$$\begin{aligned} &a^q(b^p + c^p) - (b^q c^p + c^q b^p) = a^q b^p + a^q c^p - b^q c^p - c^q b^p \\ &= (a^q c^p - b^q c^p) + (a^q b^p - c^q b^p) \\ &= c^p(a^q - b^q) + b^p(a^q - c^q), \end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{c^p(a^q - b^q)}{b^q + c^q} + \sum_{\text{cyc}} \frac{b^p(a^q - c^q)}{b^q + c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{b^p(a^q-c^q)}{b^q+c^q} \longrightarrow \frac{c^p(b^q-a^q)}{c^q+a^q} = -\frac{c^p(a^q-b^q)}{c^q+a^q}.$$

Therefore, we have

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{c^p(a^q-b^q)}{b^q+c^q} - \sum_{\text{cyc}} \frac{c^p(a^q-b^q)}{c^q+a^q} \\ &= \sum_{\text{cyc}} c^p(a^q-b^q)\left(\frac{1}{b^q+c^q} - \frac{1}{c^q+a^q}\right) \\ &= \sum_{\text{cyc}} c^p(a^q-b^q)\frac{(c^q+a^q)-(b^q+c^q)}{(a^q+c^q)(b^q+c^q)} \\ &= \sum_{\text{cyc}} \frac{c^p(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)}.\end{aligned}$$

Because $a,b,c>0$, we have $c^p>0$ and $(a^q+c^q)(b^q+c^q)>0$ (and cyclically). Moreover, we have $(a^q-b^q)^2\geq 0$ for any $q\in\mathbb{R}$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{c^p(a^q-b^q)^2}{(a^q+c^q)(b^q+c^q)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c)\geq 0$, i.e.,

$$\frac{a^q(b^p+c^p)}{b^q+c^q} + \frac{b^q(c^p+a^p)}{c^q+a^q} + \frac{c^q(a^p+b^p)}{a^q+b^q} \geq \frac{b^qc^p+c^qb^p}{b^q+c^q} + \frac{c^qa^p+a^qc^p}{c^q+a^q} + \frac{a^qb^p+b^qa^p}{a^q+b^q}.$$

The equality holds for $a=b=c$, yielding a common value of $3a^p$. □

Remark 3.10. *When $p=1$ and $q=-1$, the inequality in Theorem 3.9 simplifies to*

$$\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b}.$$

4. Conditional two-power-parameter cyclic inequalities

This section presents seventeen conditional two-power-parameter cyclic inequalities of varying types. Once again, each proof immediately follows its corresponding statement, and remarks are provided to highlight special cases of interest.

Theorem 4.1. *Let $a,b,c>0$ and $p,q\in\mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq>0$ (i.e., p and q share the same sign):*

$$\frac{a^p}{b^q+c^q} + \frac{b^p}{c^q+a^q} + \frac{c^p}{a^q+b^q} \geq \frac{1}{2}\left(\frac{b^p+c^p}{b^q+c^q} + \frac{c^p+a^p}{c^q+a^q} + \frac{a^p+b^p}{a^q+b^q}\right).$$

- *For $pq<0$ (i.e., p and q have opposite signs):*

$$\frac{a^p}{b^q+c^q} + \frac{b^p}{c^q+a^q} + \frac{c^p}{a^q+b^q} \leq \frac{1}{2}\left(\frac{b^p+c^p}{b^q+c^q} + \frac{c^p+a^p}{c^q+a^q} + \frac{a^p+b^p}{a^q+b^q}\right).$$

The equality holds for $pq\neq 0$ and $a=b=c$ (when $p=0$ or $q=0$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, multiplied by 2, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^p}{b^q + c^q} - \sum_{\text{cyc}} \frac{b^p + c^p}{b^q + c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^p - (b^p + c^p)}{b^q + c^q}.$$

Observing that

$$2a^p - (b^p + c^p) = 2a^p - b^p - c^p = (a^p - b^p) - (c^p - a^p),$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^p - b^p}{b^q + c^q} - \sum_{\text{cyc}} \frac{c^p - a^p}{b^q + c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{c^p - a^p}{b^q + c^q} \longrightarrow \frac{a^p - b^p}{c^q + a^q}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^p - b^p}{b^q + c^q} - \sum_{\text{cyc}} \frac{a^p - b^p}{c^q + a^q} \\
&= \sum_{\text{cyc}} (a^p - b^p) \left(\frac{1}{b^q + c^q} - \frac{1}{c^q + a^q} \right) \\
&= \sum_{\text{cyc}} (a^p - b^p) \frac{(c^q + a^q) - (b^q + c^q)}{(a^q + c^q)(b^q + c^q)} = \sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{(a^q + c^q)(b^q + c^q)}.
\end{aligned}$$

Because $a, b, c > 0$, we have $(a^q + c^q)(b^q + c^q) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and $(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

  $$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{(a^q + c^q)(b^q + c^q)} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

  $$\frac{a^p}{b^q + c^q} + \frac{b^p}{c^q + a^q} + \frac{c^p}{a^q + b^q} \geq \frac{1}{2} \left(\frac{b^p + c^p}{b^q + c^q} + \frac{c^p + a^p}{c^q + a^q} + \frac{a^p + b^p}{a^q + b^q} \right).$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors

will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{(a^q + c^q)(b^q + c^q)} \leq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^p}{b^q + c^q} + \frac{b^p}{c^q + a^q} + \frac{c^p}{a^q + b^q} \leq \frac{1}{2}\left(\frac{b^p + c^p}{b^q + c^q} + \frac{c^p + a^p}{c^q + a^q} + \frac{a^p + b^p}{a^q + b^q}\right).$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $(3/2)a^{p-q}$. □

Remark 4.1. *When $p = q$, the inequality in Theorem 4.1 simplifies to*

$$\frac{a^p}{b^p + c^p} + \frac{b^p}{c^p + a^p} + \frac{c^p}{a^p + b^p} \geq \frac{3}{2},$$

which is exactly the Nesbitt inequality applied to the variables $x = a^p$, $y = b^p$, and $z = c^p$. This highlights that Theorem 4.1 serves as a two-parameter polynomial power generalization of classical cyclic rational inequalities.

Remark 4.2. *Setting $p = 2$ and $q = 1$, a consequence of Theorem 4.1 is*

$$\frac{a^2}{b + c} + \frac{b^2}{c + a} + \frac{c^2}{a + b} \geq \frac{1}{2}\left(\frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + b^2}{a + b}\right) \geq \frac{a + b + c}{2},$$

providing a two-stage refinement of the Titu lemma bound for cyclic squares.

Theorem 4.2. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product $p(p + q)$.*

- *For $p(p + q) > 0$ (i.e., p lies outside the interval between 0 and $-q$):*

$$\frac{a^p b^p (a^q + b^q)}{a^p + b^p} + \frac{b^p c^p (b^q + c^q)}{b^p + c^p} + \frac{c^p a^p (c^q + a^q)}{c^p + a^p} \leq a^{p+q} + b^{p+q} + c^{p+q}.$$

- *For $p(p + q) < 0$ (i.e., p lies strictly between 0 and $-q$):*

$$\frac{a^p b^p (a^q + b^q)}{a^p + b^p} + \frac{b^p c^p (b^q + c^q)}{b^p + c^p} + \frac{c^p a^p (c^q + a^q)}{c^p + a^p} \geq a^{p+q} + b^{p+q} + c^{p+q}.$$

The equality holds for $p(p + q) \neq 0$ and $a = b = c$ (when $p = 0$ or $p = -q$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, multiplied by 2, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^p b^p (a^q + b^q)}{a^p + b^p} - 2(a^{p+q} + b^{p+q} + c^{p+q}).$$

Noting that $2(a^{p+q}+b^{p+q}+c^{p+q}) = \sum_{\text{cyc}}(a^{p+q}+b^{p+q})$ and combining the terms into the sums over a common denominator, we derive

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{2a^p b^p(a^q+b^q)}{a^p+b^p} - \sum_{\text{cyc}}(a^{p+q}+b^{p+q}) \\ &= \sum_{\text{cyc}} \frac{2a^p b^p(a^q+b^q) - (a^{p+q}+b^{p+q})(a^p+b^p)}{a^p+b^p}.\end{aligned}$$

Observing that

$$\begin{aligned}&2a^p b^p(a^q+b^q) - (a^{p+q}+b^{p+q})(a^p+b^p) \\ &= (2a^{p+q}b^p + 2a^p b^{p+q}) - (a^{2p+q} + a^{p+q}b^p + a^p b^{p+q} + b^{2p+q}) \\ &= -a^{2p+q} + a^{p+q}b^p + a^p b^{p+q} - b^{2p+q} \\ &= -(a^{2p+q} - a^p b^{p+q} - a^{p+q}b^p + b^{2p+q}) \\ &= -(a^p-b^p)(a^{p+q}-b^{p+q}),\end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = -\sum_{\text{cyc}} \frac{(a^p-b^p)(a^{p+q}-b^{p+q})}{a^p+b^p}.$$

Because $a,b,c>0$, we have $a^p+b^p>0$ (and cyclically). The sign of the entire sum relies strictly on the leading minus sign and the product of differences $(a^p-b^p)(a^{p+q}-b^{p+q})$ (and cyclically).

- Case 1 ($p(p+q)>0$): The exponents p and $p+q$ share the same sign. The power functions $f(x)=x^p$ and $g(x)=x^{p+q}$ share the exact same monotonicity. Therefore, (a^p-b^p) and $(a^{p+q}-b^{p+q})$ always have the same sign, yielding $(a^p-b^p)(a^{p+q}-b^{p+q}) \geq 0$ (and cyclically). This implies that

$$-\sum_{\text{cyc}} \frac{(a^p-b^p)(a^{p+q}-b^{p+q})}{a^p+b^p} \leq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^p b^p(a^q+b^q)}{a^p+b^p} + \frac{b^p c^p(b^q+c^q)}{b^p+c^p} + \frac{c^p a^p(c^q+a^q)}{c^p+a^p} \leq a^{p+q}+b^{p+q}+c^{p+q}.$$

- Case 2 ($p(p+q)<0$): The exponents p and $p+q$ have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p-b^p)(a^{p+q}-b^{p+q}) \leq 0$ (and cyclically). This implies that

$$-\sum_{\text{cyc}} \frac{(a^p-b^p)(a^{p+q}-b^{p+q})}{a^p+b^p} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^p b^p(a^q+b^q)}{a^p+b^p} + \frac{b^p c^p(b^q+c^q)}{b^p+c^p} + \frac{c^p a^p(c^q+a^q)}{c^p+a^p} \geq a^{p+q}+b^{p+q}+c^{p+q}.$$

The equality holds for $p(p+q)\neq 0$ and $a=b=c$, yielding a common value of $3a^{p+q}$. □

Remark 4.3. *When $p = 1$ and $q = -2$, the inequality in Theorem 4.2 reduces to*

$$\frac{a^2+b^2}{ab(a+b)} + \frac{b^2+c^2}{bc(b+c)} + \frac{c^2+a^2}{ca(c+a)} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Theorem 4.3. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^p}{b^q c^q} + \frac{b^p}{c^q a^q} + \frac{c^p}{a^q b^q} \geq \frac{1}{2}\left(\frac{b^p+c^p}{b^q c^q} + \frac{c^p+a^p}{c^q a^q} + \frac{a^p+b^p}{a^q b^q}\right).$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^p}{b^q c^q} + \frac{b^p}{c^q a^q} + \frac{c^p}{a^q b^q} \leq \frac{1}{2}\left(\frac{b^p+c^p}{b^q c^q} + \frac{c^p+a^p}{c^q a^q} + \frac{a^p+b^p}{a^q b^q}\right).$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, multiplied by 2, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^p}{b^q c^q} - \sum_{\text{cyc}} \frac{b^p + c^p}{b^q c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^p - (b^p + c^p)}{b^q c^q}.$$

Noting that

$$2a^p - (b^p + c^p) = 2a^p - b^p - c^p = (a^p - b^p) - (c^p - a^p),$$

we derive

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^p - b^p}{b^q c^q} - \sum_{\text{cyc}} \frac{c^p - a^p}{b^q c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{c^p - a^p}{b^q c^q} \longrightarrow \frac{a^p - b^p}{c^q a^q}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^p - b^p}{b^q c^q} - \sum_{\text{cyc}} \frac{a^p - b^p}{c^q a^q} \\
&= \sum_{\text{cyc}} (a^p - b^p)\left(\frac{1}{b^q c^q} - \frac{1}{c^q a^q}\right) \\
&= \sum_{\text{cyc}} (a^p - b^p)\frac{a^q - b^q}{a^q b^q c^q} = \sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^q b^q c^q}.
\end{aligned}$$

Because $a, b, c > 0$, we have $a^q b^q c^q > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and $(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^q b^q c^q} \geq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \geq 0$, i.e.,

$$\frac{a^p}{b^q c^q} + \frac{b^p}{c^q a^q} + \frac{c^p}{a^q b^q} \geq \frac{1}{2}\left(\frac{b^p + c^p}{b^q c^q} + \frac{c^p + a^p}{c^q a^q} + \frac{a^p + b^p}{a^q b^q}\right).$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^q b^q c^q} \leq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \leq 0$, i.e.,

$$\frac{a^p}{b^q c^q} + \frac{b^p}{c^q a^q} + \frac{c^p}{a^q b^q} \leq \frac{1}{2}\left(\frac{b^p + c^p}{b^q c^q} + \frac{c^p + a^p}{c^q a^q} + \frac{a^p + b^p}{a^q b^q}\right).$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $3a^{p-2q}$. □

Remark 4.4. *Setting $p = 1$ and $q = -1$, the inequality in Theorem 4.3 simplifies to*

$$3abc \leq \frac{1}{2}\left(bc(b + c) + ca(c + a) + ab(a + b)\right).$$

The theorem below differs from subsequent results by incorporating an additional power parameter $r \in \mathbb{R}$.

Theorem 4.4. *Let $a, b, c > 0$ and $p, q, r \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^p(b^r + c^r)}{b^q + c^q} + \frac{b^p(c^r + a^r)}{c^q + a^q} + \frac{c^p(a^r + b^r)}{a^q + b^q} \geq \frac{b^p c^r + c^p b^r}{b^q + c^q} + \frac{c^p a^r + a^p c^r}{c^q + a^q} + \frac{a^p b^r + b^p a^r}{a^q + b^q}.$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^p(b^r + c^r)}{b^q + c^q} + \frac{b^p(c^r + a^r)}{c^q + a^q} + \frac{c^p(a^r + b^r)}{a^q + b^q} \leq \frac{b^p c^r + c^p b^r}{b^q + c^q} + \frac{c^p a^r + a^p c^r}{c^q + a^q} + \frac{a^p b^r + b^p a^r}{a^q + b^q}.$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q,r}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q,r}(a,b,c) = \sum_{\text{cyc}} \frac{a^p(b^r+c^r)}{b^q+c^q} - \sum_{\text{cyc}} \frac{b^pc^r+c^pb^r}{b^q+c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q,r}(a,b,c) = \sum_{\text{cyc}} \frac{a^p(b^r+c^r)-(b^pc^r+c^pb^r)}{b^q+c^q}.$$

Observing that

$$\begin{aligned} a^p(b^r+c^r)-(b^pc^r+c^pb^r) &= a^pb^r+a^pc^r-b^pc^r-c^pb^r \\ &= (a^pc^r-b^pc^r)+(a^pb^r-c^pb^r) \\ &= c^r(a^p-b^p)-b^r(c^p-a^p), \end{aligned}$$

we can write

$$\mathcal{D}_{p,q,r}(a,b,c) = \sum_{\text{cyc}} \frac{c^r(a^p-b^p)}{b^q+c^q} - \sum_{\text{cyc}} \frac{b^r(c^p-a^p)}{b^q+c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{b^r(c^p-a^p)}{b^q+c^q} \longrightarrow \frac{c^r(a^p-b^p)}{c^q+a^q}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q,r}(a,b,c) &= \sum_{\text{cyc}} \frac{c^r(a^p-b^p)}{b^q+c^q} - \sum_{\text{cyc}} \frac{c^r(a^p-b^p)}{c^q+a^q} \\ &= \sum_{\text{cyc}} c^r(a^p-b^p)\left(\frac{1}{b^q+c^q}-\frac{1}{c^q+a^q}\right) \\ &= \sum_{\text{cyc}} c^r(a^p-b^p)\frac{(c^q+a^q)-(b^q+c^q)}{(b^q+c^q)(c^q+a^q)} = \sum_{\text{cyc}} \frac{c^r(a^p-b^p)(a^q-b^q)}{(b^q+c^q)(c^q+a^q)}. \end{aligned}$$

Because $a,b,c>0$, we have $c^r>0$ and $(b^q+c^q)(c^q+a^q)>0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p-b^p)(a^q-b^q)$ (and cyclically).

- Case 1 ($pq>0$): The exponents p and q share the same sign. The power functions $f(x)=x^p$ and $g(x)=x^q$ share the exact same monotonicity. Therefore, (a^p-b^p) and (a^q-b^q) always have the same sign, yielding $(a^p-b^p)(a^q-b^q)\geq 0$ (and cyclically). This implies that

  $$\sum_{\text{cyc}} \frac{c^r(a^p-b^p)(a^q-b^q)}{(b^q+c^q)(c^q+a^q)} \geq 0,$$

  so that $\mathcal{D}_{p,q,r}(a,b,c)\geq 0$, i.e.,

$$\frac{a^p(b^r+c^r)}{b^q+c^q}+\frac{b^p(c^r+a^r)}{c^q+a^q}+\frac{c^p(a^r+b^r)}{a^q+b^q} \geq \frac{b^pc^r+c^pb^r}{b^q+c^q}+\frac{c^pa^r+a^pc^r}{c^q+a^q}+\frac{a^pb^r+b^pa^r}{a^q+b^q}.$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{c^r(a^p - b^p)(a^q - b^q)}{(b^q + c^q)(c^q + a^q)} \leq 0,$$

so that $\mathcal{D}_{p,q,r}(a, b, c) \leq 0$, i.e.,

$$\frac{a^p(b^r + c^r)}{b^q + c^q} + \frac{b^p(c^r + a^r)}{c^q + a^q} + \frac{c^p(a^r + b^r)}{a^q + b^q} \leq \frac{b^p c^r + c^p b^r}{b^q + c^q} + \frac{c^p a^r + a^p c^r}{c^q + a^q} + \frac{a^p b^r + b^p a^r}{a^q + b^q}.$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $3a^{p+r-q}$. □

Remark 4.5. *Setting $r = p + q$ in Theorem 4.4 recovers a general form of cyclic rational inequalities. In particular, when $p = 1$, $q = 1$, and $r = 2$, this reduces to*

$$\frac{a(b^2 + c^2)}{b + c} + \frac{b(c^2 + a^2)}{c + a} + \frac{c(a^2 + b^2)}{a + b} \geq ab + bc + ca.$$

For another example, when $p = 1$, $q = -1$, and $r = 1$, since $pq = -1 < 0$, Theorem 4.4 yields

$$\frac{b^2c^2}{b + c} + \frac{c^2a^2}{c + a} + \frac{a^2b^2}{a + b} \geq \frac{3}{2}abc.$$

This demonstrates how Theorem 4.4 unifies a broad class of rational inequalities under a single parameterized framework.

Theorem 4.5. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^{p+q}}{b^q + c^q} + \frac{b^{p+q}}{c^q + a^q} + \frac{c^{p+q}}{a^q + b^q} \geq \frac{1}{2}\left(\frac{a^q(b^p + c^p)}{b^q + c^q} + \frac{b^q(c^p + a^p)}{c^q + a^q} + \frac{c^q(a^p + b^p)}{a^q + b^q}\right).$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^{p+q}}{b^q + c^q} + \frac{b^{p+q}}{c^q + a^q} + \frac{c^{p+q}}{a^q + b^q} \leq \frac{1}{2}\left(\frac{a^q(b^p + c^p)}{b^q + c^q} + \frac{b^q(c^p + a^p)}{c^q + a^q} + \frac{c^q(a^p + b^p)}{a^q + b^q}\right).$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, multiplied by 2, as follows:

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{2a^{p+q}}{b^q + c^q} - \sum_{\text{cyc}} \frac{a^q(b^p + c^p)}{b^q + c^q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{2a^{p+q} - a^q(b^p + c^p)}{b^q + c^q}.$$

Noting that

$$2a^{p+q} - a^q(b^p + c^p) = 2a^{p+q} - a^q b^p - a^q c^p = a^q(a^p - b^p) - a^q(c^p - a^p),$$

we derive

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^q(a^p - b^p)}{b^q + c^q} - \sum_{\text{cyc}} \frac{a^q(c^p - a^p)}{b^q + c^q}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^q(c^p - a^p)}{b^q + c^q} \longrightarrow \frac{b^q(a^p - b^p)}{c^q + a^q}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^q(a^p - b^p)}{b^q + c^q} - \sum_{\text{cyc}} \frac{b^q(a^p - b^p)}{c^q + a^q} \\
&= \sum_{\text{cyc}} (a^p - b^p) \left(\frac{a^q}{b^q + c^q} - \frac{b^q}{c^q + a^q} \right) \\
&= \sum_{\text{cyc}} (a^p - b^p) \frac{a^q(c^q + a^q) - b^q(b^q + c^q)}{(b^q + c^q)(c^q + a^q)}.
\end{aligned}$$

Observing that

$$\begin{aligned}
a^q(c^q + a^q) - b^q(b^q + c^q) &= a^q c^q + (a^q)^2 - (b^q)^2 - b^q c^q \\
&= \left((a^q)^2 - (b^q)^2 \right) + (a^q c^q - b^q c^q) \\
&= (a^q - b^q)(a^q + b^q) + c^q(a^q - b^q) \\
&= (a^q - b^q)(a^q + b^q + c^q),
\end{aligned}$$

we get

$$\mathcal{D}_{p,q}(a,b,c) = (a^q + b^q + c^q) \sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{(b^q + c^q)(c^q + a^q)}.$$

Because $a, b, c > 0$, we have $a^q + b^q + c^q > 0$ and $(b^q + c^q)(c^q + a^q) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and $(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

  $$(a^q + b^q + c^q) \sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{(b^q + c^q)(c^q + a^q)} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

  $$\frac{a^{p+q}}{b^q + c^q} + \frac{b^{p+q}}{c^q + a^q} + \frac{c^{p+q}}{a^q + b^q} \geq \frac{1}{2} \left(\frac{a^q(b^p + c^p)}{b^q + c^q} + \frac{b^q(c^p + a^p)}{c^q + a^q} + \frac{c^q(a^p + b^p)}{a^q + b^q} \right).$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$(a^q + b^q + c^q) \sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{(b^q + c^q)(c^q + a^q)} \leq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \leq 0$, i.e.,

$$\frac{a^{p+q}}{b^q + c^q} + \frac{b^{p+q}}{c^q + a^q} + \frac{c^{p+q}}{a^q + b^q} \leq \frac{1}{2}\left(\frac{a^q(b^p + c^p)}{b^q + c^q} + \frac{b^q(c^p + a^p)}{c^q + a^q} + \frac{c^q(a^p + b^p)}{a^q + b^q}\right).$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $(3/2)a^p$. □

Remark 4.6. *When $p = 2$ and $q = 1$, the inequality in Theorem 4.5 reduces to*

$$\frac{a^3}{b + c} + \frac{b^3}{c + a} + \frac{c^3}{a + b} \geq \frac{1}{2}\left(\frac{a(b^2 + c^2)}{b + c} + \frac{b(c^2 + a^2)}{c + a} + \frac{c(a^2 + b^2)}{a + b}\right).$$

Theorem 4.6. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^p + b^p}{c^q} + \frac{b^p + c^p}{a^q} + \frac{c^p + a^p}{b^q} \geq 2(a^{p-q} + b^{p-q} + c^{p-q}).$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^p + b^p}{c^q} + \frac{b^p + c^p}{a^q} + \frac{c^p + a^p}{b^q} \leq 2(a^{p-q} + b^{p-q} + c^{p-q}).$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a, b, c) = \sum_{\text{cyc}} \frac{a^p + b^p}{c^q} - 2(a^{p-q} + b^{p-q} + c^{p-q}).$$

Noting that $2(a^{p-q} + b^{p-q} + c^{p-q}) = \sum_{\text{cyc}}(a^{p-q} + b^{p-q})$ and combining the terms into the sums over a common denominator, we have

$$\begin{aligned}\mathcal{D}_{p,q}(a, b, c) &= \sum_{\text{cyc}} \frac{a^p + b^p}{c^q} - \sum_{\text{cyc}}(a^{p-q} + b^{p-q}) \\ &= \sum_{\text{cyc}} \frac{a^p + b^p - c^q(a^{p-q} + b^{p-q})}{c^q}.\end{aligned}$$

Observing that

$$\begin{aligned} a^p + b^p - c^q(a^{p-q} + b^{p-q}) &= a^p + b^p - c^q a^{p-q} - c^q b^{p-q} \\ &= a^{p-q}a^q + b^{p-q}b^q - c^q a^{p-q} - c^q b^{p-q} \\ &= (a^{p-q}a^q - a^{p-q}c^q) + (b^{p-q}b^q - b^{p-q}c^q) \\ &= a^{p-q}(a^q - c^q) + b^{p-q}(b^q - c^q), \end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p-q}(a^q - c^q)}{c^q} + \sum_{\text{cyc}} \frac{b^{p-q}(b^q - c^q)}{c^q}.$$

We shift the indices in the first and second cyclic sums to align the numerators. The main terms shift as follows:

$$\frac{a^{p-q}(a^q - c^q)}{c^q} \longrightarrow \frac{b^{p-q}(b^q - a^q)}{a^q} = -\frac{b^{p-q}(a^q - b^q)}{a^q}$$

and

$$\frac{b^{p-q}(b^q - c^q)}{c^q} \longrightarrow \frac{a^{p-q}(a^q - b^q)}{b^q}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^{p-q}(a^q - b^q)}{b^q} - \sum_{\text{cyc}} \frac{b^{p-q}(a^q - b^q)}{a^q} \\ &= \sum_{\text{cyc}} (a^q - b^q)\left(\frac{a^{p-q}}{b^q} - \frac{b^{p-q}}{a^q}\right) \\ &= \sum_{\text{cyc}} (a^q - b^q)\frac{a^{p-q}a^q - b^{p-q}b^q}{a^q b^q} = \sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^q b^q}. \end{aligned}$$

Because $a, b, c > 0$, we have $a^q b^q > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and $(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

  $$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^q b^q} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

  $$\frac{a^p + b^p}{c^q} + \frac{b^p + c^p}{a^q} + \frac{c^p + a^p}{b^q} \geq 2(a^{p-q} + b^{p-q} + c^{p-q}).$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors

will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^q b^q} \leq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^p + b^p}{c^q} + \frac{b^p + c^p}{a^q} + \frac{c^p + a^p}{b^q} \leq 2(a^{p-q} + b^{p-q} + c^{p-q}).$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $6a^{p-q}$. □

Remark 4.7. *When $p = 1$ and $q = 1$, the inequality in Theorem 4.6 reduces to*

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6.$$

Theorem 4.6 serves as a natural generalization of cyclic rational inequalities, unifying various key results under a single parametric framework.

Theorem 4.7. *Let $a, b, c > 0$ and $k, p, q \in \mathbb{R}$. The inequalities below hold based on the parameter regions of k and the sign of the product pq.*

- *For $k \geq 1$ and $pq > 0$, or for $k \leq 0$ and $pq < 0$:*

$$\frac{a^{kp}(b^{p+q} + c^{p+q})}{b^p + c^p} + \frac{b^{kp}(c^{p+q} + a^{p+q})}{c^p + a^p} + \frac{c^{kp}(a^{p+q} + b^{p+q})}{a^p + b^p} \leq a^{kp+q} + b^{kp+q} + c^{kp+q}.$$

- *For $k \geq 1$ and $pq < 0$, or for $k \leq 0$ and $pq > 0$:*

$$\frac{a^{kp}(b^{p+q} + c^{p+q})}{b^p + c^p} + \frac{b^{kp}(c^{p+q} + a^{p+q})}{c^p + a^p} + \frac{c^{kp}(a^{p+q} + b^{p+q})}{a^p + b^p} \geq a^{kp+q} + b^{kp+q} + c^{kp+q}.$$

For $0 < k < 1$, no global inequality holds. The equality holds for $pq \neq 0$ and $a = b = c$ (when $q = 0$ or $k = 1$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{k,p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{k,p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{kp}(b^{p+q} + c^{p+q})}{b^p + c^p} - \sum_{\text{cyc}} a^{kp+q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{k,p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{kp}(b^{p+q} + c^{p+q}) - a^{kp+q}(b^p + c^p)}{b^p + c^p}.$$

Noting that

$$\begin{aligned} a^{kp}(b^{p+q} + c^{p+q}) - a^{kp+q}(b^p + c^p) &= a^{kp}b^{p+q} + a^{kp}c^{p+q} - a^{kp+q}b^p - a^{kp+q}c^p \\ &= (a^{kp}b^p b^q - a^{kp}a^q b^p) + (a^{kp}c^p c^q - a^{kp}a^q c^p) \\ &= a^{kp}b^p(b^q - a^q) + a^{kp}c^p(c^q - a^q) \\ &= -a^{kp}b^p(a^q - b^q) - a^{kp}c^p(a^q - c^q), \end{aligned}$$

we derive

$$\mathcal{D}_{k,p,q}(a,b,c) = -\sum_{\text{cyc}} \frac{a^{kp}b^p(a^q-b^q)}{b^p+c^p} - \sum_{\text{cyc}} \frac{a^{kp}c^p(a^q-c^q)}{b^p+c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^{kp}c^p(a^q-c^q)}{b^p+c^p} \longrightarrow \frac{a^pb^{kp}(b^q-a^q)}{c^p+a^p} = -\frac{a^pb^{kp}(a^q-b^q)}{c^p+a^p}.$$

Therefore, we have

$$\begin{aligned}\mathcal{D}_{k,p,q}(a,b,c) &= -\sum_{\text{cyc}} \frac{a^{kp}b^p(a^q-b^q)}{b^p+c^p} + \sum_{\text{cyc}} \frac{a^pb^{kp}(a^q-b^q)}{c^p+a^p} \\ &= \sum_{\text{cyc}} (a^q-b^q)\left(\frac{a^pb^{kp}}{c^p+a^p} - \frac{a^{kp}b^p}{b^p+c^p}\right) \\ &= \sum_{\text{cyc}} (a^q-b^q)\frac{a^pb^{kp}(b^p+c^p) - a^{kp}b^p(c^p+a^p)}{(c^p+a^p)(b^p+c^p)}.\end{aligned}$$

Observing that

$$\begin{aligned}&a^pb^{kp}(b^p+c^p) - a^{kp}b^p(c^p+a^p) = a^pb^{kp}b^p + a^pb^{kp}c^p - a^{kp}b^pc^p - a^{kp}b^pa^p \\ &= a^pb^{(k+1)p} + a^pb^{kp}c^p - a^{kp}b^pc^p - a^{(k+1)p}b^p \\ &= -(a^{(k+1)p}b^p - a^pb^{(k+1)p}) + (a^pb^{kp}c^p - a^{kp}b^pc^p) \\ &= -a^pb^p(a^{kp}-b^{kp}) - c^p(a^{kp}b^p - a^pb^{kp}) \\ &= -a^pb^p(a^{kp}-b^{kp}) - a^pb^pc^p(a^{(k-1)p} - b^{(k-1)p}) \\ &= -a^pb^p\left((a^{kp}-b^{kp}) + c^p(a^{(k-1)p} - b^{(k-1)p})\right),\end{aligned}$$

we have

$$\mathcal{D}_{k,p,q}(a,b,c) = -\sum_{\text{cyc}} \frac{a^pb^p(a^q-b^q)\left((a^{kp}-b^{kp}) + c^p(a^{(k-1)p} - b^{(k-1)p})\right)}{(c^p+a^p)(b^p+c^p)}.$$

Let us set

$$F_k(x,y,z) = g_k(x,y) + zg_{k-1}(x,y),$$

where

$$g_m(x,y) = \frac{x^m-y^m}{x-y} = m\int_0^1 (tx+(1-t)y)^{m-1}\,dt.$$

Note that the sign of $g_m(x,y)$ corresponds to the sign of m. Then we have

$$\mathcal{D}_{k,p,q}(a,b,c) = -\sum_{\text{cyc}} \frac{a^pb^pF_k(a^p,b^p,c^p)(a^q-b^q)(a^p-b^p)}{(c^p+a^p)(b^p+c^p)}.$$

Because $a,b,c>0$, we have $a^pb^p>0$ and $(c^p+a^p)(b^p+c^p)>0$ (and cyclically). The sign of $\mathcal{D}_{k,p,q}(a,b,c)$ is determined by $-F_k(a^p,b^p,c^p)(a^q-b^q)(a^p-b^p)$ (and cyclically).

- Case 1 ($k \geq 1$): Here, we have $g_k(x,y)>0$ and $g_{k-1}(x,y)\geq 0$, yielding $F_k(a^p,b^p,c^p)>0$.

– If $pq > 0$, then $(a^q - b^q)(a^p - b^p) \geq 0$ (and cyclically), which implies that $\mathcal{D}_{k,p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^{kp}(b^{p+q}+c^{p+q})}{b^p+c^p} + \frac{b^{kp}(c^{p+q}+a^{p+q})}{c^p+a^p} + \frac{c^{kp}(a^{p+q}+b^{p+q})}{a^p+b^p} \leq a^{kp+q} + b^{kp+q} + c^{kp+q}.$$

– If $pq < 0$, then $(a^q - b^q)(a^p - b^p) \leq 0$, which implies that $\mathcal{D}_{k,p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^{kp}(b^{p+q}+c^{p+q})}{b^p+c^p} + \frac{b^{kp}(c^{p+q}+a^{p+q})}{c^p+a^p} + \frac{c^{kp}(a^{p+q}+b^{p+q})}{a^p+b^p} \geq a^{kp+q} + b^{kp+q} + c^{kp+q}.$$

- Case 2 ($k \leq 0$): Here, we have $g_k(x,y) \leq 0$ and $g_{k-1}(x,y) < 0$, forcing $F_k(a^p,b^p,c^p) < 0$.
  – If $pq > 0$, then $(a^q - b^q)(a^p - b^p) \geq 0$ (and cyclically), which implies that $\mathcal{D}_{k,p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^{kp}(b^{p+q}+c^{p+q})}{b^p+c^p} + \frac{b^{kp}(c^{p+q}+a^{p+q})}{c^p+a^p} + \frac{c^{kp}(a^{p+q}+b^{p+q})}{a^p+b^p} \geq a^{kp+q} + b^{kp+q} + c^{kp+q}.$$

  – If $pq < 0$, then $(a^q - b^q)(a^p - b^p) \leq 0$ (and cyclically), which implies that $\mathcal{D}_{k,p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^{kp}(b^{p+q}+c^{p+q})}{b^p+c^p} + \frac{b^{kp}(c^{p+q}+a^{p+q})}{c^p+a^p} + \frac{c^{kp}(a^{p+q}+b^{p+q})}{a^p+b^p} \leq a^{kp+q} + b^{kp+q} + c^{kp+q}.$$

- Case 3 ($0 < k < 1$): Here, we have $g_k(x,y) > 0$ while $g_{k-1}(x,y) < 0$. The sign of $F_k(a^p,b^p,c^p)$ depends on the magnitude of $z = c^p$ relative to x and y, so no global inequality holds.

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $3a^{kp+q}$. □

Remark 4.8. *When $k = -1$, $p = 1$, and $q = 2$, the inequality in Theorem 4.7 simplifies to*

$$\frac{b^3+c^3}{a(b+c)} + \frac{c^3+a^3}{b(c+a)} + \frac{a^3+b^3}{c(a+b)} \geq a+b+c.$$

When $k = -1$, $p = 1$, and $q = -1$, the inequality simplifies to

$$\frac{2}{a(b+c)} + \frac{2}{b(c+a)} + \frac{2}{c(a+b)} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

These two inequalities show the versatility of Theorem 4.7 to generate both cyclic fraction and power-sum inequalities within a unified framework.

Theorem 4.8. *Let $a,b,c > 0$ and $p,q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^{2q}(b^q+c^q)}{b^p+c^p} + \frac{b^{2q}(c^q+a^q)}{c^p+a^p} + \frac{c^{2q}(a^q+b^q)}{a^p+b^p} \geq \frac{a^q(b^{2q}+c^{2q})}{b^p+c^p} + \frac{b^q(c^{2q}+a^{2q})}{c^p+a^p} + \frac{c^q(a^{2q}+b^{2q})}{a^p+b^p}.$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^{2q}(b^q+c^q)}{b^p+c^p} + \frac{b^{2q}(c^q+a^q)}{c^p+a^p} + \frac{c^{2q}(a^q+b^q)}{a^p+b^p} \leq \frac{a^q(b^{2q}+c^{2q})}{b^p+c^p} + \frac{b^q(c^{2q}+a^{2q})}{c^p+a^p} + \frac{c^q(a^{2q}+b^{2q})}{a^p+b^p}.$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{2q}(b^q + c^q)}{b^p + c^p} - \sum_{\text{cyc}} \frac{a^q(b^{2q} + c^{2q})}{b^p + c^p}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{2q}(b^q + c^q) - a^q(b^{2q} + c^{2q})}{b^p + c^p}.$$

Noting that

$$\begin{aligned} a^{2q}(b^q + c^q) - a^q(b^{2q} + c^{2q}) &= a^{2q}b^q + a^{2q}c^q - a^q b^{2q} - a^q c^{2q} \\ &= (a^{2q}b^q - a^q b^{2q}) + (a^{2q}c^q - a^q c^{2q}) \\ &= a^q b^q(a^q - b^q) + a^q c^q(a^q - c^q), \end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^q b^q(a^q - b^q)}{b^p + c^p} + \sum_{\text{cyc}} \frac{a^q c^q(a^q - c^q)}{b^p + c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^q c^q(a^q - c^q)}{b^p + c^p} \longrightarrow \frac{b^q a^q(b^q - a^q)}{c^p + a^p} = -\frac{a^q b^q(a^q - b^q)}{c^p + a^p}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^q b^q(a^q - b^q)}{b^p + c^p} - \sum_{\text{cyc}} \frac{a^q b^q(a^q - b^q)}{c^p + a^p} \\ &= \sum_{\text{cyc}} a^q b^q(a^q - b^q)\left(\frac{1}{b^p + c^p} - \frac{1}{c^p + a^p}\right) \\ &= \sum_{\text{cyc}} a^q b^q(a^q - b^q)\frac{(c^p + a^p) - (b^p + c^p)}{(b^p + c^p)(c^p + a^p)} = \sum_{\text{cyc}} \frac{a^q b^q(a^p - b^p)(a^q - b^q)}{(b^p + c^p)(c^p + a^p)}. \end{aligned}$$

Because $a, b, c > 0$, we have $a^q b^q > 0$ and $(b^p + c^p)(c^p + a^p) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and $(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^q b^q(a^p - b^p)(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^{2q}(b^q + c^q)}{b^p + c^p} + \frac{b^{2q}(c^q + a^q)}{c^p + a^p} + \frac{c^{2q}(a^q + b^q)}{a^p + b^p} \geq \frac{a^q(b^{2q} + c^{2q})}{b^p + c^p} + \frac{b^q(c^{2q} + a^{2q})}{c^p + a^p} + \frac{c^q(a^{2q} + b^{2q})}{a^p + b^p}.$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^q b^q (a^p - b^p)(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \leq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^{2q}(b^q + c^q)}{b^p + c^p} + \frac{b^{2q}(c^q + a^q)}{c^p + a^p} + \frac{c^{2q}(a^q + b^q)}{a^p + b^p} \leq \frac{a^q(b^{2q} + c^{2q})}{b^p + c^p} + \frac{b^q(c^{2q} + a^{2q})}{c^p + a^p} + \frac{c^q(a^{2q} + b^{2q})}{a^p + b^p}.$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $3a^{3q-p}$. □

Remark 4.9. *When $p = -1$ and $q = -1$, the inequality in Theorem 4.8 reduces to*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{abc}\left(\frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + b^2}{a + b}\right).$$

When $p = -2$ and $q = 1$, the inequality simplifies to

$$abc\left(\frac{b + c}{b^2 + c^2} + \frac{c + a}{c^2 + a^2} + \frac{a + b}{a^2 + b^2}\right) \leq ab + bc + ca.$$

These examples highlight the diversity of algebraic bounds, both lower and upper, that can be generated by Theorem 4.8 by tuning the parameters p and q.

Theorem 4.9. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product $p(p + q)$.*

- *For $p(p + q) > 0$ (i.e., p lies outside the interval between 0 and $-q$):*

$$\frac{a^{p+q}(b^q + c^q)}{b^p + c^p} + \frac{b^{p+q}(c^q + a^q)}{c^p + a^p} + \frac{c^{p+q}(a^q + b^q)}{a^p + b^p} \geq a^q b^q + b^q c^q + c^q a^q.$$

- *For $p(p + q) < 0$ (i.e., p lies strictly between 0 and $-q$):*

$$\frac{a^{p+q}(b^q + c^q)}{b^p + c^p} + \frac{b^{p+q}(c^q + a^q)}{c^p + a^p} + \frac{c^{p+q}(a^q + b^q)}{a^p + b^p} \leq a^q b^q + b^q c^q + c^q a^q.$$

The equality holds for $p(p + q) \neq 0$ and $a = b = c$ (when $p = 0$ or $p = -q$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+q}(b^q + c^q)}{b^p + c^p} - \sum_{\text{cyc}} b^q c^q.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+q}(b^q + c^q) - b^q c^q (b^p + c^p)}{b^p + c^p}.$$

Observing that

$$a^{p+q}(b^q+c^q)-b^qc^q(b^p+c^p)=a^{p+q}b^q+a^{p+q}c^q-b^{p+q}c^q-b^qc^{p+q}$$
$$=(a^{p+q}c^q-b^{p+q}c^q)+(a^{p+q}b^q-b^qc^{p+q})$$
$$=c^q(a^{p+q}-b^{p+q})-b^q(c^{p+q}-a^{p+q}),$$

we derive

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{c^q(a^{p+q}-b^{p+q})}{b^p+c^p}-\sum_{\text{cyc}}\frac{b^q(c^{p+q}-a^{p+q})}{b^p+c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{b^q(c^{p+q}-a^{p+q})}{b^p+c^p}\longrightarrow\frac{c^q(a^{p+q}-b^{p+q})}{c^p+a^p}.$$

Therefore, we have

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{c^q(a^{p+q}-b^{p+q})}{b^p+c^p}-\sum_{\text{cyc}}\frac{c^q(a^{p+q}-b^{p+q})}{c^p+a^p}$$
$$=\sum_{\text{cyc}}c^q(a^{p+q}-b^{p+q})\left(\frac{1}{b^p+c^p}-\frac{1}{c^p+a^p}\right)$$
$$=\sum_{\text{cyc}}c^q(a^{p+q}-b^{p+q})\frac{(c^p+a^p)-(b^p+c^p)}{(b^p+c^p)(c^p+a^p)}=\sum_{\text{cyc}}\frac{c^q(a^{p+q}-b^{p+q})(a^p-b^p)}{(b^p+c^p)(c^p+a^p)}.$$

Because $a,b,c>0$, we have $c^q>0$ and $(b^p+c^p)(c^p+a^p)>0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^{p+q}-b^{p+q})(a^p-b^p)$ (and cyclically).

- Case 1 ($p(p+q)>0$): The exponents $p+q$ and p share the same sign. The power functions $f(x)=x^{p+q}$ and $g(x)=x^p$ share the exact same monotonicity. Therefore, $(a^{p+q}-b^{p+q})$ and (a^p-b^p) always have the same sign, yielding $(a^{p+q}-b^{p+q})(a^p-b^p)\geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}}\frac{c^q(a^{p+q}-b^{p+q})(a^p-b^p)}{(b^p+c^p)(c^p+a^p)}\geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c)\geq 0$, i.e.,

$$\frac{a^{p+q}(b^q+c^q)}{b^p+c^p}+\frac{b^{p+q}(c^q+a^q)}{c^p+a^p}+\frac{c^{p+q}(a^q+b^q)}{a^p+b^p}\geq a^qb^q+b^qc^q+c^qa^q.$$

- Case 2 ($p(p+q)<0$): The exponents $p+q$ and p have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^{p+q}-b^{p+q})(a^p-b^p)\leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}}\frac{c^q(a^{p+q}-b^{p+q})(a^p-b^p)}{(b^p+c^p)(c^p+a^p)}\leq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c)\leq 0$, i.e.,

$$\frac{a^{p+q}(b^q+c^q)}{b^p+c^p}+\frac{b^{p+q}(c^q+a^q)}{c^p+a^p}+\frac{c^{p+q}(a^q+b^q)}{a^p+b^p}\leq a^qb^q+b^qc^q+c^qa^q.$$

The equality holds for $p(p+q) \neq 0$ and $a = b = c$, yielding a common value of $3a^{2q}$. □

Remark 4.10. *When $p = -1$ and $q = 2$, the inequality in Theorem 4.9 reduces to*

$$\frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} + \frac{a^2+b^2}{a+b} \leq \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b},$$

which, to the best of our knowledge, is a new bound.

Theorem 4.10. *Let $a, b, c > 0$ and let $p, q \in \mathbb{R}$ such that $q(p+q) > 0$. The inequalities below hold based on the sign of the product $q(q-p)$.*

- *For $q(q-p) < 0$ (i.e., q lies strictly between 0 and p):*

$$\frac{a^{p+q}(b^q+c^q)}{b^p+c^p} + \frac{b^{p+q}(c^q+a^q)}{c^p+a^p} + \frac{c^{p+q}(a^q+b^q)}{a^p+b^p} \geq a^{2q} + b^{2q} + c^{2q}.$$

- *For $q(q-p) > 0$ (i.e., q lies outside the interval between 0 and p):*

$$\frac{a^{p+q}(b^q+c^q)}{b^p+c^p} + \frac{b^{p+q}(c^q+a^q)}{c^p+a^p} + \frac{c^{p+q}(a^q+b^q)}{a^p+b^p} \leq a^{2q} + b^{2q} + c^{2q}.$$

The equality holds for $q(q-p) \neq 0$ and $a = b = c$.

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+q}(b^q+c^q)}{b^p+c^p} - \sum_{\text{cyc}} a^{2q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+q}(b^q+c^q) - a^{2q}(b^p+c^p)}{b^p+c^p}.$$

Noting that

$$\begin{aligned} &a^{p+q}(b^q+c^q) - a^{2q}(b^p+c^p) = a^p a^q b^q + a^p a^q c^q - a^q a^q b^p - a^q a^q c^p \\ &= (a^p a^q b^q - a^q a^q b^p) + (a^p a^q c^q - a^q a^q c^p) \\ &= a^q(a^p b^q - a^q b^p) - a^q(c^p a^q - c^q a^p), \end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^q(a^p b^q - a^q b^p)}{b^p+c^p} - \sum_{\text{cyc}} \frac{a^q(c^p a^q - c^q a^p)}{b^p+c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^q(c^p a^q - c^q a^p)}{b^p+c^p} \longrightarrow \frac{b^q(a^p b^q - a^q b^p)}{c^p+a^p}.$$

Therefore, we have

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^q(a^pb^q - a^qb^p)}{b^p + c^p} - \sum_{\text{cyc}} \frac{b^q(a^pb^q - a^qb^p)}{c^p + a^p}$$
$$= \sum_{\text{cyc}} (a^pb^q - a^qb^p)\left(\frac{a^q}{b^p + c^p} - \frac{b^q}{c^p + a^p}\right)$$
$$= \sum_{\text{cyc}} (a^pb^q - a^qb^p)\frac{a^q(c^p + a^p) - b^q(b^p + c^p)}{(b^p + c^p)(c^p + a^p)}.$$

Observing that

$$a^q(c^p + a^p) - b^q(b^p + c^p) = a^qc^p + a^qa^p - b^qb^p - b^qc^p$$
$$= a^qc^p + a^{p+q} - b^{p+q} - b^qc^p$$
$$= (a^{p+q} - b^{p+q}) + a^qc^p - b^qc^p$$
$$= (a^{p+q} - b^{p+q}) + c^p(a^q - b^q),$$

we have

$$\mathcal{D}_{p,q}(a,b,c) = -\sum_{\text{cyc}} (ab)^p(a^{q-p} - b^{q-p})\frac{(a^{p+q} - b^{p+q}) + c^p(a^q - b^q)}{(b^p + c^p)(c^p + a^p)}.$$

Because $a, b, c > 0$, we have $(ab)^p > 0$ and $(b^p+c^p)(c^p+a^p) > 0$ (and cyclically). By the condition $q(p + q) > 0$, the powers q and $p + q$ share the same sign, so the power functions $f(x) = x^{p+q}$ and $g(x) = x^q$ share the exact same monotonicity. Hence, $(a^{p+q} - b^{p+q})$ and $(a^q - b^q)$ always have the same sign as $(a - b)$, which implies that the term $\left((a^{p+q} - b^{p+q}) + c^p(a^q - b^q)\right)$ has the same sign as $q(a - b)$ (and cyclically). On the other hand, the term $(a^{q-p} - b^{q-p})$ has the same sign as $(q - p)(a - b)$ (and cyclically).

- Case 1 ($q(q - p) < 0$): The term $-(a^{q-p} - b^{q-p})\left((a^{p+q} - b^{p+q}) + c^p(a^q - b^q)\right)$ carries the sign of $-q(q - p)(a - b)^2 \geq 0$ (and cyclically). This implies that

$$-\sum_{\text{cyc}} (ab)^p(a^{q-p} - b^{q-p})\frac{(a^{p+q} - b^{p+q}) + c^p(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a, b, c) \geq 0$, i.e.,

$$\frac{a^{p+q}(b^q + c^q)}{b^p + c^p} + \frac{b^{p+q}(c^q + a^q)}{c^p + a^p} + \frac{c^{p+q}(a^q + b^q)}{a^p + b^p} \geq a^{2q} + b^{2q} + c^{2q}.$$

- Case 2 ($q(q - p) > 0$): The term $-(a^{q-p} - b^{q-p})\left((a^{p+q} - b^{p+q}) + c^p(a^q - b^q)\right)$ carries the sign of $-q(q - p)(a - b)^2 \leq 0$ (and cyclically). This implies that

$$-\sum_{\text{cyc}} (ab)^p(a^{q-p} - b^{q-p})\frac{(a^{p+q} - b^{p+q}) + c^p(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \leq 0,$$

  so that $\mathcal{D}_{p,q}(a, b, c) \leq 0$, i.e.,

$$\frac{a^{p+q}(b^q + c^q)}{b^p + c^p} + \frac{b^{p+q}(c^q + a^q)}{c^p + a^p} + \frac{c^{p+q}(a^q + b^q)}{a^p + b^p} \leq a^{2q} + b^{2q} + c^{2q}.$$

The equality holds for $q(q - p) \neq 0$ and $a = b = c$, yielding a common value of $3a^{2q}$. □

Remark 4.11. *When $p=-2$ and $q=1$, the inequality in Theorem 4.10 reduces to*

$$\frac{bc(b+c)}{a^3(b^2+c^2)}+\frac{ca(c+a)}{b^3(c^2+a^2)}+\frac{ab(a+b)}{c^3(a^2+b^2)}\geq\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

When $p=1$ and $q=1/2$, the inequality simplifies to

$$\frac{a\sqrt{a}(\sqrt{b}+\sqrt{c})}{b+c}+\frac{b\sqrt{b}(\sqrt{c}+\sqrt{a})}{c+a}+\frac{c\sqrt{c}(\sqrt{a}+\sqrt{b})}{a+b}\geq a+b+c.$$

Theorem 4.11. *Let $a,b,c>0$ and $p,q\in\mathbb{R}$. The inequalities below hold based on the sign of the product $q(p+q)$.*

- *For $q(p+q)>0$ (i.e., q lies outside the interval between 0 and $-p$):*

$$\frac{a^{p+q}(b^p+c^p)}{b^{p+q}+c^{p+q}}+\frac{b^{p+q}(c^p+a^p)}{c^{p+q}+a^{p+q}}+\frac{c^{p+q}(a^p+b^p)}{a^{p+q}+b^{p+q}}\geq a^p+b^p+c^p.$$

- *For $q(p+q)<0$ (i.e., q lies strictly between 0 and $-p$):*

$$\frac{a^{p+q}(b^p+c^p)}{b^{p+q}+c^{p+q}}+\frac{b^{p+q}(c^p+a^p)}{c^{p+q}+a^{p+q}}+\frac{c^{p+q}(a^p+b^p)}{a^{p+q}+b^{p+q}}\leq a^p+b^p+c^p.$$

The equality holds for $q(p+q)\neq 0$ and $a=b=c$. When $q=0$ or $q=-p$, both sides are identically equal, with no restrictions on a,b,c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^{p+q}(b^p+c^p)}{b^{p+q}+c^{p+q}}-\sum_{\text{cyc}}a^p.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^{p+q}(b^p+c^p)-a^p(b^{p+q}+c^{p+q})}{b^{p+q}+c^{p+q}}.$$

Noting that

$$\begin{aligned}a^{p+q}(b^p+c^p)-a^p(b^{p+q}+c^{p+q})&=a^pa^qb^p+a^pa^qc^p-a^pb^pb^q-a^pc^pc^q\\&=(a^pa^qb^p-a^pb^pb^q)+(a^pa^qc^p-a^pc^pc^q)\\&=a^pb^p(a^q-b^q)-a^pc^p(c^q-a^q),\end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{a^pb^p(a^q-b^q)}{b^{p+q}+c^{p+q}}-\sum_{\text{cyc}}\frac{a^pc^p(c^q-a^q)}{b^{p+q}+c^{p+q}}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^pc^p(c^q-a^q)}{b^{p+q}+c^{p+q}}\longrightarrow\frac{b^pa^p(a^q-b^q)}{c^{p+q}+a^{p+q}}.$$

Therefore, we have

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)}{b^{p+q} + c^{p+q}} - \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)}{c^{p+q} + a^{p+q}} \\ &= \sum_{\text{cyc}} a^p b^p (a^q - b^q) \left(\frac{1}{b^{p+q} + c^{p+q}} - \frac{1}{c^{p+q} + a^{p+q}} \right) \\ &= \sum_{\text{cyc}} a^p b^p (a^q - b^q) \frac{(c^{p+q} + a^{p+q}) - (b^{p+q} + c^{p+q})}{(b^{p+q} + c^{p+q})(c^{p+q} + a^{p+q})} \\ &= \sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)(a^{p+q} - b^{p+q})}{(b^{p+q} + c^{p+q})(c^{p+q} + a^{p+q})}.\end{aligned}$$

Because $a, b, c > 0$, we have $a^p b^p > 0$ and $(b^{p+q} + c^{p+q})(c^{p+q} + a^{p+q}) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^q - b^q)(a^{p+q} - b^{p+q})$ (and cyclically).

- Case 1 ($q(p+q) > 0$): The exponents q and $p+q$ share the same sign. The power functions $f(x) = x^q$ and $g(x) = x^{p+q}$ share the exact same monotonicity. Therefore, $(a^q - b^q)$ and $(a^{p+q} - b^{p+q})$ always have the same sign, yielding $(a^q - b^q)(a^{p+q} - b^{p+q}) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)(a^{p+q} - b^{p+q})}{(b^{p+q} + c^{p+q})(c^{p+q} + a^{p+q})} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^{p+q}(b^p + c^p)}{b^{p+q} + c^{p+q}} + \frac{b^{p+q}(c^p + a^p)}{c^{p+q} + a^{p+q}} + \frac{c^{p+q}(a^p + b^p)}{a^{p+q} + b^{p+q}} \geq a^p + b^p + c^p.$$

- Case 2 ($q(p+q) < 0$): The exponents q and $p+q$ have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^q - b^q)(a^{p+q} - b^{p+q}) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^p b^p (a^q - b^q)(a^{p+q} - b^{p+q})}{(b^{p+q} + c^{p+q})(c^{p+q} + a^{p+q})} \leq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^{p+q}(b^p + c^p)}{b^{p+q} + c^{p+q}} + \frac{b^{p+q}(c^p + a^p)}{c^{p+q} + a^{p+q}} + \frac{c^{p+q}(a^p + b^p)}{a^{p+q} + b^{p+q}} \leq a^p + b^p + c^p.$$

The equality holds for $q(p+q) \neq 0$ and $a = b = c$, yielding a common value of $3a^p$. □

Remark 4.12. *When $p = -2$ and $q = 1$, the inequality in Theorem 4.11 reduces to*

$$\frac{1}{abc} \left(\frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} + \frac{a^2 + b^2}{a + b} \right) \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Theorem 4.12. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product $p(p+q)$.*

- *For $p(p+q) > 0$ (i.e., p lies outside the interval between 0 and $-q$):*

$$\frac{a^{p+2q}(b^q+c^q)}{b^p+c^p} + \frac{b^{p+2q}(c^q+a^q)}{c^p+a^p} + \frac{c^{p+2q}(a^q+b^q)}{a^p+b^p} \geq \frac{a^q(b^{p+2q}+c^{p+2q})}{b^p+c^p} + \frac{b^q(c^{p+2q}+a^{p+2q})}{c^p+a^p} + \frac{c^q(a^{p+2q}+b^{p+2q})}{a^p+b^p}.$$

- *For $p(p+q) < 0$ (i.e., p lies strictly between 0 and $-q$):*

$$\frac{a^{p+2q}(b^q+c^q)}{b^p+c^p} + \frac{b^{p+2q}(c^q+a^q)}{c^p+a^p} + \frac{c^{p+2q}(a^q+b^q)}{a^p+b^p} \leq \frac{a^q(b^{p+2q}+c^{p+2q})}{b^p+c^p} + \frac{b^q(c^{p+2q}+a^{p+2q})}{c^p+a^p} + \frac{c^q(a^{p+2q}+b^{p+2q})}{a^p+b^p}.$$

The equality holds for $p(p+q) \neq 0$ and $a = b = c$ (when $p = 0$ or $p = -q$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+2q}(b^q+c^q)}{b^p+c^p} - \sum_{\text{cyc}} \frac{a^q(b^{p+2q}+c^{p+2q})}{b^p+c^p}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{p+2q}(b^q+c^q) - a^q(b^{p+2q}+c^{p+2q})}{b^p+c^p}.$$

Observing that

$$\begin{aligned} &a^{p+2q}(b^q+c^q) - a^q(b^{p+2q}+c^{p+2q}) = a^{p+2q}b^q + a^{p+2q}c^q - a^q b^{p+2q} - a^q c^{p+2q} \\ &= a^q a^{p+q} b^q + a^q a^{p+q} c^q - a^q b^q b^{p+q} - a^q c^q c^{p+q} \\ &= (a^q a^{p+q} b^q - a^q b^q b^{p+q}) + (a^q a^{p+q} c^q - a^q c^q c^{p+q}) \\ &= a^q b^q (a^{p+q} - b^{p+q}) - a^q c^q (c^{p+q} - a^{p+q}), \end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^q b^q (a^{p+q} - b^{p+q})}{b^p+c^p} - \sum_{\text{cyc}} \frac{a^q c^q (c^{p+q} - a^{p+q})}{b^p+c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^q c^q (c^{p+q} - a^{p+q})}{b^p+c^p} \longrightarrow \frac{b^q a^q (a^{p+q} - b^{p+q})}{c^p+a^p}.$$

Therefore, we have

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^q b^q (a^{p+q} - b^{p+q})}{b^p + c^p} - \sum_{\text{cyc}} \frac{a^q b^q (a^{p+q} - b^{p+q})}{c^p + a^p} \\ &= \sum_{\text{cyc}} a^q b^q (a^{p+q} - b^{p+q}) \left(\frac{1}{b^p + c^p} - \frac{1}{a^p + c^p} \right) \\ &= \sum_{\text{cyc}} a^q b^q (a^{p+q} - b^{p+q}) \frac{(a^p + c^p) - (b^p + c^p)}{(b^p + c^p)(a^p + c^p)} \\ &= \sum_{\text{cyc}} \frac{a^q b^q (a^{p+q} - b^{p+q})(a^p - b^p)}{(b^p + c^p)(c^p + a^p)}.\end{aligned}$$

Because $a, b, c > 0$, we have $a^q b^q > 0$ and $(b^p + c^p)(c^p + a^p) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^{p+q} - b^{p+q})(a^p - b^p)$ (and cyclically).

- Case 1 ($p(p+q) > 0$): The exponents p and $p+q$ share the same sign. The power functions $f(x) = x^{p+q}$ and $g(x) = x^p$ share the exact same monotonicity. Therefore, $(a^{p+q} - b^{p+q})$ and $(a^p - b^p)$ always have the same sign, yielding $(a^{p+q} - b^{p+q})(a^p - b^p) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^q b^q (a^{p+q} - b^{p+q})(a^p - b^p)}{(b^p + c^p)(c^p + a^p)} \geq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\begin{aligned}&\frac{a^{p+2q}(b^q + c^q)}{b^p + c^p} + \frac{b^{p+2q}(c^q + a^q)}{c^p + a^p} + \frac{c^{p+2q}(a^q + b^q)}{a^p + b^p} \\ &\geq \frac{a^q(b^{p+2q} + c^{p+2q})}{b^p + c^p} + \frac{b^q(c^{p+2q} + a^{p+2q})}{c^p + a^p} + \frac{c^q(a^{p+2q} + b^{p+2q})}{a^p + b^p}.\end{aligned}$$

- Case 2 ($p(p+q) < 0$): The exponents p and $p+q$ have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^{p+q} - b^{p+q})(a^p - b^p) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^q b^q (a^{p+q} - b^{p+q})(a^p - b^p)}{(b^p + c^p)(c^p + a^p)} \leq 0,$$

  so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\begin{aligned}&\frac{a^{p+2q}(b^q + c^q)}{b^p + c^p} + \frac{b^{p+2q}(c^q + a^q)}{c^p + a^p} + \frac{c^{p+2q}(a^q + b^q)}{a^p + b^p} \\ &\leq \frac{a^q(b^{p+2q} + c^{p+2q})}{b^p + c^p} + \frac{b^q(c^{p+2q} + a^{p+2q})}{c^p + a^p} + \frac{c^q(a^{p+2q} + b^{p+2q})}{a^p + b^p}.\end{aligned}$$

The equality holds for $p(p+q) \neq 0$ and $a = b = c$, yielding a common value of $3a^{3q}$. □

Remark 4.13. *When $p = -2$ and $q = 1$, the inequality in Theorem 4.12 reduces to*

$$\frac{b+c}{a^2(b^2+c^2)} + \frac{c+a}{b^2(c^2+a^2)} + \frac{a+b}{c^2(a^2+b^2)} \geq \frac{2}{a(b^2+c^2)} + \frac{2}{b(c^2+a^2)} + \frac{2}{c(a^2+b^2)}.$$

Theorem 4.13. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^{p+q}+b^{p+q}}{a^p+b^p}+\frac{b^{p+q}+c^{p+q}}{b^p+c^p}+\frac{c^{p+q}+a^{p+q}}{c^p+a^p}\geq a^q+b^q+c^q.$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^{p+q}+b^{p+q}}{a^p+b^p}+\frac{b^{p+q}+c^{p+q}}{b^p+c^p}+\frac{c^{p+q}+a^{p+q}}{c^p+a^p}\leq a^q+b^q+c^q.$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, multiplied by 2, as follows:

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{2(a^{p+q}+b^{p+q})}{a^p+b^p}-2(a^q+b^q+c^q).$$

Noting that $2(a^q + b^q + c^q) = \sum_{\text{cyc}}(a^q + b^q)$ and combining the terms into the sums over a common denominator, we obtain

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c)&=\sum_{\text{cyc}}\frac{2(a^{p+q}+b^{p+q})}{a^p+b^p}-\sum_{\text{cyc}}(a^q+b^q)\\&=\sum_{\text{cyc}}\frac{2(a^{p+q}+b^{p+q})-(a^q+b^q)(a^p+b^p)}{a^p+b^p}.\end{aligned}$$

Observing that

$$\begin{aligned}&2(a^{p+q}+b^{p+q})-(a^q+b^q)(a^p+b^p)=2a^{p+q}+2b^{p+q}-(a^{p+q}+a^qb^p+a^pb^q+b^{p+q})\\&=2a^{p+q}+2b^{p+q}-a^{p+q}-a^qb^p-a^pb^q-b^{p+q}\\&=a^{p+q}+b^{p+q}-a^qb^p-a^pb^q\\&=(a^pa^q-a^pb^q)-(a^qb^p-b^pb^q)\\&=a^p(a^q-b^q)-b^p(a^q-b^q)\\&=(a^p-b^p)(a^q-b^q),\end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a,b,c)=\sum_{\text{cyc}}\frac{(a^p-b^p)(a^q-b^q)}{a^p+b^p}.$$

Because $a, b, c > 0$, we have $a^p + b^p > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and

$(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^p + b^p} \geq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \geq 0$, i.e.,

$$\frac{a^{p+q} + b^{p+q}}{a^p + b^p} + \frac{b^{p+q} + c^{p+q}}{b^p + c^p} + \frac{c^{p+q} + a^{p+q}}{c^p + a^p} \geq a^q + b^q + c^q.$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^q - b^q)}{a^p + b^p} \leq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \leq 0$, i.e.,

$$\frac{a^{p+q} + b^{p+q}}{a^p + b^p} + \frac{b^{p+q} + c^{p+q}}{b^p + c^p} + \frac{c^{p+q} + a^{p+q}}{c^p + a^p} \leq a^q + b^q + c^q.$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $3a^q$. □

Remark 4.14. *When $p = -1$ and $q = -1$, the inequality in Theorem 4.13 simplifies to*

$$\frac{a^2 + b^2}{ab(a + b)} + \frac{b^2 + c^2}{bc(b + c)} + \frac{c^2 + a^2}{ca(c + a)} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Theorem 4.14. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^p b^q + a^q b^p}{a^p + b^p} + \frac{b^p c^q + b^q c^p}{b^p + c^p} + \frac{c^p a^q + c^q a^p}{c^p + a^p} \leq a^q + b^q + c^q.$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^p b^q + a^q b^p}{a^p + b^p} + \frac{b^p c^q + b^q c^p}{b^p + c^p} + \frac{c^p a^q + c^q a^p}{c^p + a^p} \geq a^q + b^q + c^q.$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a, b, c)$ by subtracting the left-hand side of the first target inequality from the right-hand side, as follows:

$$\mathcal{D}_{p,q}(a, b, c) = a^q + b^q + c^q - \sum_{\text{cyc}} \frac{a^p b^q + a^q b^p}{a^p + b^p}.$$

Noting that $a^q+b^q+c^q = \sum_{\text{cyc}}(a^q+b^q)/2$ and combining the terms into the sums over a common denominator, we can write

$$\begin{aligned}\mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^q+b^q}{2} - \sum_{\text{cyc}} \frac{a^pb^q+a^qb^p}{a^p+b^p} \\ &= \sum_{\text{cyc}} \frac{(a^q+b^q)(a^p+b^p)-2(a^pb^q+a^qb^p)}{2(a^p+b^p)}.\end{aligned}$$

Expanding the numerator yields

$$\begin{aligned}&(a^q+b^q)(a^p+b^p)-2(a^pb^q+a^qb^p) = a^{p+q}+a^qb^p+a^pb^q+b^{p+q}-2a^pb^q-2a^qb^p \\ &= a^{p+q}+b^{p+q}-a^pb^q-a^qb^p \\ &= (a^p-b^p)(a^q-b^q).\end{aligned}$$

Therefore, we have

$$\mathcal{D}_{p,q}(a,b,c) = \frac{1}{2}\sum_{\text{cyc}} \frac{(a^p-b^p)(a^q-b^q)}{a^p+b^p}.$$

Because $a,b,c>0$, we have $a^p+b^p>0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p-b^p)(a^q-b^q)$ (and cyclically).

- Case 1 ($pq>0$): The exponents p and q share the same sign. The power functions $f(x)=x^p$ and $g(x)=x^q$ share the exact same monotonicity. Therefore, (a^p-b^p) and (a^q-b^q) always have the same sign, yielding $(a^p-b^p)(a^q-b^q)\geq 0$ (and cyclically). This implies that
$$\frac{1}{2}\sum_{\text{cyc}} \frac{(a^p-b^p)(a^q-b^q)}{a^p+b^p} \geq 0,$$
so that $\mathcal{D}_{p,q}(a,b,c)\geq 0$, i.e.,
$$\frac{a^pb^q+a^qb^p}{a^p+b^p}+\frac{b^pc^q+b^qc^p}{b^p+c^p}+\frac{c^pa^q+c^qa^p}{c^p+a^p} \leq a^q+b^q+c^q.$$
- Case 2 ($pq<0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p-b^p)(a^q-b^q)\leq 0$ (and cyclically). This implies that
$$\frac{1}{2}\sum_{\text{cyc}} \frac{(a^p-b^p)(a^q-b^q)}{a^p+b^p} \leq 0,$$
so that $\mathcal{D}_{p,q}(a,b,c)\leq 0$, i.e.,
$$\frac{a^pb^q+a^qb^p}{a^p+b^p}+\frac{b^pc^q+b^qc^p}{b^p+c^p}+\frac{c^pa^q+c^qa^p}{c^p+a^p} \geq a^q+b^q+c^q.$$

The equality holds for $pq\neq 0$ and $a=b=c$, yielding a common value of $3a^q$. □

Remark 4.15. *When $p=2$ and $q=1$, the inequality in Theorem 4.14 simplifies to*

$$\frac{ab(a+b)}{a^2+b^2}+\frac{bc(b+c)}{b^2+c^2}+\frac{ca(c+a)}{c^2+a^2} \leq a+b+c.$$

Theorem 4.15. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product $p(p-q)$.*

- *For $p(p-q) > 0$ (i.e., p lies outside the interval between 0 and q):*

$$\frac{a^p(b^q+c^q)}{b^p+c^p} + \frac{b^p(c^q+a^q)}{c^p+a^p} + \frac{c^p(a^q+b^q)}{a^p+b^p} \geq a^q + b^q + c^q.$$

- *For $p(p-q) < 0$ (i.e., p lies strictly between 0 and q):*

$$\frac{a^p(b^q+c^q)}{b^p+c^p} + \frac{b^p(c^q+a^q)}{c^p+a^p} + \frac{c^p(a^q+b^q)}{a^p+b^p} \leq a^q + b^q + c^q.$$

The equality holds for $p(p-q) \neq 0$ and $a = b = c$ (when $p = 0$ or $p = q$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^p(b^q+c^q)}{b^p+c^p} - \sum_{\text{cyc}} a^q.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^p(b^q+c^q) - a^q(b^p+c^p)}{b^p+c^p}.$$

Observing that

$$\begin{aligned} a^p(b^q+c^q) - a^q(b^p+c^p) &= a^pb^q + a^pc^q - a^qb^p - a^qc^p \\ &= (a^pb^q - a^qb^p) + (a^pc^q - a^qc^p) \\ &= a^qb^q(a^{p-q} - b^{p-q}) + a^qc^q(a^{p-q} - c^{p-q}), \end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^qb^q(a^{p-q} - b^{p-q})}{b^p+c^p} + \sum_{\text{cyc}} \frac{a^qc^q(a^{p-q} - c^{p-q})}{b^p+c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^qc^q(a^{p-q} - c^{p-q})}{b^p+c^p} \longrightarrow \frac{b^qa^q(b^{p-q} - a^{p-q})}{c^p+a^p}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^qb^q(a^{p-q} - b^{p-q})}{b^p+c^p} + \sum_{\text{cyc}} \frac{b^qa^q(b^{p-q} - a^{p-q})}{c^p+a^p} \\ &= \sum_{\text{cyc}} a^qb^q(a^{p-q} - b^{p-q}) \left(\frac{1}{b^p+c^p} - \frac{1}{c^p+a^p} \right) \\ &= \sum_{\text{cyc}} a^qb^q(a^{p-q} - b^{p-q}) \frac{(c^p+a^p) - (b^p+c^p)}{(b^p+c^p)(c^p+a^p)} = \sum_{\text{cyc}} \frac{a^qb^q(a^{p-q} - b^{p-q})(a^p - b^p)}{(b^p+c^p)(c^p+a^p)}. \end{aligned}$$

Because $a, b, c > 0$, we have $a^qb^q > 0$ and $(b^p+c^p)(c^p+a^p) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^{p-q} - b^{p-q})(a^p - b^p)$ (and cyclically).

- Case 1 ($p(p-q) > 0$): The exponents p and $p-q$ share the same sign. The power functions $f(x) = x^{p-q}$ and $g(x) = x^p$ share the exact same monotonicity. Therefore, $(a^{p-q} - b^{p-q})$ and $(a^p - b^p)$ always have the same sign, yielding $(a^{p-q} - b^{p-q})(a^p - b^p) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^q b^q (a^{p-q} - b^{p-q})(a^p - b^p)}{(b^p + c^p)(c^p + a^p)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^p(b^q + c^q)}{b^p + c^p} + \frac{b^p(c^q + a^q)}{c^p + a^p} + \frac{c^p(a^q + b^q)}{a^p + b^p} \geq a^q + b^q + c^q.$$

- Case 2 ($p(p-q) < 0$): The exponents p and $p-q$ have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^{p-q} - b^{p-q})(a^p - b^p) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{a^q b^q (a^{p-q} - b^{p-q})(a^p - b^p)}{(b^p + c^p)(c^p + a^p)} \leq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^p(b^q + c^q)}{b^p + c^p} + \frac{b^p(c^q + a^q)}{c^p + a^p} + \frac{c^p(a^q + b^q)}{a^p + b^p} \leq a^q + b^q + c^q.$$

The equality holds for $p(p-q) \neq 0$ and $a = b = c$, yielding a common value of $3a^q$. □

Remark 4.16. *When $p = 2$ and $q = 1$, the inequality in Theorem 4.15 simplifies to*

$$\frac{a^2(b+c)}{b^2 + c^2} + \frac{b^2(c+a)}{c^2 + a^2} + \frac{c^2(a+b)}{a^2 + b^2} \geq a + b + c.$$

When $p = 1$ and $q = 2$, the inequality reduces to

$$\frac{a(b^2 + c^2)}{b + c} + \frac{b(c^2 + a^2)}{c + a} + \frac{c(a^2 + b^2)}{a + b} \leq a^2 + b^2 + c^2.$$

Together, these results form a complementary pair, illustrating how exchanging the degrees in the rational terms reverses the bound.

Theorem 4.16. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the signs and relative magnitudes of p and q.*

- *For $pq \geq 0$ (i.e., p and q share the same sign, or at least one is zero):*

$$\frac{a^{p+q}}{b^p + c^p} + \frac{b^{p+q}}{c^p + a^p} + \frac{c^{p+q}}{a^p + b^p} \geq \frac{a^q + b^q + c^q}{2}.$$

- *For $pq < 0$ and $p(p+q) \leq 0$ (i.e., q has the opposite sign of p and $|q| \geq |p|$):*

$$\frac{a^{p+q}}{b^p + c^p} + \frac{b^{p+q}}{c^p + a^p} + \frac{c^{p+q}}{a^p + b^p} \leq \frac{a^q + b^q + c^q}{2}.$$

The equality holds for $p \neq 0$ and $a = b = c$ (when $p = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, multiplied by 2, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^{p+q}}{b^p + c^p} - \sum_{\text{cyc}} a^q.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{2a^{p+q} - a^q(b^p + c^p)}{b^p + c^p}.$$

Noting that

$$\begin{aligned} 2a^{p+q} - a^q(b^p + c^p) &= a^q a^p + a^q a^p - a^q b^p - a^q c^p \\ &= (a^q a^p - a^q b^p) + (a^q a^p - a^q c^p) \\ &= a^q(a^p - b^p) - a^q(c^p - a^p), \end{aligned}$$

we can write

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^q(a^p - b^p)}{b^p + c^p} - \sum_{\text{cyc}} \frac{a^q(c^p - a^p)}{b^p + c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^q(c^p - a^p)}{b^p + c^p} \longrightarrow \frac{b^q(a^p - b^p)}{c^p + a^p}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{a^q(a^p - b^p)}{b^p + c^p} - \sum_{\text{cyc}} \frac{b^q(a^p - b^p)}{c^p + a^p} \\ &= \sum_{\text{cyc}} (a^p - b^p) \left(\frac{a^q}{b^p + c^p} - \frac{b^q}{c^p + a^p} \right) \\ &= \sum_{\text{cyc}} (a^p - b^p) \frac{a^q(c^p + a^p) - b^q(b^p + c^p)}{(b^p + c^p)(c^p + a^p)}. \end{aligned}$$

Observing that

$$\begin{aligned} a^q(c^p + a^p) - b^q(b^p + c^p) &= a^q c^p + a^{p+q} - b^{p+q} - b^q c^p \\ &= (a^{p+q} - b^{p+q}) + c^p(a^q - b^q), \end{aligned}$$

we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} (a^p - b^p) \frac{(a^{p+q} - b^{p+q}) + c^p(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \\ &= \sum_{\text{cyc}} \frac{(a^p - b^p)(a^{p+q} - b^{p+q}) + c^p(a^p - b^p)(a^q - b^q)}{(b^p + c^p)(c^p + a^p)}. \end{aligned}$$

Because $a, b, c > 0$, we have $(b^p + c^p)(c^p + a^p) > 0$ (and cyclically). The sign of the entire sum relies strictly on the two difference products in the numerator, $T_1(x,y) = (x^p - y^p)(x^{p+q} - y^{p+q})$ and $T_2(x,y) = (x^p - y^p)(x^q - y^q)$.

- Case 1 ($pq \geq 0$): Since p and q share the same sign (or at least one is zero), p and $p+q$ also share the same sign. The power functions $f(x) = x^p$, $g(x) = x^q$, and $h(x) = x^{p+q}$ all share identical monotonicity. Therefore, $(a^p - b^p)$ has the same sign as both $(a^{p+q} - b^{p+q})$ and $(a^q - b^q)$, yielding $T_1(a,b) \geq 0$ and $T_2(a,b) \geq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^{p+q} - b^{p+q}) + c^p(a^p - b^p)(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \geq 0$, i.e.,

$$\frac{a^{p+q}}{b^p + c^p} + \frac{b^{p+q}}{c^p + a^p} + \frac{c^{p+q}}{a^p + b^p} \geq \frac{a^q + b^q + c^q}{2}.$$

- Case 2 ($pq < 0$ and $p(p+q) \leq 0$): The exponent q has the opposite sign of p with $|q| \geq |p|$, which forces $p+q$ to also have the opposite sign of p. Consequently, $g(x) = x^q$ and $h(x) = x^{p+q}$ have the opposite monotonicity of $f(x) = x^p$, forcing $T_1(a,b) \leq 0$ and $T_2(a,b) \leq 0$ (and cyclically). This implies that

$$\sum_{\text{cyc}} \frac{(a^p - b^p)(a^{p+q} - b^{p+q}) + c^p(a^p - b^p)(a^q - b^q)}{(b^p + c^p)(c^p + a^p)} \leq 0,$$

so that $\mathcal{D}_{p,q}(a,b,c) \leq 0$, i.e.,

$$\frac{a^{p+q}}{b^p + c^p} + \frac{b^{p+q}}{c^p + a^p} + \frac{c^{p+q}}{a^p + b^p} \leq \frac{a^q + b^q + c^q}{2}.$$

The equality holds for $p \neq 0$ and $a = b = c$, yielding a common value of $(3/2)a^q$. □

Remark 4.17. *When $p = 1$ and $q = -1$, the inequality in Theorem 4.16 simplifies to*

$$\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

When $p = 1$ and $q = -2$, the inequality reduces to

$$\frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \leq \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

Theorem 4.17. *Let $a, b, c > 0$ and $p, q \in \mathbb{R}$. The inequalities below hold based on the sign of the product pq.*

- *For $pq > 0$ (i.e., p and q share the same sign):*

$$\frac{a^{2p}(b^{p+q} + c^{p+q})}{b^p + c^p} + \frac{b^{2p}(c^{p+q} + a^{p+q})}{c^p + a^p} + \frac{c^{2p}(a^{p+q} + b^{p+q})}{a^p + b^p} \leq a^{2p+q} + b^{2p+q} + c^{2p+q}.$$

- *For $pq < 0$ (i.e., p and q have opposite signs):*

$$\frac{a^{2p}(b^{p+q} + c^{p+q})}{b^p + c^p} + \frac{b^{2p}(c^{p+q} + a^{p+q})}{c^p + a^p} + \frac{c^{2p}(a^{p+q} + b^{p+q})}{a^p + b^p} \geq a^{2p+q} + b^{2p+q} + c^{2p+q}.$$

The equality holds for $pq \neq 0$ and $a = b = c$ (when $p = 0$ or $q = 0$, both sides are identically equal, with no restrictions on a, b, c).

Proof. Let us define the difference function $\mathcal{D}_{p,q}(a,b,c)$ by subtracting the right-hand side of the first target inequality from its left-hand side, as follows:

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{2p}(b^{p+q}+c^{p+q})}{b^p+c^p} - \sum_{\text{cyc}} a^{2p+q}.$$

Combining the terms into the sums over a common denominator yields

$$\mathcal{D}_{p,q}(a,b,c) = \sum_{\text{cyc}} \frac{a^{2p}(b^{p+q}+c^{p+q}) - a^{2p+q}(b^p+c^p)}{b^p+c^p}.$$

Noting that

$$\begin{aligned} a^{2p}(b^{p+q}+c^{p+q}) - a^{2p+q}(b^p+c^p) &= a^{2p}b^pb^q + a^{2p}c^pc^q - a^{2p}a^qb^p - a^{2p}a^qc^p \\ &= (a^{2p}b^pb^q - a^{2p}a^qb^p) + (a^{2p}c^pc^q - a^{2p}a^qc^p) \\ &= -a^{2p}b^p(a^q-b^q) - a^{2p}c^p(a^q-c^q), \end{aligned}$$

we derive

$$\mathcal{D}_{p,q}(a,b,c) = -\sum_{\text{cyc}} \frac{a^{2p}b^p(a^q-b^q)}{b^p+c^p} - \sum_{\text{cyc}} \frac{a^{2p}c^p(a^q-c^q)}{b^p+c^p}.$$

We shift the indices in the second cyclic sum to align the numerators. The main term shifts as follows:

$$\frac{a^{2p}c^p(a^q-c^q)}{b^p+c^p} \longrightarrow \frac{b^{2p}a^p(b^q-a^q)}{c^p+a^p} = -\frac{b^{2p}a^p(a^q-b^q)}{c^p+a^p}.$$

Therefore, we have

$$\begin{aligned} \mathcal{D}_{p,q}(a,b,c) &= \sum_{\text{cyc}} \frac{b^{2p}a^p(a^q-b^q)}{c^p+a^p} - \sum_{\text{cyc}} \frac{a^{2p}b^p(a^q-b^q)}{b^p+c^p} \\ &= \sum_{\text{cyc}} (a^q-b^q)\left(\frac{a^pb^{2p}}{c^p+a^p} - \frac{a^{2p}b^p}{b^p+c^p}\right) \\ &= \sum_{\text{cyc}} (a^q-b^q)\frac{a^pb^{2p}(b^p+c^p) - a^{2p}b^p(c^p+a^p)}{(c^p+a^p)(b^p+c^p)}. \end{aligned}$$

Observing that

$$\begin{aligned} a^pb^{2p}(b^p+c^p) - a^{2p}b^p(c^p+a^p) &= a^pb^{3p} + a^pb^{2p}c^p - a^{2p}b^pc^p - a^{3p}b^p \\ &= (a^pb^{3p} - a^{3p}b^p) + (a^pb^{2p}c^p - a^{2p}b^pc^p) \\ &= -a^pb^p(a^{2p}-b^{2p}) - a^pb^pc^p(a^p-b^p) \\ &= -a^pb^p(a^p-b^p)(a^p+b^p) - a^pb^pc^p(a^p-b^p) \\ &= -a^pb^p(a^p-b^p)(a^p+b^p+c^p), \end{aligned}$$

we have

$$\mathcal{D}_{p,q}(a,b,c) = -(a^p+b^p+c^p)\sum_{\text{cyc}} \frac{a^pb^p(a^p-b^p)(a^q-b^q)}{(c^p+a^p)(b^p+c^p)}.$$

Because $a, b, c > 0$, we have $a^p + b^p + c^p > 0$, $a^p b^p > 0$, and $(c^p + a^p)(b^p + c^p) > 0$ (and cyclically). The sign of the entire sum relies strictly on the product of differences $(a^p - b^p)(a^q - b^q)$ (and cyclically).

- Case 1 ($pq > 0$): The exponents p and q share the same sign. The power functions $f(x) = x^p$ and $g(x) = x^q$ share the exact same monotonicity. Therefore, $(a^p - b^p)$ and $(a^q - b^q)$ always have the same sign, yielding $(a^p - b^p)(a^q - b^q) \geq 0$ (and cyclically). This implies that

$$-(a^p + b^p + c^p) \sum_{\text{cyc}} \frac{a^p b^p (a^p - b^p)(a^q - b^q)}{(c^p + a^p)(b^p + c^p)} \leq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \leq 0$, i.e.,

$$\frac{a^{2p}(b^{p+q} + c^{p+q})}{b^p + c^p} + \frac{b^{2p}(c^{p+q} + a^{p+q})}{c^p + a^p} + \frac{c^{2p}(a^{p+q} + b^{p+q})}{a^p + b^p} \leq a^{2p+q} + b^{2p+q} + c^{2p+q}.$$

- Case 2 ($pq < 0$): The exponents p and q have opposite signs. One function is strictly increasing while the other is strictly decreasing. Consequently, the two difference factors will always carry opposite signs, forcing $(a^p - b^p)(a^q - b^q) \leq 0$ (and cyclically). This implies that

$$-(a^p + b^p + c^p) \sum_{\text{cyc}} \frac{a^p b^p (a^p - b^p)(a^q - b^q)}{(c^p + a^p)(b^p + c^p)} \geq 0,$$

so that $\mathcal{D}_{p,q}(a, b, c) \geq 0$, i.e.,

$$\frac{a^{2p}(b^{p+q} + c^{p+q})}{b^p + c^p} + \frac{b^{2p}(c^{p+q} + a^{p+q})}{c^p + a^p} + \frac{c^{2p}(a^{p+q} + b^{p+q})}{a^p + b^p} \geq a^{2p+q} + b^{2p+q} + c^{2p+q}.$$

The equality holds for $pq \neq 0$ and $a = b = c$, yielding a common value of $3a^{2p+q}$. □

Remark 4.18. *When $p = -1$ and $q = 1$, the inequality in Theorem 4.17 simplifies to*

$$2\left(\frac{bc}{a^2(b + c)} + \frac{ca}{b^2(c + a)} + \frac{ab}{c^2(a + b)}\right) \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

When $p = 1$ and $q = 2$, the inequality reduces to

$$\frac{a^2(b^3 + c^3)}{b + c} + \frac{b^2(c^3 + a^3)}{c + a} + \frac{c^2(a^3 + b^3)}{a + b} \leq a^4 + b^4 + c^4.$$

5. Conclusion

In conclusion, the two-power-parameter framework developed in this study unifies and generalizes several classical cyclic inequalities. Through an analysis of cyclic difference functions, we establish unconditional inequalities valid for any $p, q \in \mathbb{R}$, and conditional inequalities whose validity strictly depends on the sign of parametric products such as pq or $p(p + q)$. This comprehensive approach offers a robust algebraic tool set for discovering, proving, and extending diverse classes of cyclic inequalities.

Regarding limitations, the equality conditions are treated as sufficient conditions rather than necessary ones. Furthermore, several of the derived inequalities could potentially be refined

using alternative analytical techniques, such as sharp identities derived from the Titu lemma or its general power form: the Radon inequality.

Future research will address these limitations by establishing exact equality characterizations, extending the framework to higher-order parameter families, exploring multiple-variable generalizations, and applying these techniques to applied problems.

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