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Convergence Analysis of Iterative Schemes for Variational Inequalities with Demi-Contractive and Quasi-Nonexpansive Mappings in Hilbert Space

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Abstract. This study investigates the convergence properties of iterative algorithms designed for addressing variational inequalities that involve demi-contractive and quasi-nonexpansive mappings within Hilbert spaces. By imposing appropriate conditions, we demonstrate the strong convergence of the developed iterative process to the unique solution of the respective variational inequality. The findings contribute novel perspectives on the convergence dynamics of iterative techniques when applied to nonlinear mathematical problems.

1. Introduction

The theoretical foundations of iterative schemes have garnered considerable focus in recent years due to their pivotal role in solving nonlinear equations and optimization problems. Specifically, the convergence analysis of iterative algorithms in Hilbert spaces has far-reaching implications in various fields, including signal processing, image reconstruction, machine learning, and game theory. This paper focuses on a novel iterative scheme that combines the merits of strongly monotone and Lipschitzian operators, demi-contractive mappings, and quasi-nonexpansive mappings. The proposed algorithm is designed to solve variational inequalities, a class of problems ubiquitous in applied mathematics, economics, and engineering (see [14,15], [2,16,17], [7–10,19], and [11,20]).

For the past few decades, variational inclusion has played a crucial role in optimization, nonlinear analysis, and fixed point theory. Recently, authors have contributed to this area of research due

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to its applications in diverse fields such as engineering, economics, physics, machine learning, signal processing, game theory, and computer science. These contributions provide a framework to model and solve various real-world problems, including equilibrium problems, optimization problems, and variational inequalities. Fixed point problems, on the other hand, deal with finding points that remain unchanged under the action of an operator. The concept of fixed points has profound implications in mathematics and its applications. A wide range of problems in analysis, differential equations, and optimization theory can be reduced to fixed point problems. They serve as powerful tools to prove the existence and uniqueness of solutions, compute approximations, and establish convergence properties of iterative algorithms.

Quasi-nonexpansive and demi-contractive operators play a pivotal role in variational inclusion and fixed point problems. These operators possess certain properties that ensure the stability and convergence of iterative algorithms. (see [1, 2], [3, 4], [18, 21], [7, 12], and [22]).

It is our purpose in this paper to develop an innovative iterative scheme that seamlessly integrates demi-contractive and quasi-nonexpansive mappings. By combining these mapping types, our algorithm leverages their complementary strengths to tackle complex variational inequalities in Hilbert spaces. A comprehensive convergence analysis is conducted under specific assumptions, including the demi-closedness of mapping, Lipschitz continuity, and strategically chosen sequence parameters. These assumptions ensure the algorithm's stability and convergence properties. Our theorems generalize and extend many other results in this directions.

2. Mathematical Preliminaries

Throughout this paper, let H denote a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. An operator $\Lambda : H \to H$ is called strongly positive bounded linear if a constant $k > 0$ exists such that

$$\langle \Lambda\xi, \xi \rangle \geq k\|\xi\|^2, \quad \forall \xi \in H.$$

An operator $\Lambda : H \to H$ is monotone if for all $\xi, \omega \in H$, $\langle \Lambda\xi - \Lambda\omega, \xi - \omega \rangle \geq 0$, and it is m-strongly monotone if there exists a constant $m > 0$ such that

$$\langle \Lambda\xi - \Lambda\omega, \xi - \omega \rangle \geq m\|\xi - \omega\|^2, \quad \forall \xi, \omega \in H.$$

Consider a nonempty subset $C \subseteq H$. A mapping $T : C \to C$ is Lipschitz continuous if a constant $L > 0$ exists with

$$\|T\xi - T\omega\| \leq L\|\xi - \omega\|, \quad \forall \xi, \omega \in C.$$

If $L < 1$, T is a contraction; if $L = 1$, it is nonexpansive. The fixed point set of T is defined as $\mathrm{Fix}(T) := \{\xi \in D(T) : \xi = T\xi\}$, which is assumed to be nonempty. For nonexpansive maps, $\mathrm{Fix}(T)$ is known to be closed and convex. A mapping T is quasi-nonexpansive if

$$\|T\xi - q\| \leq \|\xi - q\|, \quad \forall \xi \in C, q \in \mathrm{Fix}(T).$$

Further, T is firmly nonexpansive if $\|T\xi - T\omega\|^2 \leq \|\xi - \omega\|^2 - \|(\xi - \omega) - (T\xi - T\omega)\|^2$, $\forall \xi, \omega \in C$, and κ-strictly pseudo-contractive if for some $\kappa \in [0, 1)$,

$$\|T\xi - T\omega\|^2 \leq \|\xi - \omega\|^2 + \kappa\|\xi - \omega - (T\xi - T\omega)\|^2, \quad \forall \xi, \omega \in C.$$

A mapping $T : C \to C$ is κ-demi-contractive if $\text{Fix}(T) \neq \emptyset$, and

$$\|T\xi - q\|^2 \leq \|\xi - q\|^2 + \kappa\|\xi - T\xi\|^2, \quad \forall \xi \in C, q \in \text{Fix}(T).$$

Let C be a nonempty closed convex subset of H. The metric projection $P_C : H \to C$ associates with each $\xi \in H$ the unique point $P_C\xi \in C$ satisfying

$$\|\xi - P_C\xi\| \leq \|\xi - \omega\|, \quad \forall \omega \in C, \quad \text{and} \quad \langle \xi - P_C\xi, \omega - P_C\xi \rangle \leq 0, \quad \forall \omega \in C. \quad (1)$$

Lemma 1. *[13] For $\xi, \omega \in H$ and any $\lambda \in (0, 1)$:*

$$\|\xi - \omega\|^2 \leq \|\xi\|^2 + 2\langle \omega, \xi - \omega \rangle, \quad (2) \qquad (1)$$

$$\|\lambda\xi + (1 - \lambda)\omega\|^2 \leq \lambda\|\xi\|^2 + (1 - \lambda)\|\omega\|^2 - (1 - \lambda)\lambda\|\xi - \omega\|^2. \quad (3) \qquad (2)$$

Lemma 2. *[11] Suppose $\{a_n\}$ is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \beta_n)a_n + \delta_n,$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\}$ are nonnegative sequences such that $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n\to\infty} \frac{\delta_n}{\beta_n} \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 3. *(Wang [1]) Consider H a real Hilbert space and $K \subseteq H$ a nonempty closed convex set. Let $\Lambda : H \to H$ be an m-strongly monotone and L-Lipschitz operator with constants $m > 0$ and $L > 0$. For η satisfying $0 < \eta < \frac{2m}{L^2}$, and $\tau = \eta(m - \frac{L^2\eta}{2})$, it holds for all $t \in (0, \min\{1, \frac{1}{\tau}\}]$, and all $\xi, \omega \in H$:*

$$\|(I - t\eta\Lambda)\xi - (I - t\eta\Lambda)\omega\| \leq (1 - t\tau)\|\xi - \omega\|. \quad (4)$$

3. Main Results

Theorem 3.1. *Let C be a nonempty, closed, convex subset of a real Hilbert space H. Suppose:*

- *$\Lambda : C \to H$ is an m-strongly monotone and L-Lipschitzian operator,*
- *$f : C \to H$ is a b-Lipschitzian mapping with constant $b > 0$,*
- *$T_1 : C \to C$ is a κ-demi-contractive mapping,*
- *$T_2 : C \to C$ is quasi-nonexpansive,*
- *and the set $\Omega = Fix(T_1) \cap Fix(T_2) \neq \emptyset$.*

Assumptions:

(1) $0 < \eta < \frac{2m}{L^2}$
(2) $0 < \gamma b < \tau$, *where* $\tau = \eta(m - \frac{L^2\eta}{2})$
(3) $(I - T_1), (I - T_2)$ *are demi-closed at the origin*

Iterative Scheme: *Let* $\{\xi_n\}, \{\omega_n\}$ *and* $\{\varphi_n\}$ *be sequences defined iteratively from arbitrary* $x_0 \in C$ *by:*

$$\begin{cases} \omega_n = \beta_n \xi_n + (1-\beta_n) T_2 \xi_n \\ \varphi_n = \mu_n \omega_n + (1-\mu_n) T_1 \omega_n \\ \xi_{n+1} = P_C[\alpha_n \gamma f(\xi_n) + (I - \eta \alpha_n \Lambda)\varphi_n] \end{cases} \tag{3}$$

where $\mu_n \in (\kappa, 1)$*, and* $\{\alpha_n\}, \{\beta_n\}$ *are real sequences in* $(0,1)$ *satisfying:*

(1) $\lim_{n\to\infty} \alpha_n = 0$
(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(3) $\liminf_{n\to\infty} (\mu_n - \kappa)(1-\mu_n) > 0$
(4) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$

Conclusion: *Then, the sequences* $\{\xi_n\}$ *and* $\{\omega_n\}$ *generated by* (3) *converge strongly to* $\xi^* \in Fix(T_1) \cap Fix(T_2)$*, which is the unique solution of the variational inequality:*

$$\langle \eta \Lambda \xi^* - \gamma f(\xi^*), \xi^* - p \rangle \leq 0, \quad \forall p \in \Omega. \tag{4}$$

Proof. **Step 1: Uniqueness of the Variational Inequality.** Assume by contradiction that there exist two points $\xi^*, \omega^* \in \Omega$ which are solutions to the given inequality, and $\xi^* \neq \omega^*$. Then we have:

$$\langle \eta \Lambda \xi^* - \gamma f(\xi^*), \xi^* - \omega^* \rangle \leq 0 \quad (5) \tag{5}$$

$$\langle \eta \Lambda \omega^* - \gamma f(\omega^*), \omega^* - \xi^* \rangle \leq 0 \quad (6) \tag{6}$$

Adding (5) and (6) yields:

$$\langle \eta(\Lambda \xi^* - \Lambda \omega^*) - \gamma(f(\xi^*) - f(\omega^*)), \xi^* - \omega^* \rangle \leq 0.$$

Using the m-strong monotonicity of Λ and b-Lipschitz continuity of f:

$$\eta m \|\xi^* - \omega^*\|^2 - \gamma b \|\xi^* - \omega^*\|^2 \leq 0 \implies (\eta m - \gamma b)\|\xi^* - \omega^*\|^2 \leq 0.$$

Since $0 < \gamma b < \tau < \eta m$, we have $\eta m - \gamma b > 0$, which implies $\|\xi^* - \omega^*\| = 0$, hence $\xi^* = \omega^*$.

Step 2: Existence via Contraction Mapping. For a fixed constant $t \in (0, \min\{1, \frac{1}{\tau}\})$, and $\forall \xi, \omega \in H$, we have, by Lemma 3, letting $\Phi = [I + t(\gamma f - \eta\Lambda)]\xi$ and $\Theta = [I + t(\gamma f - \eta\Lambda)]\omega$:

$$\begin{aligned} \|P_C\Phi - P_C\Theta\| &\leq \|[I + t(\gamma f - \eta\Lambda)]\xi - [I + t(\gamma f - \eta\Lambda)]\omega\| \\ &\leq t\gamma \|f(\xi) - f(\omega)\| + \|(I - t\eta\Lambda)\xi - (I - t\eta\Lambda)\omega\| \\ &\leq t\gamma b\|\xi - \omega\| + (1 - t\tau)\|\xi - \omega\| \\ &\leq (1 - t(\tau - \gamma b))\|\xi - \omega\|. \end{aligned}$$

Thus, by the Banach contraction principle, the mapping $P_C[I + t(\gamma f - \eta\Lambda)]$ has a unique fixed point, which corresponds to the unique solution of the variational inequality.

Now we continue with the proof of Theorem (3.1).

$$\begin{aligned}\|\varphi_n-\xi^*\|^2 &= \|\mu_n\omega_n+(1-\mu_n)\mathcal{T}_1\omega_n-\xi^*\|^2\\ &= \|\mu_n(\omega_n-\xi^*)+(1-\mu_n)(\mathcal{T}_1\omega_n-\xi^*)\|^2 \qquad (7)\\ &= \mu_n\|\omega_n-\xi^*\|^2+(1-\mu_n)\|\mathcal{T}_1\omega_n-\xi^*\|^2-\mu_n(1-\mu_n)\|\mathcal{T}_1\omega_n-\omega_n\|^2.\end{aligned}$$

Using the fact that $\mathcal{T}_1$ is κ-demi-contractive, we have the following estimates:

$$\begin{aligned}\|\varphi_n-\xi^*\|^2 &\leq \mu_n\|\omega_n-\xi^*\|^2+(1-\mu_n)\Big[\|\omega_n-\xi^*\|^2+\kappa\|\mathcal{T}_1\omega_n-\omega_n\|^2\Big]\\ &\quad -\mu_n(1-\mu_n)\|\mathcal{T}_1\omega_n-\omega_n\|^2\\ &\leq \|\omega_n-\xi^*\|^2-(1-\mu_n)(\mu_n-\kappa)\|\mathcal{T}_1\omega_n-\omega_n\|^2.\end{aligned}$$

Since $\mu_n\in(\kappa,1)$, we have

$$\|\varphi_n-\xi^*\|^2\leq\|\omega_n-\xi^*\|^2. \qquad (8)$$

This implies that $\|\varphi_n-\xi^*\|\leq\|\omega_n-\xi^*\|$.

Again, from the algorithm (3), we have

$$\begin{aligned}\|\omega_n-\xi^*\|^2 &= \|\beta_n\xi_n+(1-\beta_n)\mathcal{T}_2\xi_n-\xi^*\|^2\\ &\leq \beta_n\|\xi_n-\xi^*\|^2+(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi^*\|^2-\beta_n(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi_n\|^2. \qquad (9)\end{aligned}$$

Since $\mathcal{T}_2$ is quasi-nonexpansive, we have the following:

$$\|\omega_n-\xi^*\|^2\leq\|\xi_n-\xi^*\|^2-\beta_n(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi_n\|^2. \qquad (10)$$

Since $\beta_n\in(0,1)$, we have

$$\|\omega_n-\xi^*\|^2\leq\|\xi_n-\xi^*\|^2. \qquad (11)$$

We can now see that

$$\|\varphi_n-\xi^*\|\leq\|\omega_n-\xi^*\|\leq\|\xi_n-\xi^*\|. \qquad (12)$$

Using (3), inequality (12), and Lemma (3), we obtain:

$$\begin{aligned}\|\xi_{n+1}-\xi^*\| &\leq \|(\alpha_n\gamma f(\xi_n)+(I-\eta\alpha_n\Lambda)\varphi_n)-\xi^*\|\\ &\leq \|\alpha_n\gamma(f(\xi_n)-f(\xi^*))\|+(1-\tau\alpha_n)\|\varphi_n-\xi^*\|+\alpha_n\|\gamma f(\xi^*)-\eta\Lambda\xi^*\|\\ &\leq \alpha_n\gamma\|f(\xi_n)-f(\xi^*)\|+(1-\tau\alpha_n)\|\xi_n-\xi^*\|+\alpha_n\|\gamma f(\xi^*)-\eta\Lambda\xi^*\|\\ &\leq \alpha_nb\gamma\|\xi_n-\xi^*\|+(1-\tau\alpha_n)\|\xi_n-\xi^*\|+\alpha_n\|\gamma f(\xi^*)-\eta\Lambda\xi^*\|\\ &\leq (1-\alpha_n(\tau-b\gamma))\|\xi_n-\xi^*\|+\alpha_n\|\gamma f(\xi^*)-\eta\Lambda\xi^*\|\\ &\leq \max\Big\{\|\xi_n-\xi^*\|,\frac{\|\gamma f(\xi^*)-\eta\Lambda\xi^*\|}{\tau-b\gamma}\Big\}.\end{aligned}$$

Therefore, by induction, it is easy to see that

$$\|\xi_{n+1}-\xi^*\|\leq\max\Big\{\|\xi_0-\xi^*\|,\frac{\|\gamma f(\xi^*)-\eta\Lambda\xi^*\|}{\tau-b\gamma}\Big\},\quad\forall n\geq1.$$

Hence $\{\xi_n\},\{f(\xi_n)\}$ and $\{\Lambda\xi_n\}$ are bounded.

Next, we show the following estimate:

$$\begin{aligned}\|\xi_{n+1}-\xi^*\|^2 &\leq \|\alpha_n(\gamma f(\xi_n)-\eta\Lambda\xi^*)+(I-\eta\alpha_n\Lambda)(\varphi_n-\xi^*)\|^2\\ &\leq \alpha_n^2\|\gamma f(\xi_n)-\eta\Lambda\xi^*\|^2+(1-\tau\alpha_n)^2\|\varphi_n-\xi^*\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(\xi_n)-\eta\Lambda\xi^*\|\|\varphi_n-\xi^*\|.\end{aligned}$$

Now using Lemma 3 with the definition of μ_n, β_n and with inequalities 8, 10, we have the following:

$$\begin{aligned}\|\xi_{n+1}-\xi^*\|^2 &\leq \alpha_n^2\|\gamma f(\xi_n)-\eta\Lambda\xi^*\|^2+(1-\tau\alpha_n)^2\|\xi_n-\xi^*\|^2\\ &- (1-\tau\alpha_n)^2\beta_n(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi_n\|^2\\ &- (1-\tau\alpha_n)^2(1-\mu_n)(\mu_n-\kappa)\|\mathcal{T}_1\omega_n-\omega_n\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(\xi_n)-\eta\Lambda\xi^*\|\|\xi_n-\xi^*\|.\end{aligned}$$

Therefore, expanding $(1-\tau\alpha_n)^2 = 1-2\tau\alpha_n+\tau^2\alpha_n^2$, we get:

$$\begin{aligned}\|\xi_{n+1}-\xi^*\|^2 &\leq \alpha_n^2\|\gamma f(\xi_n)-\eta\Lambda\xi^*\|^2+\|\xi_n-\xi^*\|^2\\ &- 2\tau\alpha_n\|\xi_n-\xi^*\|^2+\tau^2\alpha_n^2\|\xi_n-\xi^*\|^2\\ &- (1-\tau\alpha_n)^2\beta_n(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi_n\|^2\\ &- (1-\tau\alpha_n)^2(1-\mu_n)(\mu_n-\kappa)\|\mathcal{T}_1\omega_n-\omega_n\|^2\\ &+ 2\alpha_n(1-\tau\alpha_n)\|\gamma f(\xi_n)-\eta\Lambda\xi^*\|\|\xi_n-\xi^*\|.\end{aligned} \tag{13}$$

Due to the boundedness of $\{\xi_n\}$ and $\{f(\xi_n)\}$, there exists a constant $D>0$ such that:

$$\begin{aligned}(1-\tau\alpha_n)^2\beta_n(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi_n\|^2 &+ (1-\tau\alpha_n)^2(1-\mu_n)(\mu_n-\kappa)\|\mathcal{T}_1\omega_n-\omega_n\|^2\\ &\leq \|\xi_n-\xi^*\|^2-\|\xi_{n+1}-\xi^*\|^2+\alpha_n D.\end{aligned} \tag{14}$$

We now show that $\xi_n\to\xi^*$. We consider two cases.

Case 1: . Assume that the sequence $\{\|\xi_n-\xi^*\|\}$ is monotonically decreasing. Then $\{\|\xi_n-\xi^*\|\}$ must be a convergent sequence. Therefore, we have

$$\lim_{n\to\infty}[\|\xi_n-\xi^*\|^2-\|\xi_{n+1}-\xi^*\|^2]=0. \tag{15}$$

From 15 and the fact that $\alpha_n\to 0$, the right-hand side of 14 tends to 0. Thus,

$$\lim_{n\to\infty}(1-\mu_n)(\mu_n-\kappa)\|\mathcal{T}_1\omega_n-\omega_n\|^2=0.$$

Since $\mu_n\in(\kappa,1)$ and $\liminf_{n\to\infty}(\mu_n-\kappa)(1-\mu_n)>0$, we have

$$\lim_{n\to\infty}\|\mathcal{T}_1\omega_n-\omega_n\|^2=0. \tag{16}$$

Similarly, we have

$$\lim_{n\to\infty}\beta_n(1-\beta_n)\|\mathcal{T}_2\xi_n-\xi_n\|^2=0.$$

Since $\beta_n \in (0,1)$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, we obtain

$$\lim_{n\to\infty} \|\mathcal{T}_2\xi_n - \xi_n\|^2 = 0. \tag{17}$$

Next, we prove that $\lim_{n\to\infty} \|\xi_n - \xi^*\| = 0$. Using the property of the metric projection, we have:

$$\begin{aligned}
\|\xi_{n+1} - \xi^*\|^2 &\leq \langle \alpha_n\gamma f(\xi_n) + (I - \eta\alpha_n\Lambda)\varphi_n - \xi^*, \xi_{n+1} - \xi^*\rangle \\
&= \langle \alpha_n\gamma(f(\xi_n) - f(\xi^*)) + (I - \eta\alpha_n\Lambda)(\varphi_n - \xi^*) + \alpha_n(\gamma f(\xi^*) - \eta\Lambda\xi^*), \xi_{n+1} - \xi^*\rangle \\
&\leq \alpha_n\gamma b\|\xi_n - \xi^*\|\|\xi_{n+1} - \xi^*\| + (1 - \tau\alpha_n)\|\xi_n - \xi^*\|\|\xi_{n+1} - \xi^*\| \\
&+ \alpha_n\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{n+1}\rangle \\
&\leq (1 - \alpha_n(\tau - \gamma b))\|\xi_n - \xi^*\|\|\xi_{n+1} - \xi^*\| + \alpha_n\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{n+1}\rangle.
\end{aligned}$$

Since we are in Case 1, $\|\xi_{n+1} - \xi^*\| \leq \|\xi_n - \xi^*\|$, which yields:

$$\|\xi_{n+1} - \xi^*\|^2 \leq (1 - \alpha_n(\tau - \gamma b))\|\xi_n - \xi^*\|^2 + 2\alpha_n\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{n+1}\rangle.$$

Now by Lemma 2, we can conclude that the sequence ξ_n converges to ξ^*, where $\beta_n = \alpha_n(\tau - \gamma b)$ and $\delta_n = 2\alpha_n\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{n+1}\rangle$.

Finally, we prove that $\limsup_{n\to\infty}\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_n\rangle \leq 0$. Since H is reflexive and $\{\xi_n\}$ is bounded, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi_{n_k} \rightharpoonup \phi$ weakly in C.

$$\limsup_{n\to\infty}\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_n\rangle = \lim_{k\to\infty}\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{n_k}\rangle.$$

From 16, 17, and the fact that $I - \mathcal{T}_1, I - \mathcal{T}_2$ are demi-closed at the origin, we obtain $\phi \in Fix(\mathcal{T}_1) \cap Fix(\mathcal{T}_2)$. On the other hand, since ξ^* solves the variational inequality (4), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_n\rangle &= \lim_{k\to\infty}\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{n_k}\rangle \\
&= \langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \phi\rangle \leq 0.
\end{aligned}$$

Case II: . Suppose that the sequence $\{\|\xi_n - \xi^*\|\}$ is not eventually monotonically decreasing. Set $\mathbf{D}_n := \|\xi_n - \xi^*\|^2$ and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some sufficiently large n_0) by $\sigma_n := \max\{k \in \mathbb{N} : k \leq n, \mathbf{D}_k \leq \mathbf{D}_{k+1}\}$. Then, σ_n is a nondecreasing sequence such that $\sigma_n \to \infty$ as $n \to \infty$ and $\mathbf{D}_{\sigma_n} \leq \mathbf{D}_{\sigma_n+1}$ for all $n \geq n_0$.

From (14), we have:

$$\begin{aligned}
&(1 - \tau\alpha_{\sigma_n})^2\Big[\beta_{\sigma_n}(1 - \beta_{\sigma_n})\|\mathcal{T}_2\xi_{\sigma_n} - \xi_{\sigma_n}\|^2 \\
&\quad +(1 - \mu_{\sigma_n})(\mu_{\sigma_n} - \kappa)\|\mathcal{T}_1\omega_{\sigma_n} - \omega_{\sigma_n}\|^2\Big] \\
&\qquad\qquad \leq \mathbf{D}_{\sigma_n} - \mathbf{D}_{\sigma_n+1} + \alpha_{\sigma_n}D \leq \alpha_{\sigma_n}D.
\end{aligned} \tag{18}$$

Since $\alpha_{\sigma_n} \to 0$, we get

$$\lim_{n\to+\infty} \Big[\beta_{\sigma_n}(1-\beta_{\sigma_n})\|\mathcal{T}_2\xi_{\sigma_n} - \xi_{\sigma_n}\|^2 + (1-\mu_{\sigma_n})(\mu_{\sigma_n}-\kappa)\|\mathcal{T}_1\omega_{\sigma_n} - \omega_{\sigma_n}\|^2\Big] = 0.$$

Following the same argument as in Case 1, we obtain

$$\lim_{n\to\infty} \|\mathcal{T}_2\xi_{\sigma_n} - \xi_{\sigma_n}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{T}_1\omega_{\sigma_n} - \omega_{\sigma_n}\| = 0.$$

Following the same argument in Case 1, we conclude that

$$\limsup_{n\to+\infty} \langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{\sigma_n+1}\rangle \le 0.$$

Therefore, for all $n \ge n_0$, using (18) evaluated at σ_n, we have:

$$\begin{aligned} 0 \le \|\xi_{\sigma_n+1} - \xi^*\|^2 - \|\xi_{\sigma_n} - \xi^*\|^2 \quad \le \quad & \alpha_{\sigma_n}\Big[-\alpha_{\sigma_n}(\tau-\gamma b)\|\xi_{\sigma_n} - \xi^*\|^2 \\ & +2\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi^* - \xi_{\sigma_n+1}\rangle\Big]. \end{aligned}$$

Dividing by $\alpha_{\sigma_n}(\tau-\gamma b)$, we get:

$$\|\xi_{\sigma_n} - \xi^*\|^2 \le \frac{2}{\tau-\gamma b}\langle \eta\Lambda\xi^* - \gamma f(\xi^*), \xi_{\sigma_n+1} - \xi^*\rangle.$$

Since the right-hand side tends to 0, we have $\lim_{n\to\infty}\|\xi_{\sigma_n} - \xi^*\|^2 = 0$. Therefore, $\lim_{n\to\infty}\mathbf{D}_{\sigma_n} = \lim_{n\to\infty}\mathbf{D}_{\sigma_n+1} = 0$.

Furthermore, for all $n \ge n_0$, we have $\mathbf{D}_{\sigma_n} \le \mathbf{D}_{\sigma_n+1}$ if $n \ne \sigma_n$ (that is, $n > \sigma_n$), because $\mathbf{D}_j > \mathbf{D}_{j+1}$ for $\sigma_n + 1 \le j \le n$. Hence, $0 \le \mathbf{D}_{\sigma_n} \le \max\{\mathbf{D}_{\sigma_n}, \mathbf{D}_{\sigma_n+1}\} = \mathbf{D}_{\sigma_n+1}$. Therefore, $\mathbf{D}_n \to 0$ as $n \to \infty$, which implies that $\xi_n \to \xi^*$ as $n \to \infty$. This completes the proof.

□

Now using theorem 3.1, we proof strong convergency results for a given fixed point without the demiclosed principle.

Theorem 3.2. *Let C, H, Λ, f be as in Theorem 3.1. Let $\mathcal{T}_1, \mathcal{T}_2 : C \to C$ be quasi-nonexpansive mappings such that $\Omega = Fix(\mathcal{T}_1) \cap Fix(\mathcal{T}_2) \ne \emptyset$. Assume the conditions on $\eta, \gamma, \alpha_n, \beta_n, \mu_n$ hold. Then, the sequences generated by the scheme converge strongly to $\xi^* \in \Omega$, the unique solution of the variational inequality.*

Proof. Since quasi-nonexpansive mappings are a subclass of demi-contractive mappings (corresponding to the constant $\kappa = 0$), the proof follows directly from Theorem 3.1. □

Theorem 3.3. *Let C, H, Λ, f be as in Theorem 3.1. Let $\mathcal{T}_1, \mathcal{T}_2 : C \to C$ be κ-demi-contractive mappings such that $\Omega = Fix(\mathcal{T}_1) \cap Fix(\mathcal{T}_2) \ne \emptyset$. Assume the conditions on $\eta, \gamma, \alpha_n, \beta_n, \mu_n$ hold, and $(I-\mathcal{T}_1), (I-\mathcal{T}_2)$ are demi-closed at the origin. Then, the sequences generated by the scheme converge strongly to $\xi^* \in \Omega$.*

Proof. Theorem 3.3 extends the result to general demi-contractive mappings. The proof follows the same arguments as in Theorem 3.1 by utilizing the κ-demi-contractive property for both $\mathcal{T}_1$ and $\mathcal{T}_2$. □

Theorem 3.4. *Let $C, H, \Lambda, f, \mathcal{T}_1, \mathcal{T}_2$ be as defined previously. Let $g : C \to \mathbb{R}$ be a convex and differentiable constraint function with Lipschitz continuous gradient ∇g. Consider the iterative scheme:*

$$\xi_{n+1} = P_C[\alpha_n \gamma f(\xi_n) + (I - \eta\alpha_n\Lambda)\varphi_n] + \alpha_n\lambda_n\nabla g(\xi_n)$$

under appropriate parameter conditions. Then, the sequence $\{\xi_n\}$ converges strongly to $\xi^* \in \Omega$, which satisfies the optimality condition of the constrained minimization problem:

$$\min_{\xi\in\Omega}\left\{\frac{\eta}{2}\langle\Lambda\xi,\xi\rangle - h(\xi) \quad \textit{subject to} \quad g(\xi) \leq 0\right\}$$

where h is a potential function for γf.

Proof. We note that a strongly positive bounded linear operator Λ is $\|\Lambda\|$-Lipschitzian and m-strongly monotone. The proof follows the framework established in Theorem 3.1 by incorporating the penalty term $\alpha_n\lambda_n\nabla g(\xi_n)$ and applying standard properties of convex differentiable functions. □

4. Numerical Analysis

The numerical analysis of the proposed theorem was carried out through the implementation of an iterative scheme in the real Hilbert space $\mathbb{R}^2$ equipped with the standard inner product. This choice was motivated by the ease of visualization and the practicality of computations in low-dimensional Euclidean spaces. To this end, the closed unit ball

$$C = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$$

was selected as the convex constraint set, with its associated projection defined explicitly by

$$P_C(x) = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \frac{x}{\|x\|} & \text{otherwise.} \end{cases}$$

The operator Λ was chosen as a linear mapping characterized by a symmetric positive definite matrix

$$A = \begin{pmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix},$$

ensuring that $\Lambda(x) = Ax$ is m-strongly monotone and L-Lipschitzian with constants $m = 2$ and $L = 3$, respectively, such that the following holds:

$$\langle\Lambda x - \Lambda y, x - y\rangle \geq m\|x - y\|^2, \quad \text{and} \quad \|\Lambda x - \Lambda y\| \leq L\|x - y\|.$$

Similarly, the mapping f was defined as a Lipschitz continuous linear operator

$$f(x) = Bx + c,$$

with

$$B = \frac{1}{2} \cdot I, \quad c = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix},$$

where I denotes the 2×2 identity matrix. The Lipschitz constant for f is $b = \|B\|_2 = \frac{1}{2}$, satisfying the boundedness condition prescribed in the theorem.

The mappings $\mathcal{T}_1$ and $\mathcal{T}_2$ were constructed to fulfill the demi-contractive and quasi-nonexpansive conditions, respectively. Specifically, $\mathcal{T}_1$ was modeled as an averaged operator

$$\mathcal{T}_1(x) = \frac{1}{2}x + \frac{1}{2}p_0,$$

with $p_0 = (0, 0) \in C$, ensuring it is firmly nonexpansive and κ-demi-contractive with $\kappa = 0.5$. The mapping $\mathcal{T}_2$ was taken as the projection onto the half-plane

$$D = \{x \in \mathbb{R}^2 : x_1 \leq 0.5\},$$

which is a firmly nonexpansive and thus quasi-nonexpansive operator.

The iterative parameters were selected to meet the technical conditions for convergence. The sequences $\{\alpha_n\} = \dfrac{1}{n+1}, \beta_n = \dfrac{n}{(2n+1)}$, and $\mu_n = 0.75$ were chosen accordingly, ensuring that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$, and $\liminf\limits_{n\to\infty} \beta_n(1-\beta_n) > 0$. The scalar parameters $\eta = 0.1$ and $\gamma = 0.2$ satisfy

$$0 < \eta < \frac{2m}{L^2} \approx 0.44, \quad \text{and} \quad 0 < \gamma b < \tau = \eta\left(m - \frac{L^2\eta}{2}\right) = 0.155,$$

thus conforming with the assumptions underpinning the strong convergence of the iterative scheme.

The iterative updates were implemented as

$$\begin{cases} \omega_n = \beta_n\xi_n + (1-\beta_n)\mathcal{T}_2\xi_n, \\ \varphi_n = \mu_n\omega_n + (1-\mu_n)\mathcal{T}_1\omega_n, \\ \xi_{n+1} = P_C\left(\alpha_n\gamma f(\xi_n) + (I - \eta\alpha_n\Lambda)\varphi_n\right), \end{cases}$$

initialized at $\xi_0 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in C$ and iterated for $N = 500$ steps.

The numerical experiments were conducted using MATLAB, implementing the proposed algorithmic framework. The projection operators P_C and P_D, mappings Λ, f, $\mathcal{T}_1$, and $\mathcal{T}_2$, and the scalar sequences were defined strictly according to the theoretical specifications.

The convergence behavior was examined by plotting the sequence of successive iterate differences, $\|\xi_{n+1} - \xi_n\|$. The results demonstrate a clear decrease towards zero as the number of iterations increases, empirically illustrating the strong convergence of the sequence to the unique

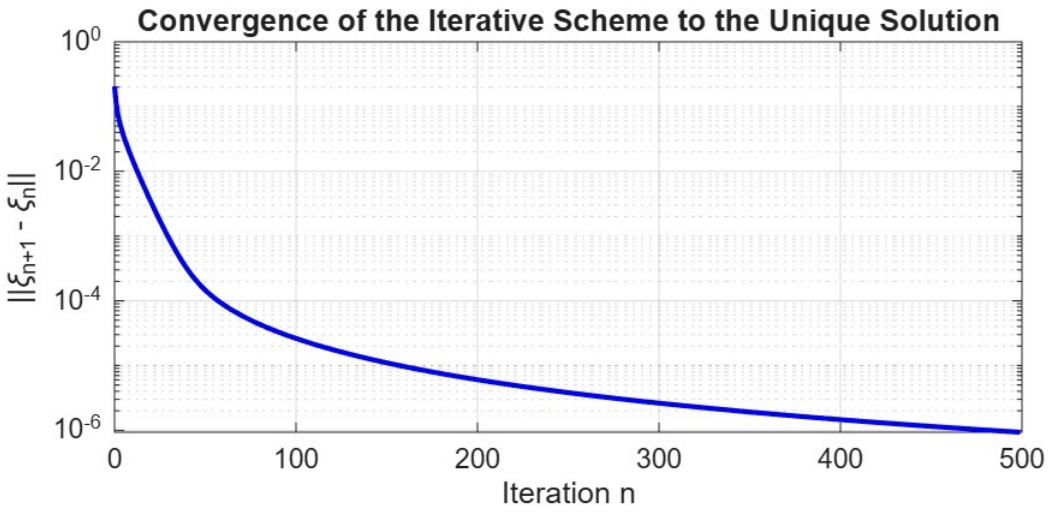


FIGURE 1. Convergence behavior of $\|\xi_{n+1} - \xi_n\|$.

solution $\xi^* \in \text{Fix}(\mathcal{T}_1) \cap \text{Fix}(\mathcal{T}_2)$ that satisfies the variational inequality. This numerical behavior is fully consistent with the established theoretical results.

This numerical analysis substantiates the validity of the proposed iterative scheme and corroborates the theoretical convergence claims. Furthermore, the MATLAB implementation provides a reproducible computational framework, offering tangible insights into the algorithm's dynamics and effectiveness.

In summary, the numerical simulation confirms that the iterative sequence converges strongly to the unique solution of the variational inequality, as predicted by the theoretical analysis, thereby providing robust computational validation of the mathematical framework developed in this paper.

5. Conclusion

This research advances the study of iterative schemes for demi-contractive and quasi-nonexpansive mappings in Hilbert spaces, extending and generalizing existing results in the field. Notably, our findings build upon the foundational work of Browder [5], Halpern [6], and Marino and Xu [3], who pioneered the development of iterative methods for nonexpansive mappings.

The strong convergence of the proposed scheme has significant implications for optimization and variational inequality problems. Future research directions may explore the application of these techniques to practical problems in signal processing, image recovery, and machine learning. The theoretical framework established herein provides a solid foundation for further investigation into the convergence properties of iterative algorithms in Hilbert spaces.

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