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Some Investigations on a Class of Analytic and Univalent Functions Involving q-Differentiation

Ayotunde Olajide Lasode*, Timothy Oloyede Opoola

Department of Mathematics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Nigeria
lasode_ayo@yahoo.com, opoola.to@unilorin.edu.ng
**Correspondence: lasode_ayo@yahoo.com*

Abstract. We use the concept of q-differentiation to define a class $\mathcal{E}_q(\beta, \delta)$ of analytic and univalent functions. The investigations thereafter includes coefficient estimates, inclusion property and some conditions for membership of some analytic functions to be in the class $\mathcal{E}_q(\beta, \delta)$. Our results generalize some known and new ones.

1. Introduction and Definitions

We let $\mathcal{UD} = \{z : z \in \mathbb{C}, \ |z| < 1\}$ represent the unit disk and $\mathcal{A}$ represent the class of normalized analytic functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in \mathcal{UD} \tag{1}$$

where $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{S}$ represent a subset of $\mathcal{A}$ containing functions univalent in $\mathcal{UD}$. A function f in $\mathcal{S}$ is a member of class $\mathcal{BT}(\delta)$ of bounded turning functions of order δ if it satisfies the geometric condition

$$\mathcal{R}e f'(z) > \delta \in [0, 1), \quad z \in \mathcal{UD}.$$

Let $\mathcal{BT}(0) = \mathcal{BT}$ represent the class of bounded turning functions. It is known (see [1]) that $f \in \mathcal{BT}$ are univalent functions. Also, a function f in $\mathcal{S}$ is a member of class $\mathcal{CV}(\delta)$ of convex functions of order δ if it satisfies the geometric condition

$$\mathcal{R}e\left(z\frac{f''(z)}{f'(z)} + 1\right) > \delta \in [0, 1), \quad z \in \mathcal{UD}.$$

Let $\mathcal{CV}(0) = \mathcal{CV}$ represent the class of convex functions.

The importance of operators in geometric function theory cannot be underrated. For instance see [2, 13, 15] for some known ones.

In 1908, Jackson [7] (see also [3, 4, 8–11]) initiated the concept of q-calculus as follows.

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Definition 1.1. For $q \in (0,1)$, the q-differentiation of function $f \in \mathcal{A}$ is defined by

$$\mathcal{D}_q f(0) = f'(0), \quad \mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{z(1-q)} \ (z \neq 0) \quad \text{and} \quad \mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)). \tag{2}$$

Obviously, applying (2) in (1) gives us

$$\mathcal{D}_q f(z) = 1 + \sum_{m=2}^{\infty} [m]_q a_m z^{m-1} \quad \text{and} \quad z\mathcal{D}_q^2 f(z) = \sum_{m=2}^{\infty} [m-1]_q [m]_q a_m z^{m-1} \tag{3}$$

where $[m]_q = \frac{1-q^m}{1-q}$ and $\lim\limits_{q \uparrow 1} [m]_q = m$.

For example if $f(z) = z^m$, then by using (2),

$$\mathcal{D}_q f(z) = \mathcal{D}_q(z^m) = \frac{1-q^m}{1-q} z^{m-1} = [m]_q z^{m-1}$$

and observe that

$$\lim_{q \uparrow 1} \mathcal{D}_q f(z) = \lim_{q \uparrow 1} \left([m]_q z^{m-1}\right) = m z^{m-1} = f'(z)$$

where $f'(z)$ is the classical differentiation.

In this work, the q-differential operator was used to define a class of analytic functions and generalize some results.

2. Relevant Lemmas

We represent by $\mathcal{P}$ the well-known class of analytic functions of the form

$$p(z) = 1 + \sum_{m=1}^{\infty} c_m z^m, \quad \mathcal{R}e\, p(z) > 0, \ z \in \mathcal{UD} \tag{4}$$

and by $\mathcal{P}(\delta) \subseteq \mathcal{P}(0) = \mathcal{P}$ the class whose members are of the form

$$p_\delta(z) = 1 + \sum_{m=1}^{\infty} (1-\delta) c_m z^m, \quad \mathcal{R}e\, p(z) > \delta \in [0,1), \ z \in \mathcal{UD}. \tag{5}$$

The following lemmas shall be required to proof our results.

Lemma 2.1 ([14]). *Let $g(z) = \sum\limits_{m=1}^{\infty} a_m z^m \prec G(z) = \sum\limits_{m=1}^{\infty} b_m z^m$, $z \in \mathcal{UD}$ where $G(z)$ is univalent in $\mathcal{UD}$ and $G(\mathcal{UD})$ is a convex domain, then $|a_m| \leq |b_1|$, $m \in \mathbb{N}$. Equality holds for the function $g(z) = G(\tau z^m)$, $|\tau| = 1$.*

The lemmas that follow are the q-analogous versions of the original ones as referenced.

Lemma 2.2 ([6]). *Let $p(z)$ be analytic in $\mathcal{UD}$ such that $p(0) = 1$. If*

$$\mathcal{R}e\left(\frac{z\mathcal{D}_q(p(z))}{p(z)} + 1\right) > \frac{3\delta - 1}{2\delta}, \ z \in \mathcal{UD},$$

then for $\alpha = (\delta - 1)/\delta$ $(\delta \in [1/2, 1))$, $\mathcal{R}e\, p(z) > 2^\alpha$. The constant 2^α is the best possible.

Lemma 2.3 ([5]). *Let $u = u_1 + u_2 i$ and $v = v_1 + v_2 i$ such that $\gamma(u,v) : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is a complex-valued function such that*

(1) *$\gamma(u,v)$ is continuous in $\Pi \subset \mathbb{C}^2$,*

(2) *$(1,0) \in \Pi$ and $\mathcal{R}e(\gamma(1,0)) > 0$ and*

(3) *$\mathcal{R}e(\gamma(\xi + (1-\xi)u_2 i, v_1)) \leq \xi$ $(0 \leq \xi < 1)$ if $(\xi + (1-\xi)u_2 i, v_1) \in \Pi$ and $v_1 \leq -\frac{1}{2}(1-\xi)(1+u_2^2)$ and $\mathcal{R}e(\gamma(\xi+(1-\xi)u_2 i, v_1)) \geq \xi$ $(\xi > 1)$ if $(\xi+(1-\xi)u_2 i, v_1) \in \Pi$ and $v_1 \geq \frac{1}{2}(1-\xi)(1+u_2^2)$.*

If $p(z) \in \mathcal{P}$ for $(p(z), z\mathcal{D}_q p(z)) \in \Pi$ and $\mathcal{R}e(\gamma(p(z),\ z\mathcal{D}_q p(z))) > \xi$, $z \in \mathcal{UD}$, then $\mathcal{R}ep(z) > \xi$ in $\mathcal{UD}$.

3. Main Results

The definition of the investigated class is as follows.

A function $f(z) \in \mathcal{A}$ is a member of the class $\mathcal{E}_q(\beta,\delta)$ if the condition

$$\mathcal{R}e\left(\mathcal{D}_q f(z) + \frac{1+e^{i\beta}}{2} z\mathcal{D}_q^2 f(z)\right) > \delta, \quad \delta \in [0,1),\ \beta \in (-\pi,\pi],\ z \in \mathcal{UD} \tag{6}$$

holds.

When parameters in (6) are varied, the class $\mathcal{E}_q(\beta,\delta)$ reduces to some well-known classes of analytic functions that have been studied by some authors. These are cited in our corollaries and remarks.

The following are the proved results.

Theorem 3.1. *Let $\beta \in (-\pi,\pi]$ and $\delta \in [0,1)$, if condition (6) holds, then*

$$\mathcal{E}_q(\beta,\delta) \subset \mathcal{BT}_q(\delta).$$

$\mathcal{BT}_q(\delta)$ is the class of q-bounded turning function of order δ.

Proof. Let $p(z) = \mathcal{D}_q f(z)$ so that $\mathcal{D}_q p(z) = \mathcal{D}_q^2 f(z)$ and for $\kappa = (1+e^{i\beta})/2$, then (6) can be expressed as

$$\mathcal{R}e(p(z) + \kappa z\mathcal{D}_q p(z)) > \delta. \tag{7}$$

In view of the conditions in Lemma 2.3 and for $p(z)$ in (7), we define the function

$$\gamma(u,\nu) = u + \kappa\nu$$

on the domain Π of $\mathbb{C}^2$, then

(i) clearly, $\gamma(u,\nu)$ satisfies the condition (1) in Lemma 2.3,

(ii) for $(1,0) \in \Pi$, $\gamma(1,0) = 1 \implies \mathcal{R}e(\gamma(1,0)) > 0$ and

(iii) $\gamma(\delta + (1-\delta)u_2 i, \nu_1) = \delta + \frac{1+\cos\delta}{2}\nu_1 + \left((1-\delta)u_2 + \frac{\sin\delta}{2}\nu_1\right) i$, thus,

$$\mathcal{R}e(\gamma(\delta + (1-\delta)u_2 i, \nu_1)) = \delta + \frac{1+\cos\beta}{2}\nu_1 \leq \delta$$

for $\nu_1 \leq -\frac{1}{2}(1-\delta)(1+u_2^2)$.

Now since $\gamma(u,\nu)$ satisfies all the conditions $(1-3)$ in Lemma 2.3, then it implies that

$$\mathcal{R}ep(z) = \mathcal{R}e(\mathcal{D}_q f(z)) > \delta, \quad z \in \mathcal{UD}$$

hence the proof is complete. □

Corollary 3.2 ([1]). *Since class $\mathcal{BT}_q(\delta)$ is well-known to consist of univalent functions, then $\mathcal{E}_q(\beta,\delta) \subset \mathcal{BT}_q(\delta)$ consists of univalent functions.*

Corollary 3.3. $\lim\limits_{q\uparrow 1} \mathcal{E}_q(\beta,\delta) \subset \mathcal{BT}(\delta),\ z \in \mathcal{UD}$.

Theorem 3.4. *If $f \in \mathcal{A}$ is such that*

$$\mathcal{R}e\left(\frac{z\mathcal{D}_q(\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z))}{\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z)}\right) > \frac{\delta-1}{2\delta}, \tag{8}$$

then

$$\mathcal{R}e(\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z)) > 2^{(\delta-1)/\delta}, \quad \delta \in [1/2,1),\ z \in \mathcal{UD}$$

and $\kappa = (1+e^{i\beta})/2$.

Proof. From (6), let $p(z) = \mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z)$, then by logarithmic q-differentiation we obtain

$$\frac{z\mathcal{D}_q p(z)}{p(z)} + 1 = \frac{z\mathcal{D}_q(\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z))}{\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z)} + 1.$$

Now applying Lemma 2.2 gives

$$\mathcal{R}e\left(\frac{z\mathcal{D}_q p(z)}{p(z)} + 1\right) = \mathcal{R}e\left(\frac{z\mathcal{D}_q(\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z))}{\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z)} + 1\right) > \frac{3\delta-1}{2\delta}$$

implies that

$$\mathcal{R}e\left(\frac{z\mathcal{D}_q(\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z))}{\mathcal{D}_q f(z) + \kappa z\mathcal{D}_q^2 f(z)}\right) > \frac{\delta-1}{2\delta}$$

and by the same Lemma 2.2 the proof in complete. □

Corollary 3.5. *If $f \in \mathcal{A}$ satisfies condition (8), then $f \in \mathcal{E}_q(\beta, 2^{(\delta-1)/\delta})$.*

Corollary 3.6. *If $f \in \lim\limits_{q\uparrow 1} \mathcal{E}_q(\beta,1/2)$ is such that*

$$\mathcal{R}e\left(\frac{z(1+\kappa)f''(z) + \kappa z^2 f'''(z)}{f'(z) + \kappa z f''(z)}\right) > -\frac{1}{2},$$

then

$$\mathcal{R}e(f'(z) + \kappa z f''(z)) > 1/2, \quad z \in \mathcal{UD}.$$

Corollary 3.7. *If $f \in \mathcal{E}_q(\pi, 1/2)$ is such that*

$$\mathcal{R}e\left(\frac{z\mathcal{D}_q(\mathcal{D}_q f(z))}{\mathcal{D}_q f(z)}\right) > -\frac{1}{2}, \tag{9}$$

then

$$\mathcal{R}e(\mathcal{D}_q f(z)) > \frac{1}{2}.$$

This means that if condition (9) holds, then f is a q-bounded turning function of order $1/2$. Now if $q \uparrow 1$, then

$$\mathcal{R}e\left(\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \tag{10}$$

implies

$$\mathcal{R}e(f'(z)) > \frac{1}{2} \quad z \in \mathcal{UD}.$$

This means that if condition (10) holds, then f is a bounded turning function of order $1/2$.

Corollary 3.8. *If $f \in \mathcal{E}_q(0, 1/2)$ is such that*

$$\mathcal{R}e\left(\frac{z\mathcal{D}_q(\mathcal{D}_q f(z) + z\mathcal{D}_q^2 f(z))}{\mathcal{D}_q f(z) + z\mathcal{D}_q^2 f(z)}\right) > -\frac{1}{2}, \tag{11}$$

then

$$\mathcal{R}e(\mathcal{D}_q f(z) + z\mathcal{D}_q^2 f(z)) > \frac{1}{2}$$

and if $q \uparrow 1$,

$$\mathcal{R}e\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)}\right) > -\frac{1}{2}$$

implies that

$$\mathcal{R}e(f'(z) + zf''(z)) > 1/2, \quad z \in \mathcal{UD}.$$

Theorem 3.9. *Let $\beta \in (-\pi, \pi]$ and $\delta \in [0, 1)$, then the function*

$$f(z) = z + a_m z^m \in \mathcal{E}_q(\beta, \delta), \quad m = \{2, 3, \ldots\} \tag{12}$$

if

$$|a_m| \leq \frac{2}{[m]_q\big\{|X_m| - ((2 + [m-1]_q)\cos\theta + [m-1]_q \cos(\beta + \theta_0))\big\}} \tag{13}$$

where

$$\left.\begin{aligned} X_m &= 2 + [m-1]_q(1 + e^{i\beta}) \\ |X_m| &= \sqrt{2\big\{2 + [m-1]_q(2 + [m-1]_q)(1 + \cos\beta)\big\}} \geq 2 \end{aligned}\right\} \tag{14}$$

and θ_0 attains minimum at

$$\theta_0 = \pi + \arctan\left(\frac{-[m-1]_q \sin\beta}{2 + [m-1]_q(1 + \cos\beta)}\right). \tag{15}$$

Proof. Firstly, applying (2) in (12) gives

$$\left.\begin{aligned} \mathcal{D}_q f(z) = 1 + [m]_q a_m z^{m-1} \\ z\mathcal{D}_q^2 f(z) = [m-1]_q [m]_q a_m z^{m-1} \end{aligned}\right\}. \tag{16}$$

Note that it suffices to study the condition that for $|z| = 1$,

$$\left|\mathcal{D}_q f(z) + \frac{1+e^{i\beta}}{2} z\mathcal{D}_q^2 f(z) - 1\right| < \mathcal{R}e\left\{\mathcal{D}_q f(z) + \frac{1+e^{i\beta}}{2} z\mathcal{D}_q^2 f(z)\right\} \tag{17}$$

so that by putting (16) into (17) we obtain

$$\left|[m]_q a_m z^{m-1} + \frac{1}{2}[m-1]_q [m]_q (1+e^{i\beta}) a_m z^{m-1}\right|$$
$$< \mathcal{R}e\left\{1 + [m]_q a_m z^{m-1} + \frac{1}{2}[m-1]_q [m]_q (1+e^{i\beta}) a_m z^{m-1}\right\}.$$

Now letting $|a_m| = r$, $a_m z^{m-1} = re^{i\theta}$ and using (14) we obtain

$$\left|\frac{1}{2}[m]_q re^{i\theta} X_m\right| \le \mathcal{R}e\left\{1 + [m]_q re^{i\theta} + \frac{1}{2}[m-1]_q [m]_q (1+e^{i\beta}) re^{i\theta}\right\} \tag{18}$$

so that

$$\frac{1}{2}[m]_q r|X_m| \le \mathcal{R}e\,\mathcal{F} \tag{19}$$

where

$$\mathcal{F} = 1 + [m]_q re^{i\theta} + \frac{1}{2}[m-1]_q [m]_q (1+e^{i\beta}) re^{i\theta}$$

in (18). Further simplification gives

$$\mathcal{F} = 1 + [m]_q r\cos\theta + \frac{1}{2}[m-1]_q [m]_q r\cos\theta + \frac{1}{2}[m-1]_q [m]_q r\cos(\beta+\theta) + \mathcal{I}m(\mathcal{F})$$

so that

$$\mathcal{R}e\,\mathcal{F} = 1 + \frac{1}{2}[m]_q r\{2\cos\theta + [m-1]_q \cos\theta + [m-1]_q \cos(\beta+\theta)\} = \psi. \tag{20}$$

Now (19) becomes

$$\frac{1}{2}[m]_q r|X_m| \le 1 + \frac{1}{2}[m]_q r\{(2 + [m-1]_q)\cos\theta + [m-1]_q \cos(\beta+\theta)\}$$

and by simplification we obtain (13).

To know the values of θ where (20) attains minimum implies that

$$\frac{\partial\psi}{\partial\theta} = -\frac{r[m]_q}{2}\left\{(2 + [m-1]_q)\sin\theta + [m-1]_q \sin(\beta+\theta)\right\}$$

implies that

$$(2 + [m-1]_q)\sin\theta + [m-1]_q \sin(\beta+\theta) = 0$$

so that

$$\tan\theta = \frac{-[m-1]_q \sin\beta}{2 + [m-1]_q (1+\cos\beta)}$$

which simplifies to (15). □

Corollary 3.10. *Let $f(z) = z + a_m z^m \in \mathcal{E}_q(0, \delta)$ and $m = \{2, 3, \dots\}$, then*

$$|a_m| \leq \frac{1}{[m]_q \left\{ \sqrt{1 + 2[m-1]_q + [m-1]_q^2} + 1 + [m-1]_q \right\}}$$

and if $q \uparrow 1$, then

$$|a_m| \leq \frac{1}{2m^2}.$$

Corollary 3.11. *Let $f(z) = z + a_m z^m \in \mathcal{E}_q(\pi, \delta)$ and $m = \{2, 3, \dots\}$, then*

$$|a_m| \leqq \frac{1}{2[m]_q}$$

and if $q \uparrow 1$, then

$$|a_m| \leq \frac{1}{2m}.$$

Remark 3.12. Let $q \uparrow 1$, then Theorem 3.9 becomes the result in [18].

Theorem 3.13 (Coefficient Estimates). *Let $\beta \in (-\pi, \pi]$, $\delta \in [0, 1)$ and let $G(z) = 1 + b_1 z + b_2 z^2 + \dots \in \mathcal{CV}(\delta)$. If $f \in \mathcal{A}$ belongs to $\mathcal{E}_q(\beta, \delta)$, then*

$$|a_m| \leq \frac{2(1-\delta)|b_1|}{[m]_q |X_m|}, \quad m = \{2, 3, \dots\} \tag{21}$$

where $|X_m|$ is defined in (14).

Proof. Let $f(z) \in \mathcal{E}_q(\beta, \delta)$, therefore from (6) and using (5),

$$\mathcal{D}_q f(z) + \frac{1 + e^{i\beta}}{2} z \mathcal{D}_q^2 f(z) = \delta + (1-\delta) p(z), \quad z \in \mathcal{UD}. \tag{22}$$

Now putting (3) and (4) into (22) and simplifying gives

$$1 + \sum_{m=2}^{\infty} \left\{ 1 + [m-1]_q \left(\frac{1 + e^{i\beta}}{2} \right) \right\} [m]_q a_m z^{m-1} = 1 + \sum_{m=2}^{\infty} (1-\delta) c_{m-1} z^{m-1}$$

which implies that

$$\{2 + [m-1]_q (1 + e^{i\beta})\} \frac{[m]_q}{2} a_m = (1-\delta) c_{m-1}, \quad m = \{2, 3, \dots\}$$

where by applying (14) we obtain

$$X_m \frac{[m]_q}{2(1-\delta)} a_m = c_{m-1}, \quad m = \{2, 3, \dots\}. \tag{23}$$

Since $G(\mathcal{UD})$ is a convex domain, then from Lemma 2.1, (23) becomes

$$\left| X_m \frac{[m]_q}{2(1-\delta)} a_m \right| = |c_{m-1}| \leq |b_1|$$

and simplifying further we obtain (21). □

Corollary 3.14. *Let $f(z) \in \mathcal{E}_q(0,\delta)$, then*

$$|a_m| \leq \frac{(1-\delta)|b_1|}{\sqrt{1+2[m-1]_q+[m-1]_q^2}}$$

and if $q \uparrow 1$, then

$$|a_m| \leq \frac{(1-\delta)|b_1|}{m}, \quad m = \{2,3,\ldots\}.$$

Corollary 3.15. *Let $f \in \mathcal{E}_q(\pi,\delta)$, then*

$$|a_m| \leq \frac{(1-\delta)|b_1|}{[m]_q}$$

and if $q \uparrow 1$, then

$$|a_m| \leq \frac{(1-\delta)|b_1|}{m}, \quad m = \{2,3,\ldots\}$$

Remark 3.16. Let $p(z) \in \mathcal{P}$ and $\phi(z) = 1 + \frac{2}{\pi^2}\left(\ln \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$. If $q \uparrow 1$,

(1) $\beta = \pi$ and $G(z) = p(z)$, then Theorem 3.13 becomes the result in [12].
(2) and $G(z) = p(z)$, then Theorem 3.13 becomes the result in [16].
(3) and $G(z) = \phi(z)$, then Theorem 3.13 becomes the result in [18].
(4) and $\beta = 0$, then Theorem 3.13 becomes the result in [17].

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