Eur. J. Math. Anal. 2 (2022) 14
doi: 10.28924/ada/ma.2.14

Analytical Approximations for the Principal Branch of the Lambert *W* Function

Roy M. Howard

School of Electrical Engineering, Computing and Mathematical Sciences, Curtin University, GPO Box U1987, Perth, 6845, Australia

Correspondence: r.howard@curtin.edu.au

ABSTRACT. A geometric based approach for specifying approximations to the Lambert *W* function, which can achieve any set relative error bound over the interval $[0, \infty)$, is detailed. Approximations that can achieve arbitrarily high accuracy for the interval $[-1/e, 0]$, based on a two point spline approximation, are specified. Iterative methods can be used to improve the accuracy of the approximations.

Applications include, first, analytical expressions, with set relative error bounds, for the Lambert *W* function over the interval $[0, \infty)$. Second, approximations, with an arbitrarily low relative error, for upper and lower bounds for the Lambert *W* function. Third, analytical expressions for the evaluation of $\lfloor W(y) \rfloor$ and the integral of $\lfloor W(y) \rfloor$, for $y \in [0, \infty)$, without knowledge of $W(y)$. Fourth, a direct approach for evaluating the Lambert *W* function to achieve a prior set error constraint.

1. INTRODUCTION

The Lambert *W* function is associated with Lambert [19] and is a multivalued complex function with the single valued function, associated with the kth branch, being denoted W_k. The principle branch of the Lambert *W* function, W_0, is the function defined by the inverse of $y = f(x) = xe^x$ for the case of $\mathrm{Re}[x] \geq -1$, i.e.

$$x = W_0(y) = f^{-1}(y). \tag{1}$$

The Lambert *W* function does not have an explicit analytical form but is of importance consistent with increasing applications as detailed in the literature, e.g. [7], [3], [25], [9], [6], [14], [26], [20], [18], [4], and [11]. Generalizations of the Lambert *W* function are also of interest, e.g. [8] and [22]. Efficient numerical methods for computing values of the Lambert *W* function have long been known, e.g. [10], and with an advance detailed by Fukushima [12].

The focus of this paper is on the principle branch and the real case which continues to receive research interest, e.g. [17]. The graph of the Lambert *W* function, for this case, is shown in Figure 1 and, for notational simplicity, is denoted W in this paper. Existing analytical approximations for the principle branch and real case of the Lambert *W* function, e.g. [5], [3] and [17], are, in general, custom and cannot be directly generalized to obtain approximations of arbitrarily high accuracy. This paper provides a geometrical basis for defining such approximations.

Received: 31 Jan 2022.

Key words and phrases. Lambert W function; two point spline approximation; upper and lower bounded functions.

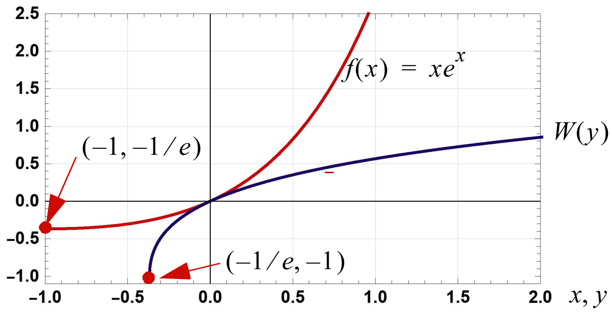

FIGURE 1. Graph of $f(x) = xe^x$ and its inverse which is the Lambert W function for the principle branch and the real case.

The advances in this paper are twofold. First, a systematic geometric approach for defining approximations to the Lambert W function, over the interval $[0, \infty)$, with quadratic convergence. Convergence is proved. The approximations can be used to specify, with an arbitrarily low relative error, upper and lower bounds for the Lambert W function. Second, a systematic method for combining series expansions at $-1/e$, and the origin, to define arbitrarily accurate approximations for the interval $[-1/e, 0]$. Applications of the approximations include analytical expressions, with set relative error bounds, for the Lambert W function over the interval $[0, \infty)$, evaluation of $\lfloor W(y) \rfloor$ and the integral of $\lfloor W(y) \rfloor$, for $y \in [0, \infty)$, without knowledge of $W(y)$, and a direct approach for evaluating the Lambert W function to achieve a prior defined error.

A review of published approximations for the Lambert W function is provided in section 2. The proposed geometric approach for establishing approximations to the Lambert W function is detailed in section 3. Convergence is discussed in section 4 and in section 5 convergent two point spline based approximations, for the interval $[-1/e, 0]$, are detailed. The use of iterative methods, to improve the accuracy of approximations, is discussed in Section 6. Applications are detailed in section 7 and conclusions are stated in section 8.

1.1. Notation and Properties.

The notation of $x = W(y)$ is used which is consistent with the principle value of the Lambert W being the inverse function of $y = f(x) = xe^x$, $x \geq -1$, $y \geq -1/e$. Relevant properties of the Lambert W function are detailed in Appendix A.

For an arbitrary function f, defined over the interval $[\alpha, \beta]$, an approximating function f_A has a relative error, at a point x_1, defined according to $\mathrm{re}(x_1) = 1 - f_A(x_1)/f(x_1)$. The relative error bound for the approximating function, over the interval $[\alpha, \beta]$, is defined according to

$$\mathrm{re}_B = \max\{|\mathrm{re}(x_1)| : x_1 \in [\alpha, \beta]\}. \quad (2)$$

The notation $f^{(k)}(x) = \frac{d^k}{dx^k} f(x)$ is used.

Mathematica has been used to facilitate analysis and to obtain numerical results. In general, relative error results, associated with approximations to the Lambert W function, have been obtained by sampling specified intervals, in either a linear or a logarithmic manner, as appropriate, with 1000 points.

2. PUBLISHED APPROXIMATIONS

A Taylor series expansion, at the origin, for the Lambert *W* function is well known, e.g. [17], eqn. 2, and yields the following approximation which has a limited region of convergence:

$$T(y) = y - y^2 + \frac{3y^3}{2} - \frac{8y^4}{3} + \frac{125y^5}{24} - \frac{54y^6}{5} + \frac{16807y^7}{720} - \frac{16384y^8}{315} + \ldots \qquad |y| < \frac{1}{e}. \tag{3}$$

The relationship $y = xe^x$ implies $x + \ln(\pm x) = \ln(\pm y)$ (positive sign for $x, y > 0$; negative sign for $x, y < 0$) and, hence:

$$x \approx \begin{cases} y & |y| \ll 1 \\ \ln(y) & y \gg 1. \end{cases} \tag{4}$$

Such approximations suggest the more general approximation for the Lambert *W* function of

$$x = W(y) \approx \ln(1+y), \qquad y > \frac{-1}{e}. \tag{5}$$

Fritsch et al., [10], eqn. 6, utilizes an initial approximation of $W(y) \approx \ln(y)$ which is suitable for $y > e$.

2.1. Published Approximations.

The following is an overview of indicative published approximations for the Lambert *W* function. Additional useful references include [26] and [14].

Boyd [5], eqn. 5-7, proposed the approximation

$$\begin{aligned} W_{Bd}(y) = &-1 + \tanh\left[\frac{\sqrt{2(1+ey)}}{\ln(10) - \ln[\ln(10)]}\right] \cdot \left[\ln(11+ey) - \ln[\ln(11+ey)]\right] \cdot \\ &\left[1 + \frac{1}{10}\left[\ln(1+ey) - \frac{7}{5}\right]\exp\left[\frac{-3}{40} \cdot \left[\ln(1+ey) - \frac{7}{5}\right]^2\right]\right] \end{aligned} \tag{6}$$

which is valid for $y \geq -1/e$. Whilst the approximation is sharp at $y = -1/e$, it is not zero at the origin and, thus, is not sharp at this point. It has a relative error bound for the interval $[1, \infty)$ of 0.0499.

Barry et al. [3], eqn. 12, proposed the approximation

$$W_{B_1}(y) = \ln\left[\frac{6}{5} \cdot \frac{y}{\ln\left[\frac{12}{5} \cdot \frac{y}{\ln[1+12y/5]}\right]}\right] \tag{7}$$

for $y > 0$. This approximation was modified, [3], eqn. 15, to be valid for $y > -1/e$ according to

$$W_{B_2}(y) = [1+\delta]\ln\left[\frac{6}{5} \cdot \frac{y}{\ln\left[\frac{12}{5} \cdot \frac{y}{\ln[1+12y/5]}\right]}\right] - \delta\ln\left[\frac{2y}{\ln[1+2y]}\right], \qquad \delta = 0.4586887. \tag{8}$$

The relative error bound, for $y > 0$, is 1.96×10^{-3}.

Iacono and Boyd, [17], eqn. 17, proposed the following approximation which is valid for $y \in [-1/e, \infty)$

$$W_{I_1}(y) = \ln\left[1 + \frac{y}{1 + \frac{\ln(1+y)}{2}}\right]. \tag{9}$$

This approximation yields a relative error bound of 3.53×10^{-2} for $y \in [0, \infty)$. An improved approximation, [17], eqn. 19, 20, is

$$\begin{aligned} W_{I_2}(y) &= -1 + a \cdot \ln\left[\frac{1 + b\sqrt{1+ey}}{1 + c\ln[1+\sqrt{1+ey}]}\right], \\ c &= \frac{e^{1/a} - 1 - \sqrt{2}/a}{1 - \ln(2)e^{1/a}}, \qquad b = \frac{\sqrt{2}}{a} + c, \end{aligned} \tag{10}$$

which yields a relative error bound for $y > 0$ of 4.53×10^{-3} (the bound occurs at y of the order of 10^{12}) for the case of a optimally chosen as $a = 2.036$. The approximation is sharp at $y = -1/e$.

2.1.1. *Padè Approximations*. Padè approximates for the Lambert *W* function for the interval $y \in [-1/e, 1]$ have been proposed, e.g. [21], eqn. 34:

$$W_L(y) = \frac{1 + \frac{123y}{40} + \frac{21y^2}{10}}{1 + \frac{143y}{40} + \frac{713y^2}{240}} \cdot \ln(1+y). \tag{11}$$

Higher order Padè approximates are detailed in Fukushima [13].

2.1.2. *Comparison of Approximations.* The relative errors in the above specified approximations are shown in Figure 2 and Figure 3.

2.2. **Classic Iterative Approximations.** The classical iterative approximation for the Lambert *W* function, e.g. [17], eqn. 8-10, is based on the fundamental relationship $x = \ln(y) - \ln(x)$, $x, y > 0$, and an initial approximation $x \approx W_0(y) = \ln(1+y)$ as specified in

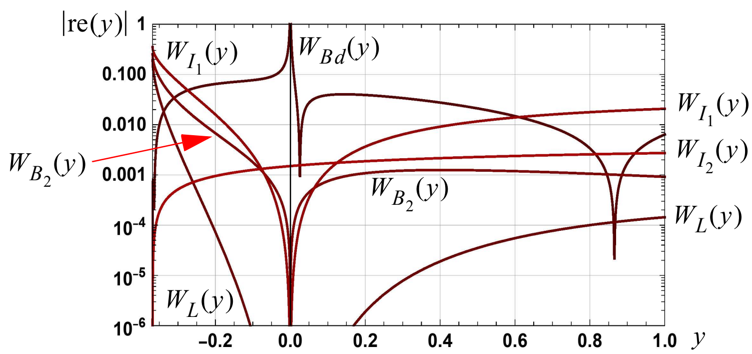

FIGURE 2. Graph of the relative errors, over the interval $[-1/e, 1]$, in published approximations to $W(y)$.

Equation 5. A first order approximation arises by substitution of this approximation into the expression $\ln(x)$ to yield

$$W_1(y) = \ln(y) - \ln[\ln(1+y)] = \ln\left[\frac{y}{\ln(1+y)}\right]. \tag{12}$$

Iteration yields

$$W_2(y) = \ln(y) - \ln\Big[\ln(y) - \ln[\ln(1+y)]\Big] = \ln\left[\frac{y}{\ln\left[\frac{y}{\ln(1+y)}\right]}\right]. \tag{13}$$

In general:

$$W_i(y) = \ln(y) - \ln[W_{i-1}(y)], \qquad i \in \{1, 2, 3, \dots\}, W_0(y) = \ln(1+y). \tag{14}$$

2.2.1. *Comparison of Approximations.* The relative error in the iterative approximations, of orders zero to three, are detailed in Figure 4 and Figure 5. The relative errors decrease, for large values of y, at an increasing rate as the order of approximation is increased. The approximations are poor for $y < 10$ which is consistent with the assumptions made in the iteration.

2.2.2. *Alternative Iterative Approximations.* An alternative iterative approach is to utilize the relationship $x = \ln(\pm y) - \ln(\pm x)$ and solve for the error given an initial approximation

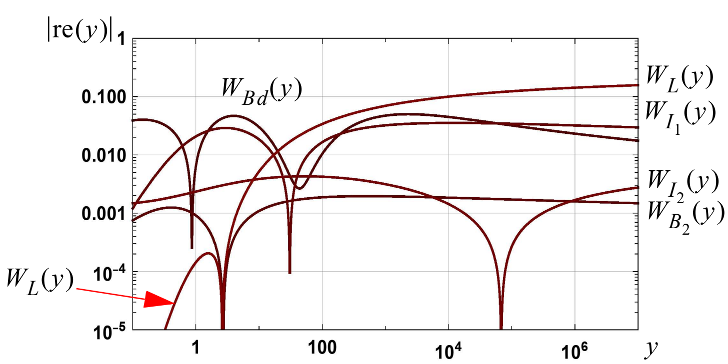

FIGURE 3. Graph of the relative errors in published approximations to $W(y)$.

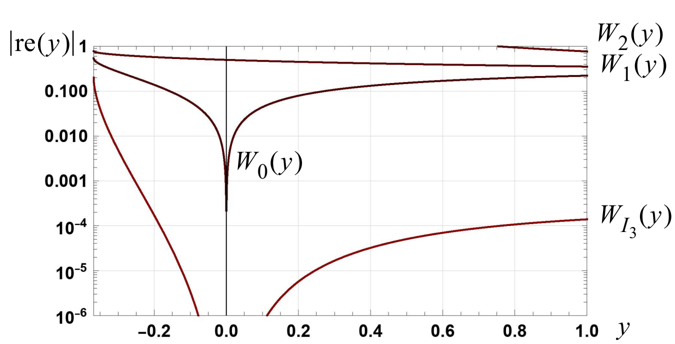

FIGURE 4. Graph of the relative errors, over the interval $[-1/e, 1]$, in iterative approximations to $W(y)$. The relative errors in the approximations for W_2 and W_3 are high.

of x_0. For y fixed, consider an initial approximation of x_0, with an error of ε_0, which implies $y = (x_0+\varepsilon_0)\exp(x_0+\varepsilon_0)$ and, thus, e.g. [17], eqn. 11:

$$x_0+\varepsilon_0 = \ln(\pm y) - \ln[\pm(x_0+\varepsilon_0)] = \ln\left[\frac{y}{x_0}\right] - \ln\left[1+\frac{\varepsilon_0}{x_0}\right]. \quad (15)$$

For the case where the error is small and $|\varepsilon_0/x_0| \ll 1$, a first order Taylor series for the logarithm function yields

$$x_0+\varepsilon_0 \approx \ln\left[\frac{y}{x_0}\right] - \frac{\varepsilon_0}{x_0}, \qquad \varepsilon_0 \approx \frac{x_0}{1+x_0}\cdot\left[\ln\left[\frac{y}{x_0}\right]-x_0\right], \quad (16)$$

and the first order approximation

$$x_1 = x_0+\varepsilon_0 \approx \frac{x_0}{1+x_0}\cdot\left[1+\ln\left[\frac{y}{x_0}\right]\right]. \quad (17)$$

The general iteration formula, e.g. [17], eqn. 12

$$x_{i+1} = \frac{x_i}{1+x_i}\cdot\left[1+\ln\left[\frac{y}{x_i}\right]\right], \quad (18)$$

then follows. Higher order iteration, based on a higher order approximation for $\ln[1+\varepsilon_0/x_0]$, is detailed in [10].

With a starting value, utilized in Fritsch et al. [10], of $x_0 = \ln(y)$ (suitable for $y > e$) it follows that a first order approximation is

$$W_F(y) = \frac{\ln(y)}{1+\ln(y)}\cdot\left[1+\ln\left[\frac{y}{\ln(y)}\right]\right]. \quad (19)$$

The relative error in this approximation is shown in Figure 5 and the relative error bound for the interval $[e,\infty)$ is 1.47×10^{-2}.

A first order iteration, based on Equation 18, with $x_0(y) = \ln\left[1+\frac{y}{1+0.5\ln(1+y)}\right]$, yields the approximation [17], eqn. 18:

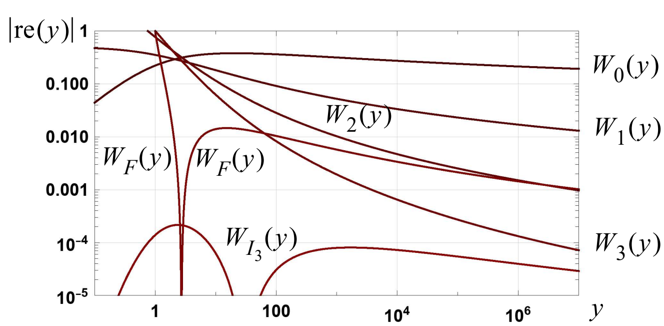

FIGURE 5. Graph of the relative errors in iterative approximations to $W(y)$.

$$W_{I_3}(y) = \frac{\ln\left[1+\frac{y}{1+0.5\ln(1+y)}\right]}{1+\ln\left[1+\frac{y}{1+0.5\ln(1+y)}\right]}\cdot\left[1+\ln\left[\frac{y}{\ln\left[1+\frac{y}{1+0.5\ln(1+y)}\right]}\right]\right]. \quad (20)$$

The relative error in this approximation is shown in Figure 4 and Figure 5. The relative error bound, over the interval $[0, \infty]$, is 2.16×10^{-4}.

2.3. **Approximations via Newton-Raphson Iteration.** Consistent with the illustration shown in Figure 6, a direct Newton-Raphson method for solving for x, in the equation $y = xe^x$, is:

$$x_i = x_{i-1} - \frac{x_{i-1}e^{x_{i-1}} - y}{e^{x_{i-1}} + x_{i-1}e^{x_{i-1}}} = x_{i-1} - \frac{x_{i-1} - ye^{-x_{i-1}}}{1 + x_{i-1}}. \quad (21)$$

A first iteration, based on a known approximating function g for W, is, e.g. [17], eqn. 15:

$$W(y) \approx g(y) - \frac{g(y)\exp[g(y)] - y}{\exp[g(y)] + g(y)\exp[g(y)]} = g(y) - \frac{g(y) - y\exp[-g(y)]}{1 + g(y)}. \quad (22)$$

Halley's method can similarly be utilized, e.g. [26].

With an initial value of $x_0 = \ln(1+y)$, the first and second order approximations, respectively, are:

$$W_{N_1}(y) = \ln(1+y) - \frac{\ln(1+y) - y/(1+y)}{1+\ln(1+y)}, \quad (23)$$

$$W_{N_2}(y) = \ln(1+y) - \frac{\ln(1+y) - y/(1+y)}{1+\ln(1+y)} - \frac{\ln(1+y) - \frac{\ln(1+y) - y/(1+y)}{1+\ln(1+y)} - \frac{y}{1+y}\cdot\exp\left[\frac{\ln(1+y) - y/(1+y)}{1+\ln(1+y)}\right]}{1+\ln(1+y) - \frac{\ln(1+y) - y/(1+y)}{1+\ln(1+y)}}. \quad (24)$$

2.3.1. *Results.* Graphs of the variation of the relative error, with iteration level, are shown in Figure 7 and Figure 8 for the case of an initial approximation of $x_0 = \ln(1+y)$.

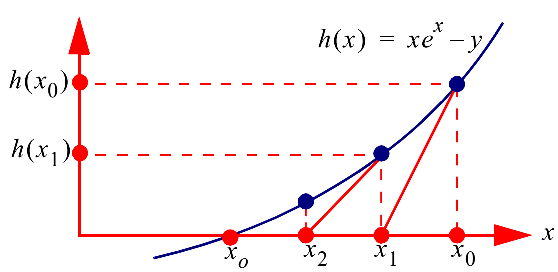

FIGURE 6. Newton-Raphson iteration for determining an approximation to the solution, denoted x_o, of $y = x\exp(x)$ for y fixed and based on an initial value of x_0.

Note, for higher order iteration, the relative error increases, from an increasingly low level, as y increases.

3. GEOMETRIC BASIS FOR ITERATIVE APPROXIMATIONS TO LAMBERT *W* FUNCTION

3.1. Geometric Basis.

To establish a systematic, geometrically based, approach for establishing approximations to the Lambert *W* function of arbitrarily high accuracy, consider a set value of y. The Lambert *W* function associated with y, denoted x_o, is such that $x_o = ye^{-x_o}$ and is the point defined by the intersection of the two curves x and ye^{-x} as illustrated in Figure 9 for the case of $y > 0$. The geometry associated with this intersection of the two curves is the basis for an initial approximation and for the iterative approximations detailed in the following theorem.

Theorem 3.1. **Iterative Approximations for Lambert *W* Function.** *For y fixed, $y > -1/e$, approximations to the Lambert W function can be iteratively defined according to*

$$\begin{aligned} W_{L_i} &= \frac{W_{L_{i-1}}(1 + W_{U_{i-1}})}{1 + W_{L_{i-1}}}, \qquad i \in \{1, 2, \ldots\}, W_{L_0} = y, \\ W_{U_i} &= \ln\left[\frac{y}{W_{L_i}}\right], \qquad i \in \{1, 2, \ldots\}, W_{U_0} = 0. \end{aligned} \tag{25}$$

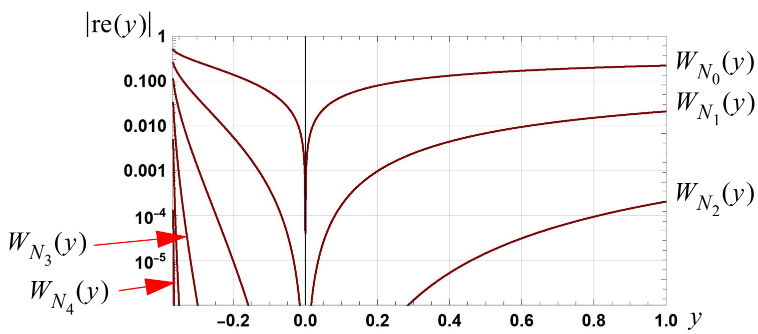

FIGURE 7. Graph of the relative errors, in the zero to fourth order iterative approximations to $W(y)$, based on Newton-Raphson iteration with an initial approximation of $W_{N_0}(y) = \ln(1 + y)$.

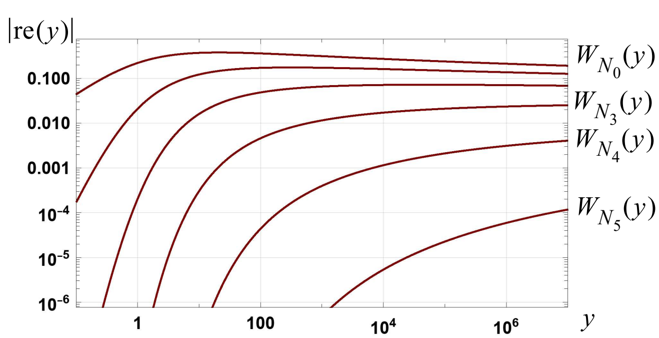

FIGURE 8. Graph of the relative errors, in the zero to fifth order iterative approximations to $W(y)$, based on Newton-Raphson iteration with an initial approximation of $W_{N_0}(y) = \ln(1 + y)$.

Proof. Consider the geometry detailed in Figure 9, for the case of $y > 0$, and an initial approximation to x_o of $W_{L_1} = y/(1+y)$ which is based the intersection of the first order Taylor series for ye^{-x} at the origin, i.e. $y - yx$, and x. An associated approximation, denoted W_{U_1}, arises from the intersection of the level defined by W_{L_1} and the curve ye^{-x}. The solution is

$$W_{U_1} = \ln\left[\frac{y}{W_{L_1}}\right] = \ln(1+y). \tag{26}$$

Second order approximations follow from the intersection of x with a first order Taylor series of ye^{-x}, based on the point W_{U_1}, i.e. $W_{L_1} - (x - W_{U_1})W_{L_1}$, to yield

$$W_{L_2} = \frac{W_{L_1}[1 + W_{U_1}]}{1 + W_{L_1}}, \tag{27}$$

and by the intersection of this level with the curve ye^{-x} to yield

$$W_{U_2} = \ln\left[\frac{y}{W_{L_2}}\right] = \ln\left[\frac{1+2y}{1+\ln(1+y)}\right]. \tag{28}$$

The general iterative form:

$$W_{L_i} = \frac{W_{L_{i-1}}(1 + W_{U_{i-1}})}{1 + W_{L_{i-1}}}, \qquad W_{U_i} = \ln\left[\frac{y}{W_{L_i}}\right], \qquad i \in \{2, 3, \ldots\}, \tag{29}$$

then follows. Whilst the geometry, and the defined approximations, are clearly defined for the case of $y > 0$, the approximations are also valid for $y \in (-1/e, 0)$ as simulation results, shown in Figure 10, demonstrate.

3.1.1. *Explicit Approximations*. Approximations to the Lambert *W* function, of orders one to five, are:

$$W_{L_1}(y) = \frac{y}{1+y}, \qquad W_{U_1}(y) = \ln(1+y) \tag{30}$$

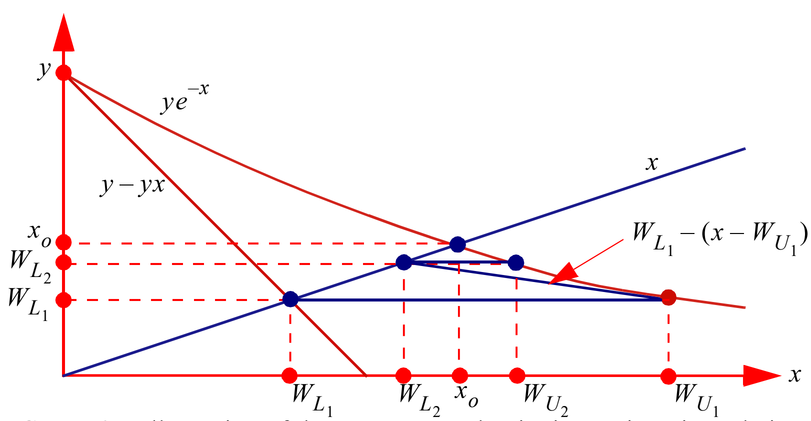

FIGURE 9. Illustration of the geometry underpinning an iterative relationship to find an approximation to x_o which is the solution of $x = y\exp(-x)$ for y fixed, $y > 0$.

$$W_{L_2}(y) = \frac{y[1+\ln(1+y)]}{1+2y}, \qquad W_{U_2}(y) = \ln\left[\frac{1+2y}{1+\ln(1+y)}\right] \tag{31}$$

$$W_{L_3}(y) = \frac{y[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}{1+3y+y\ln(1+y)} \tag{32}$$

$$W_{U_3}(y) = \ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right] \tag{33}$$

$$W_{L_4}(y) = \frac{y[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]\left[1+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]\right]}{1+4y+y\ln\left[\frac{1+2y}{1+\ln[1+y]}\right]+y\ln(1+y)\left[2+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]} \tag{34}$$

$$W_{U_4}(y) = \ln\left[\frac{1+4y+y\ln\left[\frac{1+2y}{1+\ln[1+y]}\right]+y\ln(1+y)\left[2+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]\left[1+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]\right]}\right] \tag{35}$$

$$W_{L_5}(y) = \frac{yn_5(y)}{d_5(y)}, \qquad W_{U_5}(y) = \ln\left[\frac{d_5(y)}{n_5(y)}\right] \tag{36}$$

where

$$n_5(y) = [1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]\left[1+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]\right]\cdot$$
$$\left[1+\ln\left[\frac{1+4y+y\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]+y\ln(1+y)\left[2+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]\left[1+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]\right]}\right]\right] \tag{37}$$

$$d_5(y) = 1 + 5y + y\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right] +$$
$$y\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\left[2+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]\right] + \quad (38)$$
$$y\ln[1+y]\left[\begin{array}{l} 3+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right] + \\ \ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\left[2+\ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]\right]\end{array}\right]$$

3.1.2. *Notes.* The approximation $W_{U_1}(y) = \ln(1+y)$ is the approximation stated in Equation 5 and is the basis for the Newton-Raphson iteration leading to Equation 23 and Equation 24.

Consider the iteration formula specified in Equation 18 ([17], eqn. 12). With a starting value of $x_0 = y/(1+y)$, it follows that a first order iteration leads to the approximation

$$W_1(y) = \frac{y}{1+2y} \cdot [1+\ln(1+y)], \quad (39)$$

which is $W_{L_2}(y)$ as specified by Equation 31.

3.1.3. *Results.* The relative errors associated with the approximations, specified in Theorem 3.1, are shown in Figure 10 and Figure 11, whilst the relative error bounds, for the interval $[0, \infty]$, are detailed in Table 1. Note the quadratic convergence.

TABLE 1. Relative error bounds, over the interval $[0, \infty]$, for the iterative approximations to the Lambert W function defined in Theorem 3.1.

Iteration order: *i*	**Relative error bound for W_{L_i}**	**Relative error bound for W_{U_i}**
1	increasing	0.381
2	increasing	0.0569
3	8.32×10^{-3}	1.33×10^{-3}
4	3.88×10^{-6}	7.23×10^{-7}
5	1.08×10^{-12}	2.15×10^{-13}
6	9.33×10^{-26}	1.90×10^{-26}

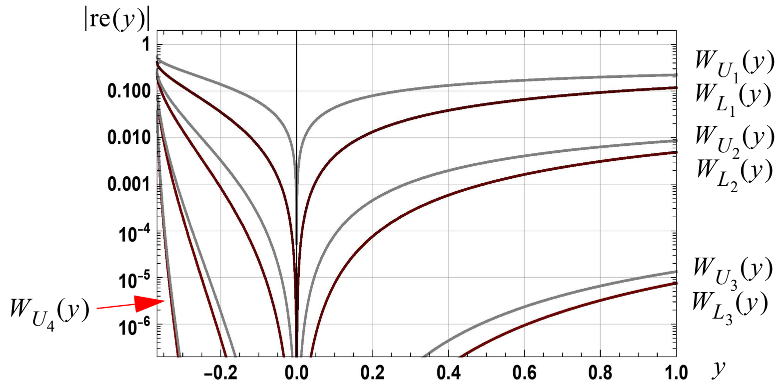

FIGURE 10. Graph of the relative errors in the iterative approximations to the Lambert W function defined in Theorem 3.1.

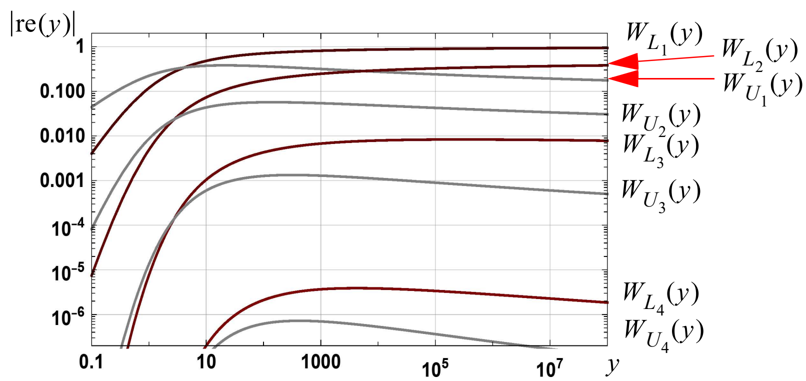

FIGURE 11. Graph of the relative errors in the iterative approximations to the Lambert W function defined in Theorem 3.1.

3.2. **Alternative Iterative Formulas.** The approximations W_{L_i} and W_{U_i} for the Lambert W function, as specified in Theorem 3.1, can be also be defined via iterative formulas based on numerator and denominator expressions.

Theorem 3.2. **Alternative Iterative Formulas for Lambert *W*.** *The approximations* W_{L_i} *and* W_{U_i}, $i \in \{1, 2, \dots\}$, *for the Lambert W function can be specified according to*

$$W_{L_i}(y) = \frac{n_i(y)}{d_i(y)}, \qquad W_{U_i}(y) = \ln[d_i(y)] - \ln\left[\frac{n_i(y)}{y}\right], \qquad i \in \{1, 2, \dots\}, \tag{40}$$

where

$$\begin{aligned} n_i(y) &= n_{i-1}(y)\left[1 + \ln[d_{i-1}(y)] - \ln\left[\frac{n_{i-1}(y)}{y}\right]\right], \qquad n_0(y) = y,\, d_0(y) = 1, \\ d_i(y) &= n_{i-1}(y) + d_{i-1}(y). \end{aligned} \tag{41}$$

Proof. The proof is detailed in Appendix B.

3.2.1. *Explicit Formulas.* Explicit formulas, for the first to fourth order approximations, are:

$$n_1(y) = y, \qquad d_1(y) = 1+y, \tag{42}$$

$$n_2(y) = y[1+\ln(1+y)], \qquad d_2(y) = 1+2y, \tag{43}$$

$$\begin{aligned} n_3(y) &= y[1+\ln(1+y)]\Big[1+\ln[1+2y]-\ln[1+\ln(1+y)]\Big], \\ d_3(y) &= 1+3y+y\ln(1+y), \end{aligned} \tag{44}$$

$$\begin{aligned} n_4(y) = {} & y[1+\ln(1+y)]\Big[1+\ln[1+2y]-\ln[1+\ln(1+y)]\Big]\cdot \\ & \Bigg[1+\ln[1+3y+y\ln(1+y)]-\ln\Bigg[[1+\ln(1+y)]\cdot\Big[1+\ln[1+2y]-\ln[1+\ln(1+y)]\Big]\Bigg]\Bigg], \end{aligned} \tag{45}$$

$$d_4(y) = 1+3y+y\ln(1+y)+y[1+\ln(1+y)]\cdot\Big[1+\ln(1+2y)-\ln[1+\ln(1+y)]\Big]. \tag{46}$$

3.3. **Alternative Geometrical Approach.** An alternative approach that leads to the approximations W_{U_1}, W_{U_2}, ..., as specified in Theorem 3.1, is to utilize the transformation $x = \ln(z)$, $x \geq 0$, $z \geq 1$ in the relationship $y = xe^x$ which implies $y = z\ln(z)$. The geometry underpinning the iteration that leads to the approximations is illustrated in Figure 12.

Theorem 3.3. Alternative Geometrical Iteration. *For y fixed, $y > 0$, iterative approximations for the Lambert W function can be defined according to*

$$\begin{aligned} z_i &= z_{i-1} - \frac{z_{i-1}\ln[z_{i-1}]-y}{\ln[z_{i-1}]+1}, \qquad z_1 = 1+y, \\ x_i &= \ln(z_i), \end{aligned} \tag{47}$$

and it is the case that $x_i = W_{U_i}(y)$ as specified in Theorem 3.1.

Proof. Consider the case of y fixed and the geometry illustrated in Figure 12 which is based on affine approximations to find, iteratively, the solution z_o to $y = z\ln(z)$. The ini-

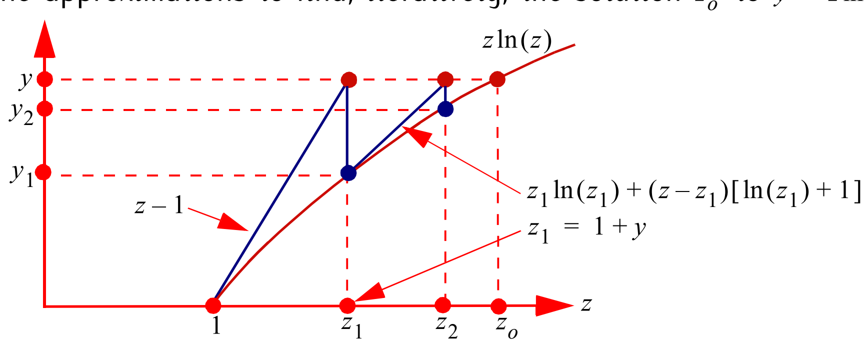

FIGURE 12. Illustration of the geometry underpinning the iterative relationship to find approximations to z_o which is the solution of $z\ln(z) = y$ for y fixed.

tial value is established by the intersection of a first order Taylor series for $z\ln(z)$ at the point $z = 1$, which is $z-1$, and the level y. The solution is $z_1 = 1+y$. A first order Taylor series for $z\ln(z)$ at this point is $z_1\ln(z_1)+(z-z_1)[\ln(z_1)+1]$ and the intersection of this approximation with the level y is the point z_2 defined according to

$$z_2 = z_1 - \frac{z_1\ln[z_1]-y}{\ln[z_1]+1}. \tag{48}$$

Iteration in this manner leads to the stated general iteration formulas.

3.3.1. *Explicit Formulas*. Approximations, of orders one to three, are:

$$z_1(y) = 1+y, \qquad x_1 = W_{U_1}(y) = \ln(1+y), \tag{49}$$

$$z_2(y) = \frac{1+2y}{1+\ln(1+y)}, \qquad x_2 = W_{U_2}(y) = \ln\left[\frac{1+2y}{1+\ln(1+y)}\right], \tag{50}$$

$$\begin{aligned} z_3(y) &= \frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}, \\ x_3 = W_{U_3}(y) &= \ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]. \end{aligned} \tag{51}$$

3.4. **Iterative Algorithm for (-1/e,0].** For the case of $y \in (-1/e, 0]$, the geometric approach, illustrated in Figure 13, can be utilized to establish an algorithm for determining approximations to the Lambert *W* function.

Theorem 3.4. Iterative Algorithm for (-1/*e*,0]. *An iterative algorithm for defining approximations to* $x = W(y)$, *for* $y \in (-1/e, 0]$, *is*:

$$\begin{aligned} x_i &= \frac{y_{i-1}(1+x_{i-1})}{1+y_{i-1}}, \qquad x_0 = 0, y_0 = y, \\ y_i &= ye^{-x_i}. \end{aligned} \tag{52}$$

Proof. Consider the illustration shown in Figure 13. With an initial approximation for x_o of $x_0 = 0$, a first order Taylor series for ye^{-x}, based on the point (x_0, y_0), with $y_0 = y$, is $y_0[1-(x-x_0)]$. The intersection of this Taylor series with x, at the point x_1, leads to

$$x_1 = \frac{y_0(1+x_0)}{1+y_0}. \tag{53}$$

The value of ye^{-x} associated with x_1 is $y_1 = ye^{-x_1}$. A first order Taylor series approximation for ye^{-x} at the point (x_1, y_1) is $y_1[1-(x-x_1)]$ and the intersection of this approximation with x, at the point x_2, leads to

$$x_2 = \frac{y_1(1+x_1)}{1+y_1}. \tag{54}$$

The value of ye^{-x} associated with this value is $y_2 = ye^{-x_2}$. Iteration in this manner leads to the general formula as stated in the theorem.

3.4.1. *Explicit Approximations.* Approximations, for orders one to four, are:

$$W_{E_1}(y) = \frac{y}{1+y}, \qquad W_{E_2}(y) = \frac{y(1+2y)}{(1+y)[y+e^{y/(1+y)}]} \tag{55}$$

$$W_{E_3}(y) = \frac{y\left[y(2+3y)+(1+y)e^{y/(1+y)}\right]}{(1+y)[y+e^{y/(1+y)}]\left[y+\exp\left[\frac{y(1+2y)}{(1+y)[y+e^{y/(1+y)}]}\right]\right]} \tag{56}$$

$$W_{E_4}(y) = \frac{y\left[\begin{array}{c} y^2(3+4y)+2y(1+y)e^{y/(1+y)}+y(1+y)\exp\left[\frac{y(1+2y)}{(1+y)[y+e^{y/(1+y)}]}\right]+ \\ (1+y)\exp\left[\frac{y[1+3y+e^{y/(1+y)}]}{(1+y)[y+e^{y/(1+y)}]}\right] \end{array}\right]}{(1+y)[y+e^{y/(1+y)}]\left[y+\exp\left[\frac{y(1+2y)}{(1+y)[y+e^{y/(1+y)}]}\right]\right]\cdot \left[y+\exp\left[\frac{y[y(2+3y)+(1+y)e^{y/(1+y)}]}{(1+y)[y+e^{y/(1+y)}]\left[y+\exp\left[\frac{y(1+2y)}{(1+y)[y+e^{y/(1+y)}]}\right]\right]}\right]\right]} \tag{57}$$

3.4.2. *Results.* The relative error in the iterative approximations specified in Theorem 3.4 are shown in Figure 14. For orders two, and higher, the approximation are more accurate than the approximations detailed in Theorem 3.1.

3.5. **Improved Approximations for [0,∞).** Improved approximations, for the interval $[0, \infty)$, can be established by using the iteration formula specified in Theorem 3.3 and by using

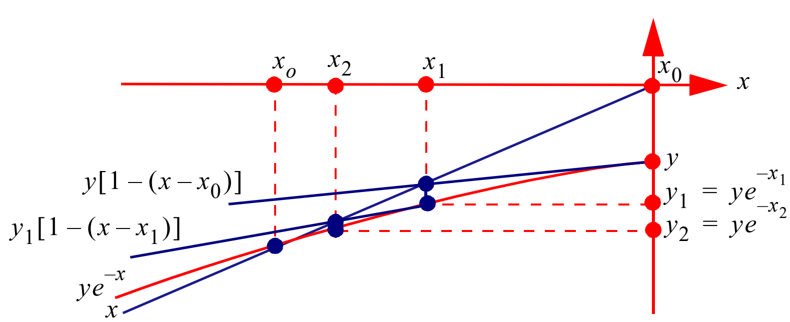

FIGURE 13. Illustration of the geometry underpinning the iterative relationship to find approximations to x_o, the solution of $x = ye^{-x}$, for the case of y fixed and $y \in (-1/e, 0]$.

an initial approximation of $z_1(y) = 1+ky$ rather than $z_1(y) = 1+y$. The resulting approximations, of orders one to four, are:

$$z_1(y) = 1+ky, \qquad W_{k_1}(y) = \ln(1+ky), \tag{58}$$

$$z_2(y) = \frac{1+(1+k)y}{1+\ln(1+ky)}, \qquad W_{k_2}(y) = \ln\left[\frac{1+(1+k)y}{1+\ln(1+ky)}\right], \tag{59}$$

$$\begin{aligned} z_3(y) &= \frac{1+(2+k)y+y\ln(1+ky)}{[1+\ln(1+ky)]\left[1+\ln\left[\frac{1+(1+k)y}{1+\ln(1+ky)}\right]\right]}, \\ W_{k_3}(y) &= \ln[z_3(y)], \end{aligned} \tag{60}$$

$$\begin{aligned} &z_4(y) = \\ &\frac{1+(3+k)y+y\ln\left[\frac{1+(1+k)y}{1+\ln[1+ky]}\right]+y\ln(1+ky)\left[2+\ln\left[\frac{1+(1+k)y}{1+\ln(1+ky)}\right]\right]}{[1+\ln(1+ky)]\left[1+\ln\left[\frac{1+(1+k)y}{1+\ln(1+ky)}\right]\right]\left[1+\ln\left[\frac{1+(2+k)y+y\ln(1+ky)}{[1+\ln(1+ky)]\left[1+\ln\left[\frac{1+(1+k)y}{1+\ln(1+ky)}\right]\right]}\right]\right]} \\ &W_{k_4}(y) = \ln[z_4(y)]. \end{aligned} \tag{61}$$

The improvement in the relative error bounds for the interval $[0,\infty)$, for values of k close to optimum, are detailed in Table 2.

For the case of a third order approximation, and for $k = \frac{4465}{10000}$, the maximum relative error is 1.023×10^{-4}. Thus, the approximation

$$W_{k_3}(y) = \ln\left[\frac{1+(2+k)y+y\ln(1+ky)}{[1+\ln(1+ky)]\left[1+\ln\left[\frac{1+(1+k)y}{1+\ln(1+ky)}\right]\right]}\right], \qquad k = \frac{4465}{10000}, \tag{62}$$

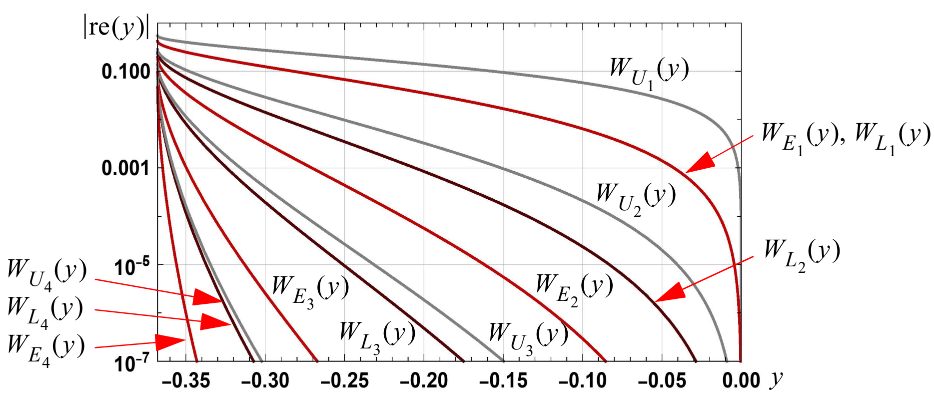

FIGURE 14. Graph of the relative error in approximations to $W(y)$, for the interval $[-1/e, 0]$, based on the iterative algorithms specified in Theorem 3.4 and Theorem 3.1.

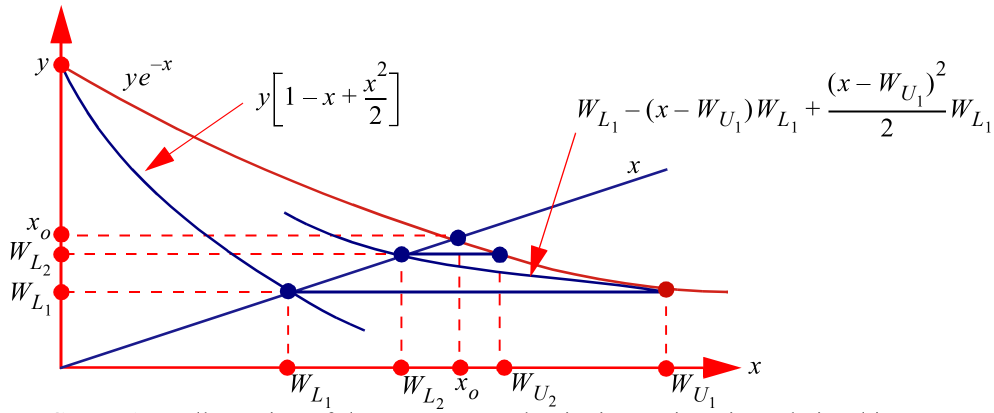

FIGURE 15. Illustration of the geometry underpinning an iterative relationship, based on quadratic approximations, to find an approximation to x_o which is the solution of $x = y\exp(-x)$ for y fixed, $y > 0$.

represents a good compromise between accuracy and complexity for the interval $[0, \infty)$ with a relative error bound of close to 10^{-4}.

TABLE 2. Relative error bounds, over the interval $[0, \infty)$, in the approximations $W_{k_i}(y)$, $i \in \{1, 2, \ldots, 6\}$, to the Lambert W function.

Iteration order: i	**Relative error bound: $k = 1$**	**Relative error bound: $k = 0.4$**	**Relative error bound: $k = 0.45$**	**Relative error bound: $k = 0.5$**
1	0.381	0.570	0.518	0.465
2	0.0569	0.0335	0.0250	0.0184
3	1.33×10^{-3}	1.87×10^{-4}	1.05×10^{-4}	1.50×10^{-4}
4	7.23×10^{-7}	6.46×10^{-9}	4.96×10^{-9}	9.97×10^{-9}
5	2.15×10^{-13}	7.98×10^{-18}	1.11×10^{-17}	4.43×10^{-17}
6	1.90×10^{-26}	1.24×10^{-35}	5.53×10^{-35}	8.78×10^{-34}

3.6. **Higher Accuracy via Iterative Quadratic Approximations.** The iterative approximation detailed in Theorem 3.1 can be improved upon by utilizing quadratic, rather than affine, approximations as illustrated in Figure 15.

Theorem 3.5. Iterative Quadratic Approximations for Lambert *W* Function. *An iterative formula, based on quadratic approximations, for the Lambert W function, and valid for* $y > -1/e$, *is*

$$W_{L_i} = \frac{1}{W_{L_{i-1}}}\left[1 + W_{L_{i-1}}[1 + W_{U_{i-1}}] - \sqrt{1 - W_{L_{i-1}}^2 + 2W_{L_{i-1}}[1 + W_{U_{i-1}}]}\ \right],$$
$$W_{U_i} = \ln\left[\frac{y}{W_{L_i}}\right], \qquad W_{L_1} = \frac{y}{1+y}, \qquad W_{U_1} = \ln(1+y). \tag{63}$$

Proof. The proof is detailed in Appendix C.

3.6.1. *Explicit Approximations.* Approximations, of orders one to three, are:

$$W_{L_1}(y) = \frac{y}{1+y}, \qquad W_{U_1}(y) = \ln(1+y) \tag{64}$$

$$W_{L_2}(y) = \frac{1}{y}\cdot\left[1+2y+y\ln(1+y)-\sqrt{1+4y+2y^2+2y(1+y)\ln(1+y)}\right]$$

$$W_{U_2}(y) = \ln\left[\frac{y^2}{1+2y+y\ln(1+y)-\sqrt{1+4y+2y^2+2y(1+y)\ln(1+y)}}\right] \quad (65)$$

$$W_{L_3}(y) = \frac{1}{1+2y+y\ln(1+y)-q_2(y)}\cdot$$

$$\left[\begin{array}{l} y+\left[1+2y+y\ln(1+y)-q_2(y)\right]\left[1+\ln\left[\frac{y^2}{1+2y+y\ln(1+y)-q_2(y)}\right]\right]- \\ \sqrt{\begin{array}{l} y^2-[1+2y+y\ln(1+y)-q_2(y)]^2+ \\ 2y\left[1+2y+y\ln(1+y)-q_2(y)\right]\left[1+\ln\left[\frac{y^2}{1+2y+y\ln(1+y)-q_2(y)}\right]\right]\end{array}}\end{array}\right] \quad (66)$$

$$q_2(y) = \sqrt{1+4y+2y^2+2y(1+y)\ln(1+y)}$$

$$W_{U_3}(y) = \ln\left[\frac{y}{W_{L_3}[i]}\right]. \quad (67)$$

3.6.2. *Results.* The relative errors in the approximations specified in Theorem 3.5 are shown in Figure 16 and Figure 17. The relative error bounds, for the interval $[0, \infty]$, are detailed in Table 3 and the convergence is cubic in nature. The approximation W_{U_3} has a relative error bound for the interval $[0, \infty]$ of 1.26×10^{-6}.

TABLE 3. Relative error bounds, over the interval $[0, \infty]$, for the approximations to the Lambert W function detailed in Theorem 3.5.

Iteration order *i*	**Maximum relative error in W_{L_i}**	**Maximum relative error in W_{U_i}**
1	increasing	0.381
2	increasing	0.0122
3	1.02×10^{-5}	1.26×10^{-6}
4	1.66×10^{-17}	2.04×10^{-18}
5	7.88×10^{-53}	9.63×10^{-54}

4. CONVERGENCE

Consider the case of $y > 0$, $x_o = W(y)$, and the error definitions

$$\varepsilon_{L_i} = x_o - W_{L_i}, \qquad \varepsilon_{U_i} = W_{U_i} - x_o, \quad (68)$$

associated with the ith upper and lower approximations detailed in Theorem 3.1 and as illustrated in Figure 18. The following results hold:

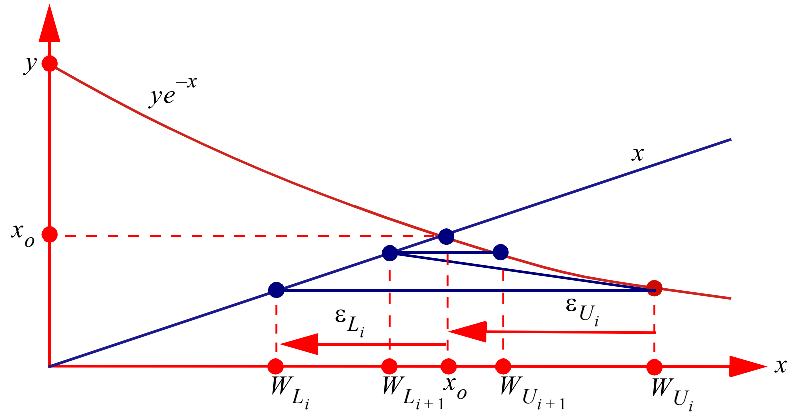

FIGURE 18. Error definitions associated with the ith upper and lower approximations to $x_o = W(y)$.

Theorem 4.1. **Convergence of Iterative Algorithm.** *For the case of y fixed and $x_o = W(y)$, the errors associated with the lower approximations W_{L_i}, $i \in \{1, 2, \ldots\}$, detailed in Theorem 3.1, are*

$$\varepsilon_{L_i} = \varepsilon_{L_{i-1}} - \frac{W_{L_{i-1}}[W_{U_{i-1}} - W_{L_{i-1}}]}{1 + W_{L_{i-1}}} = \varepsilon_{L_{i-1}} - \frac{W_{L_{i-1}}[\varepsilon_{L_{i-1}} + \varepsilon_{U_{i-1}}]}{1 + W_{L_{i-1}}}, \qquad \varepsilon_{L_1} = x_o - \frac{y}{1+y}. \quad (69)$$

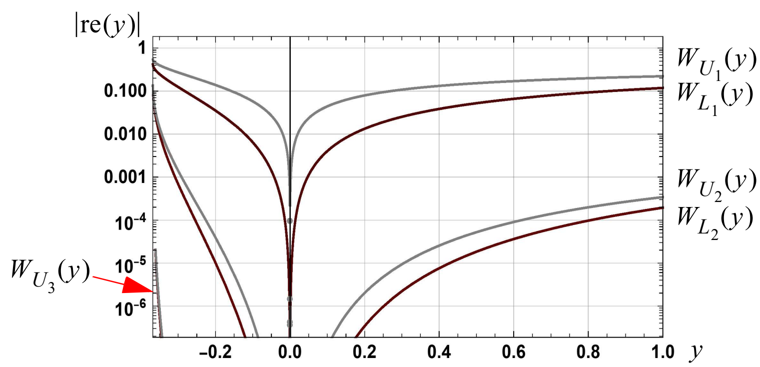

FIGURE 16. Graph of the relative errors in approximations to $W(y)$ as specified in Theorem 3.5.

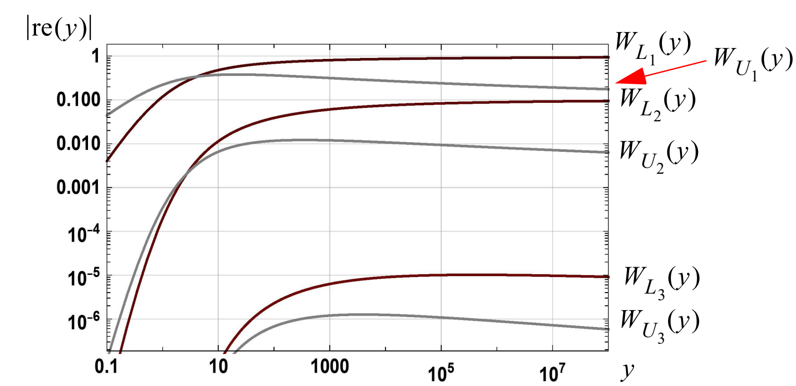

FIGURE 17. Graph of the relative errors in approximations to $W(y)$ as specified in Theorem 3.5.

The following results hold: First, the sequence W_{L_i} *is a monotonically increasing sequence, i.e.* $W_{L_i} > W_{L_{i-1}}$. *Second, the errors* ε_{L_i}, $i \in \{1, 2, \ldots\}$, *define a monotonically decreasing sequence, i.e.* $\varepsilon_{L_{i+1}} < \varepsilon_{L_i}$. *Third, convergence is guaranteed, i.e.* $\lim_{i \to \infty} \varepsilon_{L_i} = 0$, $\lim_{i \to \infty} W_{L_i} = x_o$ *and* $\lim_{i \to \infty} W_{U_i} = x_o$.

Proof. The proof of these results is detailed in Appendix D.

5. SPLINE BASED APPROXIMATION FOR [-1/e,0]

The approximations, detailed above in Theorem 3.1 and Theorem 3.4, are sharp at the origin but not at the point $-1/e$. It is useful to have approximations that are sharp at both points and which converge throughout the interval $[-1/e, 0]$, e. g. [3], eqn. 7 and [17], eqn. 20. The latter approximation is sharp at $-1/e$ but not at the origin. One approach, with potential, is to utilize the two point spline approximation for a function f as specified by Howard, [16], eqn. 40. For an interval $[\alpha, \beta]$, the nth order approximation can be written in the form (see Appendix E)

$$f_n(x) = (\beta - x)^{n+1} \sum_{r=0}^{n} a_{n,r}(x-\alpha)^r + (x-\alpha)^{n+1} \sum_{r=0}^{n} b_{n,r}(\beta - x)^r, \qquad x \in [\alpha, \beta], \tag{70}$$

where

$$\begin{aligned} a_{n,r} &= \frac{1}{(\beta-\alpha)^{n+1}} \cdot \sum_{u=0}^{r} \frac{f^{(r-u)}(\alpha)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta-\alpha)^u}, \\ b_{n,r} &= \frac{1}{(\beta-\alpha)^{n+1}} \cdot \sum_{u=0}^{r} \frac{(-1)^{r-u} f^{(r-u)}(\beta)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta-\alpha)^u}. \end{aligned} \tag{71}$$

5.1. Spline Based Approximations for [-1/e,0].

The spline approximation detailed in Equation 70 has the potential to provide approximations for the Lambert *W* function in the interval $[-1/e, 0]$ with exact values at the end points of this interval. However, an initial problem is that the derivatives of the Lambert *W* function are undefined at the point $-1/e$. This problem can be overcome by utilizing two suitable transformations.

Lemma 1. Transformations. *With* $f(x) = xe^x$, $x \geq -1$, *the first transformation*

$$y_1 = g_1(x_1) = \frac{1}{e} + f(x_1 - 1), \tag{72}$$

with $x = x_1 - 1$, $x_1 \geq 0$ *and* $y = y_1 - \frac{1}{e}$, $y \geq \frac{-1}{e}$, $y_1 \geq 0$ *yields*

$$g_1(x_1) = \frac{1}{e} \cdot [1 + (x_1 - 1)e^{x_1}], \qquad x_1 \geq 0, \tag{73}$$

$$W(y) = f^{-1}(y) = g_1^{-1}\left[y + \frac{1}{e}\right] - 1, \qquad y \geq \frac{-1}{e}. \tag{74}$$

With the second transformation of $g(x_1) = \sqrt{g_1(x_1)}$, *it follows that*

$$g(x_1) = \sqrt{\frac{1}{e} \cdot [1 + (x_1 - 1)e^{x_1}]}, \qquad x_1 \geq 0, \tag{75}$$

$$W(y) = g^{-1}\left[\sqrt{y + \frac{1}{e}}\right] - 1, \qquad y \geq -\frac{1}{e}. \tag{76}$$

Proof. The proofs for these results are detailed in Appendix F.

5.1.1. *Graphs and Values.* Consistent with Equation 76, the Lambert *W* function is defined in terms of g^{-1} for $y \in [-1/e, 0]$. Relevant values associated with the points $y = -1/e$ and $y = 0$ are specified in Table 4. The graph of $g(x_1)$, and its inverse $g^{-1}(y_2)$, are shown in Figure 19.

TABLE 4. Values associated with the points $y = -1/e$ and $y = 0$.

y	$y_1 = y + 1/e$	$x = W(y)$	$x_1 = x + 1$	$y_2 = g(x_1)$
$-1/e$	0	-1	0	0
0	$1/e$	0	1	$1/\sqrt{e}$

5.1.2. *Spline Based Approximations.* Consistent with Equation 76, and the values tabulated in Table 4, an approximation for the Lambert W, over the interval $[-1/e, 0]$, requires an approximation to the inverse of $g(x_1)$, $x_1 \in [0, 1]$, i.e. $g^{-1}(y_2)$, $y_2 \in [0, 1/\sqrt{e}]$, to be determined. A spline approximation for g^{-1}, of order n, requires derivatives, of orders zero to n, at the points 0 and $1/\sqrt{e}$ to be determined. Such values can be determined from the derivatives of g at the points 0 and 1 as detailed in Appendix G. The following approximations result.

Theorem 5.1. Spline Based Approximations for the Lambert *W* Function. *The* kth *order spline based approximation for the Lambert W function, based on the transformation* g *and the points* $[-1/e]$ *and* 0, *is*

$$W(k, y) = -1 + \sqrt{2e}\sqrt{y + \frac{1}{e}}\left[1 + \alpha_1\sqrt{y + \frac{1}{e}} + \alpha_2\left[y + \frac{1}{e}\right] + \alpha_3\left[y + \frac{1}{e}\right]^{3/2} + \ldots + \alpha_{2k}\left[y + \frac{1}{e}\right]^k\right], \tag{77}$$

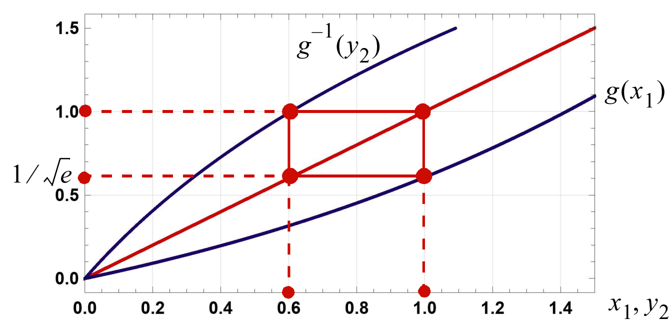

FIGURE 19. Graph of $y_2 = g(x_1)$ and its inverse $g^{-1}(y_2)$.

for $k \in \{1, 2, \ldots\}$ *and for appropriately defined constants.*

Proof. The proof is detailed in Appendix G.

5.1.3. *Explicit Approximations.* Explicit approximations, of orders one to four, are:

$$W(1,y) = -1+\sqrt{2e}\sqrt{y+\frac{1}{e}}\cdot\left[1-2\sqrt{e}\left[1+\frac{1}{\sqrt{2}e}-\frac{3}{2\sqrt{2}}\right]\sqrt{y+\frac{1}{e}}+e\left[1-\sqrt{2}+\frac{\sqrt{2}}{e}\right]\left[y+\frac{1}{e}\right]\right] \tag{78}$$

$$W(2,y) = -1+\sqrt{2e}\sqrt{y+\frac{1}{e}}\cdot\left[\begin{array}{l}1-\frac{\sqrt{2e}}{3}\sqrt{y+\frac{1}{e}}+\left[6e(\sqrt{2}-1)-\frac{7}{\sqrt{2}}-\frac{2\sqrt{2}}{e}\right]\left[y+\frac{1}{e}\right]-\\ \left[e^{3/2}\left[\frac{17}{\sqrt{2}}-8\right]-6\sqrt{2e}-\frac{4\sqrt{2}}{\sqrt{e}}\right]\left[y+\frac{1}{e}\right]^{3/2}+\\ \left[e^{2}\left[\frac{10\sqrt{2}}{3}-3\right]-\frac{5e}{\sqrt{2}}-2\sqrt{2}\right]\left[y+\frac{1}{e}\right]^{2}\end{array}\right] \tag{79}$$

$$W(3,y) = -1+\sqrt{2e}\sqrt{y+\frac{1}{e}}\cdot\left[\begin{array}{l}1-\frac{\sqrt{2e}}{3}\sqrt{y+\frac{1}{e}}+\frac{11e}{36}\left[y+\frac{1}{e}\right]-\\ \left[e^{3/2}\left[\frac{191}{9}-\frac{125}{3\sqrt{2}}\right]+\frac{25\sqrt{e}}{\sqrt{2}}+\frac{8\sqrt{2}}{\sqrt{e}}+\frac{6\sqrt{2}}{e^{3/2}}\right]\left[y+\frac{1}{e}\right]^{3/2}+\\ \left[e^{2}\left[\frac{281}{6}-\frac{146\sqrt{2}}{3}\right]+32\sqrt{2}e+22\sqrt{2}+\frac{18\sqrt{2}}{e}\right]\left[y+\frac{1}{e}\right]^{2}-\\ \left[e^{5/2}\left[\frac{335}{9}-40\sqrt{2}\right]+\frac{55e^{3/2}}{\sqrt{2}}+20\sqrt{2}\sqrt{e}+\frac{18\sqrt{2}}{\sqrt{e}}\right]\left[y+\frac{1}{e}\right]^{5/2}+\\ \left[e^{3}\left[\frac{371}{36}-\frac{34\sqrt{2}}{3}\right]+8\sqrt{2}e^{2}+6\sqrt{2}e+6\sqrt{2}\right]\left[y+\frac{1}{e}\right]^{3}\end{array}\right] \tag{80}$$

$$W(4,y) = -1+ \sqrt{2e}\sqrt{y+\frac{1}{e}}\cdot\left[\begin{array}{l}1-\frac{\sqrt{2e}}{3}\sqrt{y+\frac{1}{e}}+\frac{11e}{36}\left[y+\frac{1}{e}\right]-\frac{43e^{3/2}}{135\sqrt{2}}\left[y+\frac{1}{e}\right]^{3/2}+\\ \left[e^{2}\left[\frac{4075}{27\sqrt{2}}-\frac{895}{12}\right]-\frac{91e}{\sqrt{2}}-\frac{61}{\sqrt{2}}-\frac{27\sqrt{2}}{e}-\frac{64\sqrt{2}}{3e^{2}}\right]\left[y+\frac{1}{e}\right]^{2}-\\ \left[e^{5/2}\left[\frac{6658\sqrt{2}}{27}-\frac{2126}{9}\right]-\frac{315e^{3/2}}{\sqrt{2}}-110\sqrt{2e}-\frac{102\sqrt{2}}{\sqrt{e}}-\frac{256\sqrt{2}}{3e^{3/2}}\right]\left[y+\frac{1}{e}\right]^{5/2}+\\ \left[e^{3}\left[\frac{8467\sqrt{2}}{27}-\frac{1175}{4}\right]-207\sqrt{2}e^{2}-149\sqrt{2}e-144\sqrt{2}-\frac{128\sqrt{2}}{e}\right]\left[y+\frac{1}{e}\right]^{3}-\\ \left[e^{7/2}\left[\frac{4904\sqrt{2}}{27}-\frac{502}{3}\right]-\frac{245e^{5/2}}{\sqrt{2}}-90\sqrt{2}e^{3/2}-90\sqrt{2e}-\frac{256\sqrt{2}}{3\sqrt{e}}\right]\left[y+\frac{1}{e}\right]^{7/2}+\\ \left[e^{4}\left[\frac{10843}{135\sqrt{2}}-\frac{1315}{36}\right]-\frac{55e^{3}}{\sqrt{2}}-\frac{41e^{2}}{\sqrt{2}}-21\sqrt{2}e-\frac{64\sqrt{2}}{3}\right]\left[y+\frac{1}{e}\right]^{4}\end{array}\right] \tag{81}$$

5.1.4. *Results*. The relative errors in the spline based approximations to the Lambert *W* function, as specified in Theorem 5.1, of orders one to five, are shown in Figure 20. The relative error bounds over the interval $[-1/e, 0]$, for first to fifth order approximations, respectively, are: 1.27×10^{-2}, 9.68×10^{-4}, 8.69×10^{-5}, 8.46×10^{-6} and 8.65×10^{-7}. By construction, the approximations are sharp at the points $-1/e$ and 0. The results shown in Figure 20 indicate that the sequence of approximations have good convergence to the Lambert *W* function over the interval $[-1/e, 0]$ and modest convergence over the interval $[0, 1]$.

6. IMPROVED APPROXIMATIONS VIA ITERATION

Iteration is, potentially, effective in improving the accuracy of an initial approximation. One potential approach is to utilize the iteration potential in the fundamental relationship for the Lambert *W* function as specified by Equation 116. An alternative approach is to utilized the Newton-Raphson method. These two approaches are detailed below.

6.1. **Inherent Iteration.** The basis for an iterative relationship for the Lambert *W* function is Equation 116, i.e.

$$W(y) = \ln(y) - \ln[W(y)]. \tag{82}$$

It then follows that an approximation, $W_0(y)$, for $W(y)$, can potentially be improved upon according to

$$W_1(y) = \ln(y) - \ln[W_0(y)] \tag{83}$$

$$\begin{aligned} W_2(y) &= \ln(y) - \ln\Big[\ln(y) - \ln[W_0(y)]\Big] \\ W_{21}(y) &= \ln(y) - \ln\Big[\ln(y) - \ln[W_1(y)]\Big] \end{aligned} \tag{84}$$

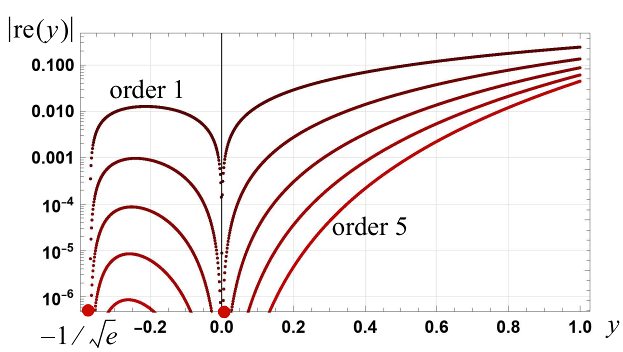

FIGURE 20. Graph of the relative errors in the spline based approximations, of orders one to five, to the Lambert W function as specified in Theorem 5.1.

$$W_3(y) = \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_0(y)]\right]\right]$$

$$W_{31}(y) = \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_1(y)]\right]\right]$$

$$W_{32}(y) = \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_2(y)]\right]\right] \quad (85)$$

$$W_{3,21}(y) = \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_{21}(y)]\right]\right]$$

$$W_4(y) = \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_0(y)]\right]\right]\right] \quad (86)$$

etc. The number of possible permutations for approximations to the Lambert W function is clearly large as the iteration order increases. Representative relative error bounds, for the interval $[0, \infty)$, are tabulated in Table 5 for the base approximations of $W_0(y) = W_{U_3}(y)$, $W_0(y) = W_{U_4}(y)$ and $W_0(y) = W_{U_5}(y)$ as specified in Theorem 3.1. Graphs of the relative errors, based on the approximations $W_0(y) = W_{U_4}(y)$ and $W_0(y) = W_{U_5}(y)$, are shown, respectively, in Figure 21 and Figure 22.

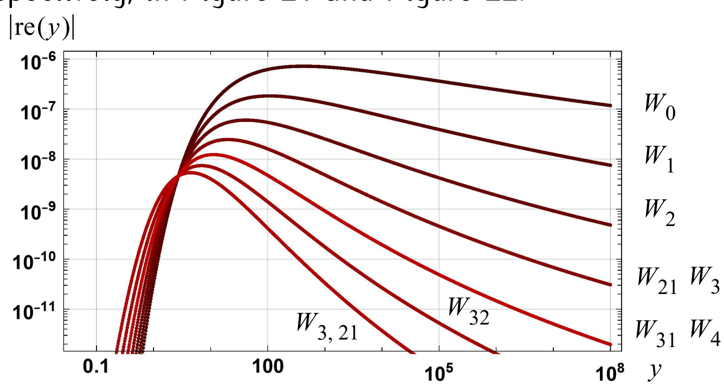

FIGURE 21. Graph of the relative errors in approximations to $W(y)$, based on $W_0(y) = W_{U_4}(y)$ (Equation 35).

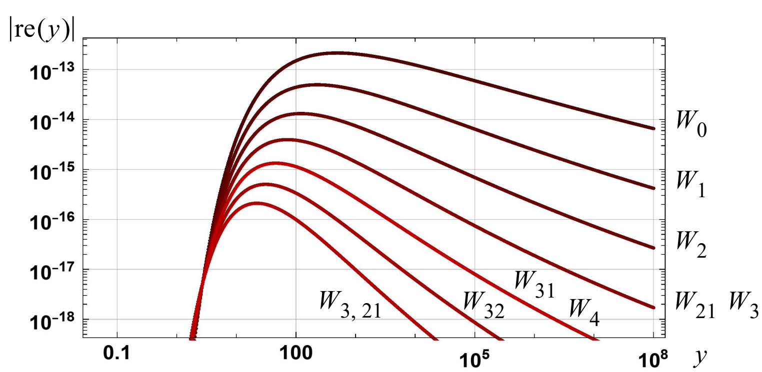

FIGURE 22. Graph of the relative errors in approximations to $W(y)$, based on $W_0(y) = W_{U_5}(y)$ (Equation 36).

TABLE 5. Relative error bounds, for the interval $[0, \infty)$, based on iteration and for the specified base approximations of $W_{U_3}(y)$ (Equation 33), $W_{U_4}(y)$ (Equation 35) and $W_{U_5}(y)$ (Equation 36).

Iteration Form	$W_0(y) = W_{U_3}(y)$	$W_0(y) = W_{U_4}(y)$	$W_0(y) = W_{U_5}(y)$
W_0	1.33×10^{-3}	7.23×10^{-7}	2.15×10^{-13}
W_1	4.08×10^{-4}	1.84×10^{-7}	4.95×10^{-14}
W_2	1.97×10^{-4}	6.03×10^{-8}	1.31×10^{-14}
W_{21}	1.40×10^{-4}	2.47×10^{-8}	3.95×10^{-15}
W_3	1.40×10^{-4}	2.47×10^{-8}	3.95×10^{-15}
W_{31}	1.46×10^{-4}	1.23×10^{-8}	1.34×10^{-15}
W_{32}	2.33×10^{-4}	7.43×10^{-9}	5.04×10^{-16}
$W_{3,21}$	6.65×10^{-4}	5.36×10^{-9}	2.11×10^{-16}
W_4	1.46×10^{-4}	1.23×10^{-8}	1.34×10^{-15}

6.1.1. *Explicit Approximations.* The approximation W_3, based on $W_0(y) = W_{U_3}(y)$, is

$$\begin{aligned} W_{3,3}(y) &= \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_{U_3}(y)]\right]\right] \\ &= \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln\left[\ln\left[\frac{1 + 3y + y\ln(1+y)}{[1 + \ln(1+y)]\left[1 + \ln\left[\frac{1+2y}{1 + \ln(1+y)}\right]\right]}\right]\right]\right]\right] \end{aligned} \quad (87)$$

and has a maximum relative error bound, over the interval $[0, \infty)$, of 1.40×10^{-4} which is a factor of 9.5 lower that the original approximation $W_{U_3}(y)$ whose relative error bound is 1.33×10^{-3}.

The approximation W_2, based on $W_0(y) = W_{U_4}(y)$, is

$$W_{2,4}(y) = \ln(y) - \ln\left[\ln(y) - \ln[W_{U_4}(y)]\right], \quad (88)$$

where $W_{U_4}(y)$ is specified by Equation 35. The maximum relative error bound, over the interval $[0, \infty)$, is 6.03×10^{-8} which is a factor of 12 lower that the original approximation $W_{U_4}(y)$ whose relative error bound is 7.23×10^{-7}.

The approximation W_3, based on $W_0(y) = W_{U_4}(y)$, is

$$W_{3,4}(y) = \ln(y) - \ln\left[\ln(y) - \ln\left[\ln(y) - \ln[W_{U_4}(y)]\right]\right], \quad \textbf{(89)}$$

and has a maximum relative error bound, over the interval $[0, \infty)$, of 2.47×10^{-8} which is a factor of 29.3 lower that the original approximation $W_{U_4}(y)$ whose relative error bound is 7.23×10^{-7}.

6.2. **Newton-Raphson Iteration.** Consistent with Equation 22, the approximations stated in Theorem 3.1, Theorem 3.2 and Theorem 3.5 can be utilized as the basis for iterative approximations based on the Newton-Raphson method. Results are detailed in Table 6.

TABLE 6. Relative error bounds, for the interval $[0, \infty)$, based on Newton-Raphson iteration and for the specified base approximations, $W_0(y)$, of $W_{U_2}(y)$, $W_{U_3}(y)$, $W_{U_4}(y)$ and $W_{U_5}(y)$.

Iteration Order	$W_0(y) = W_{U_2}(y)$ **(Equation 31)**	$W_0(y) = W_{U_3}(y)$ **(Equation 33)**	$W_0(y) = W_{U_4}(y)$ **(Equation 35)**	$W_0(y) = W_{U_5}(y)$ **(Equation 36)**
0	5.69×10^{-2}	1.33×10^{-3}	7.23×10^{-7}	2.15×10^{-13}
1	8.28×10^{-3}	5.12×10^{-6}	1.49×10^{-12}	1.30×10^{-25}
2	3.48×10^{-4}	9.61×10^{-11}	6.98×10^{-24}	5.02×10^{-50}
3	9.95×10^{-7}	3.91×10^{-20}	1.62×10^{-46}	7.67×10^{-99}
4	8.21×10^{-12}	7.08×10^{-39}	9.04×10^{-92}	1.81×10^{-196}
5	5.60×10^{-22}	2.43×10^{-76}	2.85×10^{-182}	1.02×10^{-391}

As an example, the approximation arising from a first order iteration of $W_{U_3}(y)$ (Equation 33), has a maximum relative error bound, over the interval $[0, \infty)$, of 5.12×10^{-6} which is a factor of 260 lower that the original approximation whose relative error bound is 1.33×10^{-3}. The approximation is:

$$W_{1,3}(y) = \ln\left[\frac{1 + 3y + y\ln(1+y)}{[1 + \ln(1+y)]\left[1 + \ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right] - \frac{\ln\left[\frac{1 + 3y + y\ln(1+y)}{[1 + \ln(1+y)]\left[1 + \ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right] - \frac{y[1+\ln(1+y)]\left[1 + \ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}{[1+3y+y\ln(1+y)]}}{1 + \ln\left[\frac{1 + 3y + y\ln(1+y)}{[1 + \ln(1+y)]\left[1 + \ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right]} \quad \textbf{(90)}$$

7. APPLICATIONS

7.1. **Approximation with Fixed Relative Error Bound.** There are many applications where the Lambert W function is used to model the physical nature/characteristics of an entity which are positive in nature. In many of these cases the parameter values are not known with high accuracy. For such cases, highly accurate computation of the Lambert W function is not required and a fixed approximation, with a set relative error bound, is useful rather than relying on, for example, iterative approximations where the relative error achieved depends on the initial approximate used. The relatively simple approximation for the Lambert W function, as specified by Equation 33, i.e.

$$W_{U_3}(y) = \ln\left[\frac{1+3y+y\ln(1+y)}{[1+\ln(1+y)]\left[1+\ln\left[\frac{1+2y}{1+\ln(1+y)}\right]\right]}\right], \quad (91)$$

with a relative error bound of 1.33×10^{-3} over the interval $[0,\infty)$, is likely to be useful. One application, for example, is in the evaluation of the collector current in a common emitter circuit, e.g. [2], eqn. 21.

For the interval $[-1/e, 0]$, the approximation $W(2,y)$ detailed in Equation 79, is of modest complexity, is sharp at the points $-1/e$ and 0 and has a relative error bound of 9.68×10^{-4}.

For highly accurate approximations over the interval $[0,\infty)$, an explicit analytical approximation, with a relative error bound of 7.98×10^{-18}, can be specified by utilizing the fifth order iterative approximation, denoted W_{k_5}, arising from the iteration specified by Equation 47 and with $z_1 = 1+ky$, $k = 0.4$ (see Table 2).

An alternative analytical expression, with a relative error bound of 1.30×10^{-25} over the interval $[0,\infty)$, is

$$W_{1,5}(y) = \ln\left[\frac{d_5(y)}{n_5(y)}\right] - \frac{\ln\left[\frac{d_5(y)}{n_5(y)}\right] - \frac{y n_5(y)}{d_5(y)}}{1+\ln\left[\frac{d_5(y)}{n_5(y)}\right]}, \quad (92)$$

where n_5 and d_5 are defined, respectively, in Equation 37 and Equation 38. This expression arises from a first iteration of the Newton-Raphson method (Equation 22) utilizing the fifth order approximation $W_{U_5}(y)$ specified in Equation 36. The relative error bound is specified in Table 6.

7.2. **Upper/Lower Bounds for Lambert W.** There is interest in upper/lower bounds for the Lambert W function, e.g. [15] and [24]. Alzahrani and Salem, [1], detail bounds for W_{-1}. The following bounds were proposed by Hoorfar (see, [17], eqn. 21)

$$\ln\left[\frac{y}{\ln(y)}\right] + \frac{\ln[\ln(y)]}{2\ln(y)} < W(y) < \ln\left[\frac{y}{\ln(y)}\right] + \frac{e}{e-1}\cdot\frac{\ln[\ln(y)]}{\ln(y)}. \quad (93)$$

For the interval $[e,\infty)$, the relative error bound associated with the lower bounded function is 0.0568; the relative error bound for the upper bounded function is 0.207.

The bounds proposed in [17], eqn. 25, 27, have modest relative errors for $y > e$ and are:

$$\ln\left[\frac{y}{\ln(y)}\right] - \frac{\ln\left[\frac{y}{\ln(y)}\right]}{1 + \ln\left[\frac{y}{\ln(y)}\right]} \cdot \ln\left[1 - \frac{\ln[\ln(y)]}{\ln(y)}\right] < W(y) <$$
$$\ln\left[\frac{y}{\ln(y)}\right] - \ln\left[\left[1 - \frac{\ln[\ln(y)]}{\ln[y]}\right] \cdot \left[1 - \frac{\ln\left[1 - \frac{\ln[\ln(y)]}{\ln(y)}\right]}{1 + \ln\left[\frac{y}{\ln(y)}\right]}\right]\right]. \quad \mathbf{(94)}$$

The relative error bounds in the lower and upper bounded functions, over the interval $[e, \infty)$, respectively, are 5.96×10^{-3} and 4.10×10^{-3}. The relative errors decrease for $y \gg 10$.

By construction, $W_{L_i}(y)$, $i \in \{1, 2, \ldots\}$, as defined in Theorem 3.1, are a sequence of increasingly accurate lower bounds for $W(y)$. Similarly, $W_{U_i}(y)$, $i \in \{1, 2, \ldots\}$, is a sequence of increasingly accurate upper bounds for $W(y)$. For example:

$$W_{L_3}(y) = \frac{y[1 + \ln(1 + y)]\left[1 + \ln\left[\frac{1 + 2y}{1 + \ln(1 + y)}\right]\right]}{1 + 3y + y\ln(1 + y)} < W(y) <$$
$$W_{U_3}(y) = \ln\left[\frac{1 + 3y + y\ln(1 + y)}{[1 + \ln(1 + y)]\left[1 + \ln\left[\frac{1 + 2y}{1 + \ln(1 + y)}\right]\right]}\right], \qquad y > 0, \quad \mathbf{(95)}$$

with relative error bounds for the interval $[0, \infty)$ of, respectively 8.32×10^{-3} and 1.33×10^{-3} (see Table 1). Higher order approximations lead to lower relative bounds and these can be made arbitrarily small. The relative error bounds associated with $W_{L_4}(y) < W(y) < W_{U_4}(y)$ over the interval $[0, \infty)$ are, respectively, 3.88×10^{-6} and 7.23×10^{-7}. The relative error bounds associated with $W_{L_5}(y) < W(y) < W_{U_5}(y)$ over the interval $[0, \infty)$ are, respectively, 1.08×10^{-12} and 2.15×10^{-13}.

One potential application for the upper bound is a bound for the prime counting function, e.g. [27].

7.3. **Spline Approximations Based on Upper/Lower Bounds.** Consider the ith upper, W_{U_i}, and lower, W_{L_i}, bounded functions for the Lambert *W* function as illustrated in Figure 23 and as defined in Theorem 3.1. For y fixed at y_o, a spline approximation, as specified by Equation 70 and based on the points $(u_o \exp(u_o), u_o)$, $u_o = W_{L_i}(y_o)$ and $(v_o \exp(v_o), v_o)$, $v_o = W_{U_i}(y_o)$, can readily be determined. From such an approximation, an approximation to $x_o = W(y_o)$ can then be specified.

Theorem 7.1. Spline Approximations Based on Upper/Lower Bounds. *Consider the* ith *lower and upper bounded approximations,* W_{L_i} *and* W_{U_i}*, defined in Theorem 3.1. The zero* *order spline approximation for the Lambert W function, based on the* ith *approximations* W_{L_i} *and* W_{U_i}*, is*

$$W_{0,i}(y) = \frac{W_{L_i}(y)W_{U_i}(y)\cdot\left[\exp[W_{U_i}(y)] - \exp[W_{L_i}(y)]\right] + y[W_{U_i}(y) - W_{L_i}(y)]}{W_{U_i}(y)\exp[W_{U_i}(y)] - W_{L_i}(y)\exp[W_{L_i}(y)]}. \tag{96}$$

The nth *order spline approximation for the Lambert W function, based on the* ith *approximations* W_{L_i} *and* W_{U_i}*, is*

$$\begin{aligned} W_{n,i}(y) = {} & \frac{[v_o\exp(v_o) - y]^{n+1}}{[v_o\exp(v_o) - u_o\exp(u_o)]^{n+1}}\cdot \\ & \sum_{r=0}^{n}[y - u_o\exp(u_o)]^r\cdot\left[\begin{array}{l}\dfrac{(n+r)!u_o}{r!n![v_o\exp(v_o) - u_o\exp(u_o)]^r} + \\ \displaystyle\sum_{u=0}^{r-1}\frac{p_{r-u}(u_o)e^{-(r-u)u_o}}{(r-u)![1+u_o]^{2(r-u)-1}}\cdot\frac{(n+u)!}{u!n!}\cdot\frac{1}{[v_o\exp(v_o) - u_o\exp(u_o)]^u}\end{array}\right] + \\ & \frac{[y - u_o\exp(u_o)]^{n+1}}{[v_o\exp(v_o) - u_o\exp(u_o)]^{n+1}}\cdot \\ & \sum_{r=0}^{n}[v_o\exp(v_o) - y]^r\cdot\left[\begin{array}{l}\dfrac{(n+r)!v_o}{r!n![v_o\exp(v_o) - u_o\exp(u_o)]^r} + \\ \displaystyle\sum_{u=0}^{r-1}\frac{(-1)^{r-u}p_{r-u}(v_o)e^{-(r-u)v_o}}{(r-u)![1+v_o]^{2(r-u)-1}}\cdot\frac{(n+u)!}{u!n!}\cdot\frac{1}{[v_o\exp(v_o) - u_o\exp(u_o)]^u}\end{array}\right] \end{aligned} \tag{97}$$

where $u_o = W_{L_i}(y)$, $v_o = W_{U_i}(y)$ *and* p_k *is defined by Equation 120.*

Proof. The proof is detailed in Appendix H.

7.3.1. *Results.* Results are detailed in Table 7 and clearly show the high accuracy of the approximations. For example, the zero order spline approximations, as specified by Equation 96, yields relative error bounds, for the interval $[0, \infty)$, of 3.84×10^{-5}, 8.56×10^{-12} and 6.80×10^{-25}, respectively, based on third, fourth and fifth order approximations for the

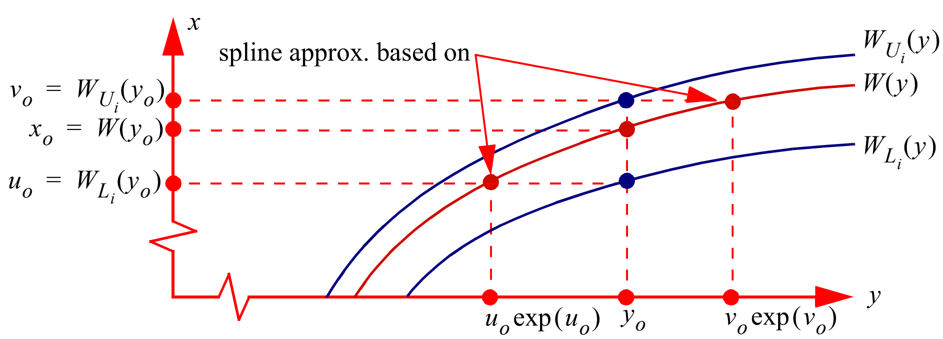

FIGURE 23. Illustration of upper and lower bounded approximations to the Lambert W function and the points $(u_o\exp(u_o), u_o)$, $(v_o\exp(v_o), v_o)$, which are the basis for spline based approximations.

Lambert W function detailed in Equation 3.1 Such convergence is approximately quadratic.

7.3.2. *Application*. The Omega constant, defined as $W(1)$, can be evaluated by using Equation 96 with relative errors, respectively, of 1.94×10^{-5}, 4.71×10^{-11}, 2.76×10^{-22} and 9.48×10^{-45} for the case of upper and lower bounded approximations of orders two to five, i.e. $W_{0,2}(1)$, $W_{0,3}(1)$, $W_{0,4}(1)$ and $W_{0,5}(1)$.

TABLE 7. Relative error bounds, over the interval $[0, \infty)$, for spline approximations to the Lambert W function based on upper and lower bounded functions.

Upper/lower bounded functions	Spline order	Approximation	Relative error bound
$W_{L_2}(y)$, $W_{U_2}(y)$	1	$W_{1,2}$	increasing re as $y \to \infty$
(Equation 31)	2	$W_{2,2}$	increasing re as $y \to \infty$
$W_{L_3}(y)$, $W_{U_3}(y)$	0	$W_{0,3}$	3.84×10^{-5}
(Equation 32, Equation 33)	1	$W_{1,3}$	1.92×10^{-8}
	2	$W_{2,3}$	1.46×10^{-11}
	3	$W_{3,3}$	1.31×10^{-14}
	4	$W_{4,3}$	1.27×10^{-17}
$W_{L_4}(y)$, $W_{U_4}(y)$	0	$W_{0,4}$	8.56×10^{-12}
(Equation 34, Equation 35)	1	$W_{1,4}$	5.60×10^{-22}
	2	$W_{2,4}$	5.05×10^{-32}
	3	$W_{3,4}$	5.18×10^{-42}
	4	$W_{4,4}$	5.68×10^{-52}
$W_{L_5}(y)$, $W_{U_5}(y)$	0	$W_{0,5}$	6.80×10^{-25}
(Equation 36)	1	$W_{1,5}$	3.01×10^{-48}

7.4. **Asymptotic Approximations.** As is evident in the results shown in Figure 11, apart from the results for $W_{L_1}(y)$ and $W_{L_2}(y)$, the relative errors in the approximations defined in Theorem 3.1, for a set order, decrease as their argument increases, i.e. for a set order, the approximations asymptotically approach the Lambert W function as their arguments become unbounded. Thus:

$$\begin{aligned} W_{L_i}(y) &\sim W(y), \qquad i \in \{3, \ldots\}, \\ W_{U_i}(y) &\sim W(y), \qquad i \in \{1, 2, 3, \ldots\}. \end{aligned} \tag{98}$$

7.4.1. *Lambert W Function and Prime Counting Function*. The prime number theorem states that the relative error between the prime counting function $\pi(y)$ and $y/\ln(y)$ decreases to zero as $y \to \infty$, i.e.

$$\pi(y) \sim y/\ln(y). \tag{99}$$

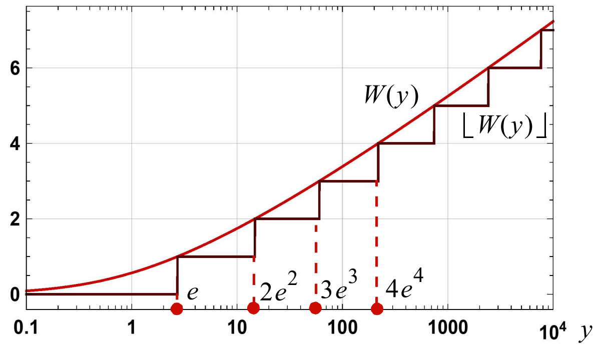

FIGURE 24. Graph of $W(y)$ and $\lfloor W(y) \rfloor$.

As $W[z\ln(z)] = \ln(z)$ (Equation 115), an equivalent statement for the prime number theorem is

$$\pi(y) \sim y / W[y\ln(y)]. \tag{100}$$

With the manipulation of

$$W[y\ln(y)] = \ln(y) = \ln(xe^x) = x + \ln(x) = W(y) + \ln[W(y)], \tag{101}$$

and with $W(y) \gg \ln[W(y)]$ for y large, it follows that

$$\pi(y) \sim y / W(y), \tag{102}$$

which has been proposed by Visser [27] and briefly discussed by Iacono and Boyd, [17], section 4.4. Visser has proved that $y/W(y)$ is an upper bound for the prime counting function whilst $y/\ln(y)$ is a lower bound for y large. The magnitude of the relative error in the approximation of $\pi(y) \approx y/W(y)$ is lower than the magnitude of the relative error in the approximation $\pi(y) \approx y/W[y\ln(y)]$ for y greater than around 5000 with the relative error decreasing as y increases. However, the magnitude of the relative error in the approximation $\pi(y) \approx y/W(y)$ is of the order of 0.05 for $y = 10^9$.

7.5. Floor and Integral of Floor of Lambert *W*.

It is possible to specify approximations for the Lambert *W* function, with a set accuracy bound, if an explicit expression for the floor of the Lambert function can be specified. The graph of $\lfloor W(y) \rfloor$ is shown in Figure 24.

Theorem 7.2. Floor of Lambert *W*. *The floor of the Lambert W function can be ascertained, without knowledge of the function itself, according to*

$$\lfloor W(y) \rfloor = \sum_{k=1}^{\infty} u(y - ke^k) = \sum_{k=1}^{1+\lfloor W_{U_i}(y) \rfloor} u(y - ke^k), \qquad y \geq 0, \tag{103}$$

where, $i \in \{1, 2, \ldots\}$ is fixed, u is the unit step function and $W_{U_i}(y)$ is an upper bound for the Lambert W function as specified in Theorem 3.1.

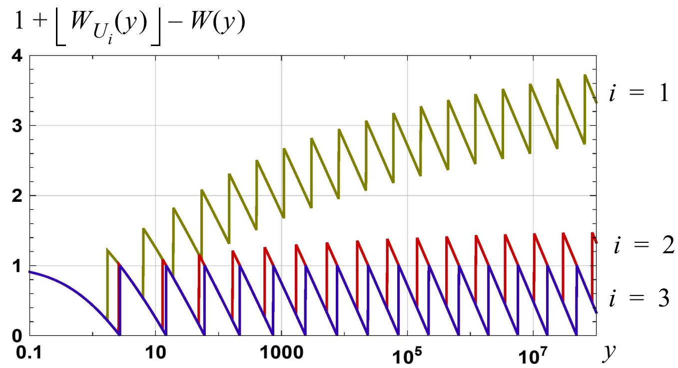

FIGURE 25. Graph of $1+\left\lfloor W_{U_i}(y)\right\rfloor - W(y)$ for $i \in 1, 2, 3$.

Proof. This result follows from the fact that $W[ke^k] = k$ and $W_{U_i}(y)$, $i \in \{1, 2, \ldots\}$, i fixed, is an upper bound for $W(y)$, i.e. $\lfloor W(y)\rfloor \le \left\lfloor W_{U_i}(y)\right\rfloor$. It then follows that the upper limit of the summation can be specified as $1+\left\lfloor W_{U_i}(y)\right\rfloor$.

7.5.1. *Notes*. The simplest upper bound, specified in Theorem 3.1, for the Lambert *W* function is $W_{U_1}(y) = \ln(1+y)$ and, thus:

$$\lfloor W(y)\rfloor = \sum_{k=1}^{1+\lfloor \ln(1+y)\rfloor} u(y-ke^k), \qquad y \ge 0. \tag{104}$$

The graphs of $1+\left\lfloor W_{U_i}(y)\right\rfloor - W(y)$, for $i \in \{1, 2, 3\}$, are shown in Figure 25 and it follows that the use of $1+W_{U_1}(y) = 1+\lfloor \ln(1+y)\rfloor$ as the upper limit for the summation results in an increasing small number of additional zero terms in the summation defining $\lfloor W(y)\rfloor$ as y increases. The use of $1+\left\lfloor W_{U_2}(y)\right\rfloor$ results, depending on the value of y, in an additional zero term in the summation defining $\lfloor W(y)\rfloor$.

7.5.2. *Integral of Floor of Lambert W*. Using the result for the floor of the Lambert *W* function, as specified in Theorem 7.2, it is possible to explicitly specify the integral of the floor of the Lambert *W* function.

Theorem 7.3. Integral of Floor of Lambert *W*. *The integral of the floor of the Lambert W function can be explicitly specified according to*

$$\int_0^y \lfloor W(\lambda)\rfloor d\lambda = \frac{e}{(e-1)^2}\cdot\left[\begin{array}{l} -1+e^{\lfloor W(y)\rfloor}[1+\lfloor W(y)\rfloor-2\lfloor W(y)\rfloor^2]+ \\ \lfloor W(y)\rfloor[-1+\lfloor W(y)\rfloor]e^{1+\lfloor W(y)\rfloor}+\lfloor W(y)\rfloor^2 e^{\lfloor W(y)\rfloor-1} \end{array}\right] + \lfloor W(y)\rfloor[y-\lfloor W(y)\rfloor e^{\lfloor W(y)\rfloor}] \tag{105}$$

where $\lfloor W(y)\rfloor$ *is specified in Theorem 7.2.*

Proof. The required result follows, consistent with the graph of $\lfloor W(y)\rfloor$ shown in Figure 24, according to

$$\int_0^y \lfloor W(\lambda) \rfloor d\lambda = \sum_{k=0}^{\max\{0, \lfloor W(y) \rfloor - 1\}} k[(k+1)e^{k+1} - ke^k] + \lfloor W(y) \rfloor [y - \lfloor W(y) \rfloor e^{\lfloor W(y) \rfloor}]$$

$$= \frac{e}{(e-1)^2} \cdot \begin{bmatrix} -1 + e^{\lfloor W(y) \rfloor}[1 + \lfloor W(y) \rfloor - 2\lfloor W(y) \rfloor^2] + \\ \lfloor W(y) \rfloor [-1 + \lfloor W(y) \rfloor] e^{1 + \lfloor W(y) \rfloor} + \lfloor W(y) \rfloor^2 e^{\lfloor W(y) \rfloor - 1} \end{bmatrix} + \lfloor W(y) \rfloor [y - \lfloor W(y) \rfloor e^{\lfloor W(y) \rfloor}] \tag{106}$$

where the following result has been used:

$$\sum_{k=1}^{n-1} k[(k+1)e^{k+1} - ke^k] = \frac{e}{(e-1)^2} \cdot \left[-1 + e^n[1 + n - 2n^2] + ne^{1+n}[-1+n] + n^2 e^{-1+n} \right]. \tag{107}$$

7.6. **Set Accuracy Approximation for Lambert *W*.** Consider a set accuracy limit of Δ required for the evaluation of $W(y)$. This can be achieved by a step approximation, with a resolution of Δ, and such an approximation is:

$$W_\Delta(y) = \Delta \sum_{k=1}^{1 + \left\lfloor \frac{1}{\Delta} \cdot W_{U_i}(y) \right\rfloor} u[y - k\Delta e^{k\Delta}], \qquad y \geq 0. \tag{108}$$

where W_{U_i}, $i \in \{1, 2, \ldots\}$, i fixed, is a set function defined in Theorem 3.1. As an example, the error in the approximation to $W(y)$, with a resolution of $\Delta = 1/10$, is shown in Figure 26.

7.6.1. *Computationally Efficient Implementation.* The direct approximation detailed in Equation 108, for a set error level of $\Delta = 10^{-q}$, requires, approximately, a summation of $10^q \lfloor W(y) \rfloor$ terms. The approach detailed below requires, approximately, the summation of $\lfloor W(y) \rfloor + 11q - 1$ terms which represents a significant reduction for q modest to large. For example, for $\lfloor W(y) \rfloor = 10$ and $q = 6$, the direct approach requires a summation of approximately 10^7 terms whilst the approach detailed below requires close to 75 terms.

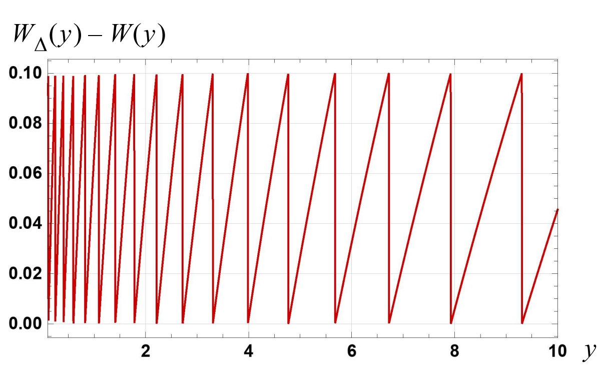

FIGURE 26. Graph in the error in $W_\Delta(y)$ for the case of $\Delta = 0.1$.

Theorem 7.4. Direct Evaluation of Lambert W with Specified Resolution. *The Lambert W function can be evaluated, with a maximum error of $\Delta = 10^{-q}$, according to*

$$W(y) = \lfloor W(y) \rfloor + d_1(y) + \ldots + d_q(y), \tag{109}$$

where $\lfloor W(y) \rfloor$ is defined in Theorem 7.2 and $10^q d_q(y)$ is the qth digit in the decimal expansion of $W(y)$:

$$\begin{aligned} d_1(y) &= \frac{1}{10}\sum_{k=1}^{10} u\left[y - \left[\lfloor W(y) \rfloor + \frac{k}{10}\right]\exp\left[\lfloor W(y) \rfloor + \frac{k}{10}\right]\right], \\ d_2(y) &= \frac{1}{100}\sum_{k=1}^{10} u\left[y - \left[\lfloor W(y) \rfloor + d_1(y) + \frac{k}{100}\right]\exp\left[\lfloor W(y) \rfloor + d_1(y) + \frac{k}{100}\right]\right], \\ &\ldots \\ d_q(y) &= \frac{1}{10^q}\sum_{k=1}^{10} u\left[y - \left[\lfloor W(y) \rfloor + \sum_{i=1}^{q-1} d_i(y) + \frac{k}{10^q}\right]\exp\left[\lfloor W(y) \rfloor + \sum_{i=1}^{q-1} d_i(y) + \frac{k}{10^q}\right]\right]. \end{aligned} \tag{110}$$

Proof. The floor of the Lambert W function has been defined in Theorem 7.2. Consider the illustration shown in Figure 27. With

$$W(y) = \lfloor W(y) \rfloor + d_1(y) + d_2(y) + \ldots \tag{111}$$

and with $d_q(y)$ being the qth digit to the right of the decimal point, it follows that

$$d_1(y) = \frac{1}{10}\sum_{k=1}^{10} u\left[y - \left[\lfloor W(y) \rfloor + \frac{k}{10}\right]\exp\left[\lfloor W(y) \rfloor + \frac{k}{10}\right]\right]. \tag{112}$$

Iteration with finer resolution, and from the point defined by $\lfloor W(y) \rfloor + d_1(y)$, yields $d_2(y)$ etc.

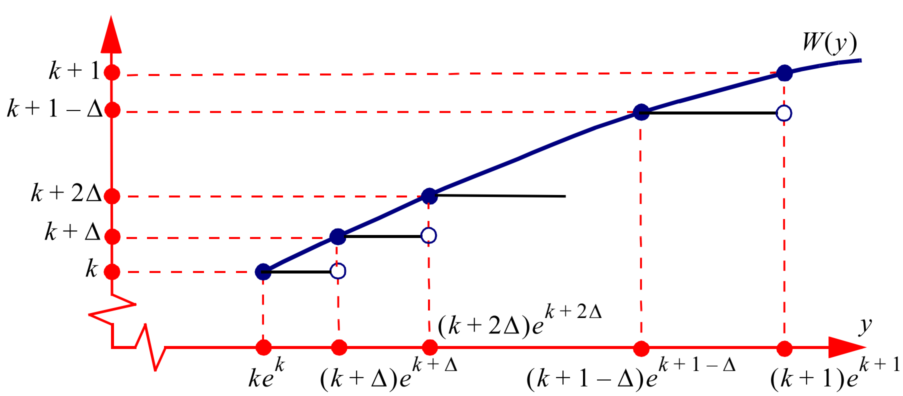

FIGURE 27. Illustration of demarcation that underpins determination of $W(y)$, to a set resolution of Δ, between ke^k and $(k+1)e^{k+1}$.

The number of terms in the summation defined by Equation 109 comprises approximately $\lfloor W(y) \rfloor$ terms for the evaluation of $\lfloor W(y) \rfloor$, plus $10q$ terms for the summations comprising $d_1(y), d_2(y), \ldots, d_q(y)$ and $q-1$ terms for the summation of $d_1(y)+d_2(y)$, …, $d_1(y)+d_2(y)+\ldots+d_q(y)$.

7.6.2. *Note*. Theorem 7.4 defines a series for the Lambert *W* function, which, by construction is convergent, i.e.

$$W(y) = \lfloor W(y) \rfloor + d_1(y) + d_2(y) + \ldots \qquad y > 0, \tag{113}$$

and is such that

$$\left| W(y) - \left[\lfloor W(y) \rfloor + d_1(y) + \ldots + d_q(y) \right] \right| < 10^{-q}. \tag{114}$$

8. CONCLUSION

A geometric based approach for iteratively specifying approximations to the Lambert *W* function, which can achieve any set relative error bound over the interval $[0, \infty)$, was detailed. The approximations are also valid for the interval $(-1/e, 0]$ but are not sharp at the point $-1/e$. Convergence was proved. For the interval $[-1/e, 0]$, arbitrarily accurate approximations, based on a two point spline approximation, were specified. Iteration, either by using the iteration structure inherent in the definition of the Lambert *W* function, or via the Newton-Raphson method, leads to significantly improved approximations albeit with increasing complex functional forms.

Applications of the approximations were detailed and include, first, analytical expressions for the Lambert *W* function that achieve set relative error bounds over the interval $[0, \infty)$. Second, based on the geometry inherent in the approximations, upper and lower bounds for the Lambert *W* function that can be made arbitrary accurate. Third, higher accuracy spline based approximations for the Lambert *W* function based on the defined upper and lower bounded functions. Fourth, analytical expressions for the evaluation of $\lfloor W(y) \rfloor$, and the integral of $\lfloor W(y) \rfloor$, without knowledge of $W(y)$ for $y \in [0, \infty)$. Finally, a direct approach for evaluating the Lambert *W* function to achieve a prior defined error.

Acknowledgement: The author is pleased to acknowledge the support of Prof. A. Zoubir, SPG, Technische Universität Darmstadt, Darmstadt, Germany, who hosted a visit where the research, underpinning this paper, was completed.

APPENDIX A. PROPERTIES OF LAMBERT *W* FUNCTION

The following are useful properties of the Lambert *W* function:

$$W[z\ln(z)] = \ln(z), \qquad z > 0, \tag{115}$$

$$W(y) = \ln\left[\frac{y}{W(y)}\right] = \ln(y) - \ln[W(y)], \qquad y > 0. \tag{116}$$

The latter formula underpins the iteration:

$$\begin{aligned} W(y) &= \ln(y) - \ln[\ln(y) - \ln[W(y)]], \\ W(y) &= \ln(y) - \ln[\ln(y) - \ln[\ln(y) - \ln[W(y)]]], \\ &\ldots \end{aligned} \tag{117}$$

To prove that $W[z\ln(z)] = \ln(z)$, consider $f(x) = xe^x$ which implies $f[\ln(z)] = z\ln(z)$ and, thus, $\ln(z) = f^{-1}[z\ln(z)] = W[z\ln(z)]$.

The relationship $W(y) = \ln(y) - \ln[W(y)]$ follows from the definitions $y = xe^x$, $x = W(y)$ which implies $\ln(y) = x + \ln(x)$ and, thus, $x = \ln(y) - \ln[W(y)]$.

A.1. **Differentiation.** The derivatives of the Lambert W function are defined according to

$$W^{(1)}(y) = \frac{e^{-W(y)}}{1 + W(y)} = \frac{W(y)}{y[1 + W(y)]}, \tag{118}$$

$$W^{(k)}(y) = \frac{W^k(y)p_k[W(y)]}{y^k[1 + W(y)]^{2k-1}} = \frac{p_k[W(y)]e^{-kW(y)}}{[1 + W(y)]^{2k-1}}, \tag{119}$$

where the second inequalities follow from the relationship $y = W(y)e^{W(y)}$ and the polynomial p_k is defined according to

$$p_k(r) = c_{k,0} + c_{k,1}r + c_{k,2}r^2 + \ldots + c_{k,k-1}r^{k-1}. \tag{120}$$

The coefficients in this expression are defined according to (https://oeis.org/A042977; [7], eqn 3.4; [23], p. 1370):

$$c_{n,k} = \begin{cases} 0 & k < 0 \\ (-1)^{n+1}n^{n-1} & k = 0 \\ -(n-1)c_{n-1,k-1} - [3(n-1) - (k+1)]c_{n-1,k} + (k+1)c_{n-1,k+1} & 1 \le k \le n-3 \\ -(n-1)c_{n-1,k-1} - [3(n-1) - (k+1)]c_{n-1,k} & k = n-2 \\ -(n-1)c_{n-1,k-1} = (n-1)! & k = n-1 \end{cases} \tag{121}$$

Explicit expressions are:

$$\begin{aligned} &p_1(r) = 1, \qquad p_2(r) = -(2 + r), \qquad p_3(r) = 9 + 8r + 2r^2, \\ &p_4(r) = -[64 + 79r + 36r^2 + 6r^3], \qquad p_5(r) = 625 + 974r + 622r^2 + 192r^3 + 24r^4, \\ &p_6(r) = -[7776 + 14543r + 11758r^2 + 5126r^3 + 1200r^4 + 120r^5]. \end{aligned} \tag{122}$$

APPENDIX B. PROOF of Theorem 3.2

The iteration formula, as specified in Theorem 3.1, yields the first order approximations as stated in Equation 30:

$$\begin{aligned} W_{L_1}(y) &= \frac{y}{1 + y} = \frac{n_1(y)}{d_1(y)}, \\ W_{U_1}(y) &= \ln(1 + y) - \ln(1) = \ln[d_1(y)] - \ln\left[\frac{n_1(y)}{y}\right], \end{aligned} \tag{123}$$

where $n_1(y) = y$ and $d_1(y) = 1 + y$. The second order approximations, as specified by Equation 31, can be written in the form

$$W_{L_2}(y) = \frac{y[1+\ln(1+y)]}{1+2y} = \frac{n_2(y)}{d_2(y)},$$
$$W_{U_2}(y) = \ln[1+2y] - \ln[1+\ln(1+y)] = \ln[d_2(y)] - \ln\left[\frac{n_2(y)}{y}\right], \tag{124}$$

where

$$n_2(y) = y[1+\ln(1+y)] = n_1(y)[1+\ln[d_1(y)]],$$
$$d_2(y) = 1+2y = n_1(y)+d_1(y). \tag{125}$$

The third order approximations, as specified by Equation 32 and Equation 33, can be written in the form:

$$W_{L_3}(y) = \frac{y[1+\ln(1+y)]\Big[1+\ln[1+2y]-\ln[1+\ln(1+y)]\Big]}{1+3y+y\ln(1+y)} = \frac{n_3(y)}{d_3(y)}, \tag{126}$$

$$W_{U_3}(y) = \ln[1+3y+y\ln(1+y)] - \ln\Big[[1+\ln(1+y)]\Big[1+\ln[1+2y]-\ln[1+\ln(1+y)]\Big]\Big]$$
$$= \ln[d_3(y)] - \ln\left[\frac{n_3(y)}{y}\right], \tag{127}$$

where

$$n_3(y) = y[1+\ln(1+y)]\Big[1+\ln[1+2y]-\ln[1+\ln(1+y)]\Big]$$
$$= n_2(y)\left[1+\ln[d_2(y)]-\ln\left[\frac{n_2(y)}{y}\right]\right], \tag{128}$$
$$d_3(y) = 1+3y+y\ln(1+y) = n_2(y)+d_2(y).$$

Thus, iteration yields the general formulas:

$$W_{L_i} = \frac{n_i(y)}{d_i(y)}, \qquad W_{U_i} = \ln[d_i(y)] - \ln\left[\frac{n_i(y)}{y}\right], \tag{129}$$

where

$$n_i(y) = n_{i-1}(y)\left[1+\ln[d_{i-1}(y)]-\ln\left[\frac{n_{i-1}(y)}{y}\right]\right], \qquad n_0(y) = y,\ d_0(y) = 1,$$
$$d_i(y) = n_{i-1}(y)+d_{i-1}(y). \tag{130}$$

APPENDIX C. PROOF of Theorem 3.5

Consider the geometry illustrated in Figure 15 and an initial approximation for x_o, based on the intersection of the second order Taylor series for ye^{-x} at the origin, i.e.

$y[1-x+x^2/2]$, and x which is $W_{L_1} = \frac{1}{y}\cdot\left[1+y\pm\sqrt{1+2y-y^2}\right]$. The problem with such an approximation is that it only yields a real solution for $1-\sqrt{2}<y<1+\sqrt{2}$.

A practical approach is to utilize an affine approximation at the origin of $y(1-x)$ leading to the first approximation of $W_{L_1} = y/(1+y)$. To establish a further approximation, consider the point where the level W_{L_1} intersects ye^{-x} which is

$$W_{U_1} = \ln\left[\frac{y}{W_{L_1}}\right] = \ln(1+y). \tag{131}$$

A second level approximation follows by finding the intersection of a second order Taylor series at the point W_{U_1}, which is

$$W_{L_1} - [x - W_{U_1}]W_{L_1} + \frac{(x-W_{U_1})^2 W_{L_1}}{2}, \tag{132}$$

with x to yield

$$W_{L_2} = \frac{1}{W_{L_1}}\left[1+W_{L_1}[1+W_{U_1}] - \sqrt{1-W_{L_1}^2+2W_{L_1}[1+W_{U_1}]}\ \right] \tag{133}$$

and

$$W_{U_2} = \ln\left[\frac{y}{W_{L_2}}\right]. \tag{134}$$

Iteration in this manner leads to the general iteration formulas:

$$\begin{aligned} W_{L_i} &= \frac{1}{W_{L_{i-1}}}\left[1+W_{L_{i-1}}[1+W_{U_{i-1}}] - \sqrt{1-W_{L_{i-1}}^2+2W_{L_{i-1}}[1+W_{U_{i-1}}]}\ \right], \\ W_{U_i} &= \ln\left[\frac{y}{W_{L_i}}\right]. \end{aligned} \tag{135}$$

Simulation results (see Figure 16) indicate that the approximations also have good convergence for the interval $(-1/e, 0)$ but the approximations are not sharp at $y = -1/e$.

APPENDIX D. PROOF of Theorem 4.1

Consider the case of y fixed, $y>0$, and the illustration shown in Figure 18. By construction, $\varepsilon_{L_i}>0$, $\varepsilon_{U_i}>0$ and $W_{U_i}>W_{L_i}$ for $i\in\{1,2,\ldots\}$. Further, (see Equation 25)

$$W_{L_{i+1}} = W_{L_i}\cdot\frac{1+W_{U_i}}{1+W_{L_i}}, \tag{136}$$

and it follows that W_{L_i}, $i\in\{1,2,\ldots\}$, is a monotonically increasing sequence.

Using Equation 136 it follows that

$$\varepsilon_{L_i} - \varepsilon_{L_{i+1}} = [x_o - W_{L_i}] - \left[x_o - \frac{W_{L_i}[1 + W_{U_i}]}{1 + W_{L_i}}\right] = \frac{W_{L_i}[W_{U_i} - W_{L_i}]}{1 + W_{L_i}}. \quad (137)$$

As $W_{U_i} - W_{L_i} = \varepsilon_{L_i} + \varepsilon_{U_i}$, and $\varepsilon_{U_i} > 0$, it follows that

$$\varepsilon_{L_{i+1}} = \varepsilon_{L_i} - \frac{W_{L_i}}{1 + W_{L_i}} \cdot [\varepsilon_{L_i} + \varepsilon_{U_i}] < \left[1 - \frac{W_{L_i}}{1 + W_{L_i}}\right] \cdot \varepsilon_{L_i} = r_i \varepsilon_{L_i}, \quad (138)$$

where $r_i = \frac{1}{1 + W_{L_i}}$. Thus, as W_{L_i} monotonically increases with i, it follows that r_i monotonically decreases with i, i.e. $0 < r_{i+1} < r_i < 1$. It then follows that

$$0 < \varepsilon_{L_{i+1}} < \varepsilon_{L_1} \prod_{k=1}^{i} r_k < \varepsilon_{L_1} r_1^i. \quad (139)$$

Hence, convergence is guaranteed as $0 < r_1 = \frac{1}{1 + y/(1+y)} < 1$. Thus: $\lim_{i \to \infty} \varepsilon_{L_i} = 0$ and $\lim_{i \to \infty} W_{L_i} = x_o$. The result $\varepsilon_{L_{i+1}} = \varepsilon_{L_i} - \frac{W_{L_i}}{1 + W_{L_i}} \cdot [\varepsilon_{L_i} + \varepsilon_{U_i}]$ implies that $\lim_{i \to \infty} \varepsilon_{U_i} = 0$ and, thus, $\lim_{i \to \infty} W_{U_i} = x_o$.

APPENDIX E. ALTERNATIVE FORM FOR SPLINE APPROXIMATION

The general form for a nth order, two point, spline approximation for a function f, over the interval $[\alpha, \beta]$, has been detailed in Howard, [16], eqn. 40. The assumption is that the function f is at least nth order differentiable over the interval $[\alpha, \beta]$. The approximation, denoted f_n, can be written in the modified form:

$$f_n(x) = \frac{(\beta - x)^{n+1}}{(\beta - \alpha)^{n+1}} \cdot \sum_{k=0}^{n} \left[\sum_{i=0}^{n-k} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{(n+i)!}{i!n!} \cdot \frac{(x - \alpha)^{k+i}}{(\beta - \alpha)^i}\right] + \frac{(x - \alpha)^{n+1}}{(\beta - \alpha)^{n+1}} \cdot \sum_{k=0}^{n} \left[\sum_{i=0}^{n-k} \frac{(-1)^k f^{(k)}(\beta)}{k!} \cdot \frac{(n+i)!}{i!n!} \cdot \frac{(\beta - x)^{k+i}}{(\beta - \alpha)^i}\right]. \quad (140)$$

In this equation, the double summation can be rewritten by utilizing the transformations $r = i + k$ and $u = i$, $k \in \{0, 1, \dots, n\}$, $i \in \{0, 1, \dots, n - k\}$. The possible values of $r = i + k$ are detailed in Table 8 and for r fixed, the valid values for i are from the set $\{0, 1, \dots, r\}$.

TABLE 8. Valid values of $r = i + k$ for $k \in \{0, 1, \dots, n\}$, $i \in \{0, 1, \dots, n - k\}$.

				i				
k	**0**	**1**	**2**	**3**	**...**	**n-2**	**n-1**	**n**
0	0	1	2	3		n-2	n-1	n
1	1	2	3	4		n-1	n	
2	2	3	4	5		n		
3	3	4	5	6				

TABLE 8. Valid values of $r = i+k$ for $k \in \{0, 1, \ldots, n\}$, $i \in \{0, 1, \ldots, n-k\}$.

				i				
k	**0**	**1**	**2**	**3**	**...**	**n-2**	**n-1**	**n**
…								
n-2	n-2	n-1	n					
n-1	n-1	n						
n	n							

With $r \in \{0, 1, \ldots, n\}$, $u \in \{0, 1, \ldots, r\}$ and $i = u$, $k = r-u$, Equation 140 can be written as

$$f_n(x) = \frac{(\beta-x)^{n+1}}{(\beta-\alpha)^{n+1}} \cdot \sum_{r=0}^{n} (x-\alpha)^r \left[\sum_{u=0}^{r} \frac{f^{(r-u)}(\alpha)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta-\alpha)^u} \right] + \frac{(x-\alpha)^{n+1}}{(\beta-\alpha)^{n+1}} \cdot \sum_{r=0}^{n} (\beta-x)^r \left[\sum_{u=0}^{r} \frac{(-1)^{r-u} f^{(r-u)}(\beta)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta-\alpha)^u} \right]. \quad (141)$$

Thus:

$$f_n(x) = (\beta-x)^{n+1} \sum_{r=0}^{n} a_{n,r}(x-\alpha)^r + (x-\alpha)^{n+1} \sum_{r=0}^{n} b_{n,r}(\beta-x)^r \quad (142)$$

where

$$a_{n,r} = \frac{1}{(\beta-\alpha)^{n+1}} \cdot \sum_{u=0}^{r} \frac{f^{(r-u)}(\alpha)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta-\alpha)^u},$$
$$b_{n,r} = \frac{1}{(\beta-\alpha)^{n+1}} \cdot \sum_{u=0}^{r} \frac{(-1)^{r-u} f^{(r-u)}(\beta)}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{(\beta-\alpha)^u}. \quad (143)$$

APPENDIX F. PROOF of Lemma 1

First, the definitions of f, g_1 and $x = x_1 - 1$ with $x \geq -1$, imply:

$$y_1 = g_1(x_1) = \frac{1}{e} + (x_1-1)e^{x_1-1}, \qquad x_1 \geq 0. \quad (144)$$

It then follows that $y_1 - 1/e = f(x_1-1)$ which implies $f^{-1}[y_1 - 1/e] = x_1 - 1$. With $x_1 = g_1^{-1}(y_1)$ and $y = y_1 - 1/e$, the required result of $f^{-1}(y) = g_1^{-1}[y + 1/e] - 1$ then follows. Second, the transformation of $y_2 = g(x_1) = \sqrt{g_1(x_1)} = \sqrt{y_1}$, $x_1 \geq 0$, results in

$$g^{-1}(y_2) = x_1 = g^{-1}[\sqrt{y_1}]. \quad (145)$$

As $x_1 = g_1^{-1}(y_1)$ it then follows that $g_1^{-1}(y_1) = g^{-1}[\sqrt{y_1}]$ and the final result follows:

$$f^{-1}(y) = g_1^{-1}\left[y + \frac{1}{e}\right] - 1 = g^{-1}\left[\sqrt{y + \frac{1}{e}}\right] - 1. \quad (146)$$

APPENDIX G. PROOF of Theorem 5.1

With D denoting the differentiation operator, the following well known results apply for an arbitrary function f:

$$D[f^{-1}(z)] = \left.\frac{1}{f^{(1)}(x)}\right|_{x=f^{-1}(z)} \qquad D^{(2)}[f^{-1}(z)] = \left.\frac{-f^{(2)}(x)}{[f^{(1)}(x)]^3}\right|_{x=f^{-1}(z)} \tag{147}$$

$$D^{(3)}[f^{-1}(z)] = \left.\frac{-f^{(3)}(x)}{[f^{(1)}(x)]^4} + \frac{3[f^{(2)}(x)]^2}{[f^{(1)}(x)]^5}\right|_{x=f^{-1}(z)} \tag{148}$$

$$D^{(4)}[f^{-1}(z)] = \left.\frac{-f^{(4)}(x)}{[f^{(1)}(x)]^5} + \frac{10f^{(3)}(x)f^{(2)}(x)}{[f^{(1)}(x)]^6} - \frac{15[f^{(2)}(x)]^3}{[f^{(1)}(x)]^7}\right|_{x=f^{-1}(z)} \tag{149}$$

etc.

Consider g, as defined by Equation 75, over the interval $[0, 1]$. A spline approximation (see Equation 70) for g^{-1}, of order n, requires derivatives, of orders zero to n at the points 0 and $1/\sqrt{e}$, to be determined. Using the above formulas, such values can be determined from the derivatives of g at the points 0 and 1 and values of these derivatives are tabulated in Table 9. To determine the derivative values at zero, the standard Taylor series expansion for the exponential function can be used to yield the alternative form for g of

$$g(x_1) = \frac{x_1}{\sqrt{2e}} \cdot \left[\sqrt{1 + 2\sum_{i=1}^{\infty} \frac{(i+1)x_1^i}{(i+2)!}}\right]. \tag{150}$$

Using Equation 70, and the derivative values given in Table 9, the spline approximations for $g^{-1}(y_2)$, based on the points 0 and $1/\sqrt{e}$ and for orders one to four, are:

$$g_1^{-1}(y_2) = \sqrt{2e}y_2\left[1 - 2\sqrt{e}y_2\left[1 + \frac{1}{\sqrt{2e}} - \frac{3}{2\sqrt{2}}\right] + ey_2^2\left[1 - \sqrt{2} + \frac{\sqrt{2}}{e}\right]\right] \tag{151}$$

$$g_2^{-1}(y_2) = \sqrt{2e}y_2\left[\begin{matrix} 1 - \frac{\sqrt{2e}y_2}{3} + \left[6e(\sqrt{2}-1) - \frac{7}{\sqrt{2}} - \frac{2\sqrt{2}}{e}\right]y_2^2 - \\ \left[e^{3/2}\left[\frac{17}{\sqrt{2}} - 8\right] - 6\sqrt{2e} - \frac{4\sqrt{2}}{\sqrt{e}}\right]y_2^3 + \left[e^2\left[\frac{10\sqrt{2}}{3} - 3\right] - \frac{5e}{\sqrt{2}} - 2\sqrt{2}\right]y_2^4 \end{matrix}\right] \tag{152}$$

TABLE 9. Values of the derivatives of $g(x_1)$ at the points zero and one.

order	$g^{(i)}(0)$	$g^{(i)}(1)$
0	0	$\frac{1}{\sqrt{e}}$
1	$\frac{1}{\sqrt{2e}}$	$\frac{\sqrt{e}}{2}$
2	$\frac{\sqrt{2}}{3\sqrt{e}}$	$\sqrt{e}\left[1-\frac{e}{4}\right]$
3	$\frac{5}{12\sqrt{2e}}$	$\frac{3\sqrt{e}}{2}\left[1-e+\frac{e^2}{4}\right]$
4	$\frac{11}{45\sqrt{2e}}$	$2\sqrt{e}\left[1-3e+\frac{9e^2}{4}-\frac{15e^3}{32}\right]$
5	$\frac{59}{432\sqrt{2e}}$	$\frac{5\sqrt{e}}{2}\left[1-8e+\frac{27e^2}{2}-\frac{15e^3}{2}+\frac{21e^4}{16}\right]$

$$g_3^{-1}(y_2) = \sqrt{2e}y_2\left[\begin{array}{l} 1-\frac{\sqrt{2e}y_2}{3}+\frac{11ey_2^2}{36}-\left[e^{3/2}\left[\frac{191}{9}-\frac{125}{3\sqrt{2}}\right]+\frac{25\sqrt{e}}{\sqrt{2}}+\frac{8\sqrt{2}}{\sqrt{e}}+\frac{6\sqrt{2}}{e^{3/2}}\right]y_2^3+ \\ \left[e^2\left[\frac{281}{6}-\frac{146\sqrt{2}}{3}\right]+32\sqrt{2}e+22\sqrt{2}+\frac{18\sqrt{2}}{e}\right]y_2^4- \\ \left[e^{5/2}\left[\frac{335}{9}-40\sqrt{2}\right]+\frac{55e^{3/2}}{\sqrt{2}}+20\sqrt{2}\sqrt{e}+\frac{18\sqrt{2}}{\sqrt{e}}\right]y_2^5+ \\ \left[e^3\left[\frac{371}{36}-\frac{34\sqrt{2}}{3}\right]+8\sqrt{2}e^2+6\sqrt{2}e+6\sqrt{2}\right]y_2^6 \end{array}\right] \tag{153}$$

$$g_4^{-1}(y_2) = \sqrt{2e}y_2\left[\begin{array}{l} 1-\frac{\sqrt{2e}y_2}{3}+\frac{11ey_2^2}{36}-\frac{43e^{3/2}y_2^3}{135\sqrt{2}}+ \\ \left[e^2\left[\frac{4075}{27\sqrt{2}}-\frac{895}{12}\right]-\frac{91e}{\sqrt{2}}-\frac{61}{\sqrt{2}}-\frac{27\sqrt{2}}{e}-\frac{64\sqrt{2}}{3e^2}\right]y_2^4- \\ \left[e^{5/2}\left[\frac{6658\sqrt{2}}{27}-\frac{2126}{9}\right]-\frac{315e^{3/2}}{\sqrt{2}}-110\sqrt{2}e-\frac{102\sqrt{2}}{\sqrt{e}}-\frac{256\sqrt{2}}{3e^{3/2}}\right]y_2^5+ \\ \left[e^3\left[\frac{8467\sqrt{2}}{27}-\frac{1175}{4}\right]-207\sqrt{2}e^2-149\sqrt{2}e-144\sqrt{2}-\frac{128\sqrt{2}}{e}\right]y_2^6- \\ \left[e^{7/2}\left[\frac{4904\sqrt{2}}{27}-\frac{502}{3}\right]-\frac{245e^{5/2}}{\sqrt{2}}-90\sqrt{2}e^{3/2}-90\sqrt{2}e-\frac{256\sqrt{2}}{3\sqrt{e}}\right]y_2^7+ \\ \left[e^4\left[\frac{10843}{135\sqrt{2}}-\frac{1315}{36}\right]-\frac{55e^3}{\sqrt{2}}-\frac{41e^2}{\sqrt{2}}-21\sqrt{2}e-\frac{64\sqrt{2}}{3}\right]y_2^8 \end{array}\right] \tag{154}$$

In general:

$$g_k^{-1}(y_2) \approx \sqrt{2e}y_2[1+\alpha_1 y_2+\alpha_2 y_2^2+\alpha_3 y_2^3+\alpha_4 y_2^4+\ldots+\alpha_{2k}y_2^{2k}], \tag{155}$$

for appropriately defined constants and, thus:

$$\begin{aligned} W(y) &= g^{-1}\left[\sqrt{y+\frac{1}{e}}\right]-1 \\ &\approx \sqrt{2e}\sqrt{y+\frac{1}{e}}\left[1+\alpha_1\sqrt{y+\frac{1}{e}}+\alpha_2\left[y+\frac{1}{e}\right]+\alpha_3\left[y+\frac{1}{e}\right]^{3/2}+\ldots+\alpha_{2k}\left[y+\frac{1}{e}\right]^k\right]-1. \end{aligned} \tag{156}$$

APPENDIX H. PROOF of Theorem 7.1

A zero order spline approximation is simply an affine approximation between the two specified points. Consistent with Figure 23, the zero order spline approximation to $W(y)$, denoted f_0, is an affine approximation between the points $(u_o\exp(u_o), u_o)$ and $(v_o\exp(v_o), v_o)$. Thus:

$$f_0(y) = u_o+[y-u_o\exp(u_o)]\cdot\frac{v_o-u_o}{v_o\exp(v_o)-u_o\exp(u_o)}, \qquad y\in[u_o\exp(u_o), v_o\exp(v_o)]. \tag{157}$$

With the approximation $x_o = W(y_o)\approx f_0(y_o)$ it follows that

$$x_o\approx u_o+[y_o-u_o\exp(u_o)]\cdot\frac{v_o-u_o}{v_o\exp(v_o)-u_o\exp(u_o)}. \tag{158}$$

Simplification yields

$$x_o\approx\frac{u_ov_o[\exp(v_o)-\exp(u_o)]+y_o[v_o-u_o]}{v_o\exp(v_o)-u_o\exp(u_o)}. \tag{159}$$

Substitution of $u_o = W_{L_i}(y_o)$ and $v_o = W_{U_i}(y_o)$ yields the required result.

H.1. **General Result.**

The general result arises from the spline approximation, specified by Equation 70 with $f(y) = W(y)$, based on the points $(u_o\exp(u_o), u_o)$ and $(v_o\exp(v_o), v_o)$ where $u_o = W_{L_i}(y_o)$, $v_o = W_{U_i}(y_o)$, and with

$$W^{(1)}(y) = \frac{e^{-W(y)}}{1+W(y)}, \qquad W^{(k)}(y) = \frac{p_k[W(y)]e^{-kW(y)}}{[1+W(y)]^{2k-1}}, \qquad k\in\{1, 2, \ldots\}. \tag{160}$$

Here p_k is defined by Equation 120. The nth order spline approximation, for $y\in[u_o\exp(u_o), v_o\exp(v_o)]$, is

$$f_n(y) = \frac{[v_o\exp(v_o) - y]^{n+1}}{[v_o\exp(v_o) - u_o\exp(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} (y - u_o\exp(u_o))^r \left[\sum_{u=0}^{r} \frac{W^{(r-u)}[u_o\exp(u_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[v_o\exp(v_o) - u_o\exp(u_o)]^u}\right] + \frac{[y - u_o\exp(u_o)]^{n+1}}{[v_o\exp(v_o) - u_o\exp(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} (v_o\exp(v_o) - y)^r \left[\sum_{u=0}^{r} \frac{(-1)^{r-u} W^{(r-u)}[v_o\exp(v_o)]}{(r-u)!} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[v_o\exp(v_o) - u_o\exp(u_o)]^u}\right] \tag{161}$$

The results $W(u_o\exp(u_o)) = u_o$, $W(v_o\exp(v_o)) = v_o$ imply that

$$W^{(r-u)}[u_o\exp(u_o)] = \frac{p_{r-u}(u_o)e^{-(r-u)u_o}}{[1+u_o]^{2(r-u)-1}}, \qquad W^{(r-u)}[v_o\exp(v_o)] = \frac{p_{r-u}(v_o)e^{-(r-u)v_o}}{[1+v_o]^{2(r-u)-1}}, \tag{162}$$

assuming $r > u$. Hence:

$$f_n(y) = \frac{[v_o\exp(v_o) - y]^{n+1}}{[v_o\exp(v_o) - u_o\exp(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} [y - u_o\exp(u_o)]^r \cdot \left[\frac{(n+r)!u_o}{r!n![v_o\exp(v_o) - u_o\exp(u_o)]^r} + \sum_{u=0}^{r-1} \frac{p_{r-u}(u_o)e^{-(r-u)u_o}}{(r-u)![1+u_o]^{2(r-u)-1}} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[v_o\exp(v_o) - u_o\exp(u_o)]^u}\right] + \frac{[y - u_o\exp(u_o)]^{n+1}}{[v_o\exp(v_o) - u_o\exp(u_o)]^{n+1}} \cdot \sum_{r=0}^{n} [v_o\exp(v_o) - y]^r \cdot \left[\frac{(n+r)!v_o}{r!n![v_o\exp(v_o) - u_o\exp(u_o)]^r} + \sum_{u=0}^{r-1} \frac{(-1)^{r-u} p_{r-u}(v_o)e^{-(r-u)v_o}}{(r-u)![1+v_o]^{2(r-u)-1}} \cdot \frac{(n+u)!}{u!n!} \cdot \frac{1}{[v_o\exp(v_o) - u_o\exp(u_o)]^u}\right] \tag{163}$$

The required result follows: the approximation for $W(y_o)$, denoted $W_{n,i}(y_o)$, arises for the case of $y = y_o$. Thus, $W_{n,i}(y_o) = f_n(y_o)$.

References

[1] F. Alzahrani, A. Salem, Sharp bounds for the Lambert W function, Integral Transforms Spec. Funct. 29 (2018) 971–978. https://doi.org/10.1080/10652469.2018.1528247.

[2] T.C. Banwell, Bipolar transistor circuit analysis using the Lambert W-function, IEEE Trans. Circuits Syst. I. 47 (2000) 1621–1633. https://doi.org/10.1109/81.895330.

[3] D.A. Barry, J.-Y. Parlange, L. Li, H. Prommer, C.J. Cunningham, F. Stagnitti, Analytical approximations for real values of the Lambert W-function, Math. Computers Simul. 53 (2000) 95–103. https://doi.org/10.1016/S0378-4754(00)00172-5.

[4] D. Belkić, The Euler T and Lambert W functions in mechanistic radiobiological models with chemical kinetics for repair of irradiated cells, J. Math. Chem. 56 (2018) 2133–2193. https://doi.org/10.1007/s10910-018-0932-3.

[5] J.P. Boyd, Global approximations to the principal real-valued branch of the Lambert W-function, Appl. Math. Lett. 11 (1998) 27–31. https://doi.org/10.1016/S0893-9659(98)00097-4.

[6] P.B. Brito, F. Fabião, A. Staubyn, Euler, Lambert and the Lambert W function today, Math. Scientist, 33 (2008), 127-133.

[7] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, On the Lambert W function, Adv. Comput. Math. 5 (1996) 329–359. https://doi.org/10.1007/BF02124750.

[8] C.B. Corcino, R.B. Corcino, Logarithmic generalization of the Lambert W function and its applications to adiabatic thermostatistics of the three-parameter entropy, Adv. Math. Phys. 2021 (2021) 1–16. https://doi.org/10.1155/2021/6695559.

[9] A.E. Dubinov, I.D. Dubinova, How can one solve exactly some problems in plasma theory, J. Plasma Phys. 71 (2005) 715. https://doi.org/10.1017/S0022377805003788.

[10] F.N. Fritsch, R.E. Shafer, W.P. Crowley, Algorithm 443: Solution of the transcendental equation wexp(w) = x', Commun. ACM, 16 (1973), 123–124.

[11] S. Foschi, D. Ritelli, The Lambert function, the quintic equation and the proactive discovery of the Implicit Function Theorem, Open J. Math. Sci. 5 (2021) 94–114. https://doi.org/10.30538/oms2021.0149.

[12] T. Fukushima, Precise and fast computation of Lambert W-functions without transcendental function evaluations, J. Comput. Appl. Math. 244 (2013) 77–89. https://doi.org/10.1016/j.cam.2012.11.021.

[13] T. Fukushima, Precise and fast computation of Lambert W function by piecewise minimax rational function approximation with variable transformation, (2020), https://scholar.google.com/citations?user=invpkckAAAAJ&hl=en&oi=sra (accessed 12 October 2021).

[14] M. Goličnik, On the Lambert W function and its utility in biochemical kinetics, Biochem. Eng. J. 63 (2012) 116–123. https://doi.org/10.1016/j.bej.2012.01.010.

[15] A. Hoorfar, M. Hassani, Inequalities on the Lambert W function and hyperpower function, J. Inequal. Pure Appl. Math, 9 (2008) 51.

[16] R.M. Howard, Dual Taylor Series, Spline Based function and integral approximation and applications, Math. Comput. Appl. 24 (2019) 35. https://doi.org/10.3390/mca24020035.

[17] R. Iacono, J.P. Boyd, New approximations to the principal real-valued branch of the Lambert W-function, Adv. Comput. Math. 43 (2017) 1403–1436. https://doi.org/10.1007/s10444-017-9530-3.

[18] A.M. Ishkhanyan, The Lambert- W step-potential – an exactly solvable confluent hypergeometric potential, Phys. Lett. A. 380 (2016) 640–644. https://doi.org/10.1016/j.physleta.2015.12.004.

[19] J.H. Lambert, Observationes variae in mathesin puram', Acta Helveticae physico-mathematico-anatomico-botanico-medica, Band III, (1758) pp. 128–168.

[20] J. Lehtonen, The Lambert W function in ecological and evolutionary models, Methods Ecol. Evol. 7 (2016) 1110–1118. https://doi.org/10.1111/2041-210X.12568.

[21] Q. Luo, Z. Wang, J. Han, A Padé approximant approach to two kinds of transcendental equations with applications in physics, Eur. J. Phys. 36 (2015) 035030. https://doi.org/10.1088/0143-0807/36/3/035030.

[22] I. Mező, Á. Baricz, On the generalization of the Lambert W function with applications in theoretical physics, ArXiv:1408.3999 [Math]. (2015). http://arxiv.org/abs/1408.3999.

[23] A.G. Pakes, Lambert's W meets Kermack–McKendrick Epidemics, IMA J. Appl. Math. 80 (2015) 1368–1386. https://doi.org/10.1093/imamat/hxu057.

[24] S.M. Stewart, On certain inequalities involving the Lambert W function, J. Inequal. Pure Appl. Math. 10 (2009), 96.

[25] S.R. Valluri, D.J. Jeffrey, R.M. Corless, Some applications of the Lambert W function to physics, Can. J. Phys. 78 (2000) 823–831. https://doi.org/10.1139/p00-065.

[26] D. Veberič, Lambert W function for applications in physics, Computer Phys. Commun. 183 (2012) 2622–2628. https://doi.org/10.1016/j.cpc.2012.07.008.

[27] M. Visser, Primes and the Lambert W function, Mathematics. 6 (2018) 56. https://doi.org/10.3390/math6040056.