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Developments of Newton's Method under Hölder Conditions

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Abstract. The semi-local convergence criteria for Newton's method are weakened without new conditions. Moreover, tighter error distances are provided as well as a more precise information on the location of the solution.

1. Introduction

The computation of a solution x_* of nonlinear equation

$$F(x) = 0 \tag{1.1}$$

is important in computational sciences, since many applications can be written as (1.1). Here $F : \Omega \subseteq X \longrightarrow Y$ is Fréchet-differentiable operator, X, Y are Banach spaces and $\Omega \neq \emptyset$ is a convex and open set. But this can be attained only in special cases. That explains why most solution methods for (1.1) are iterative. There is a plethora of methods for solving (1.1) [1–14]. Among them Newton's method (NM) defined by

$$x_0 \in \Omega, \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1.2}$$

seems to be the most popular [2,4]. But the convergence domain is small, limiting the applicability of NM. That is why we have developed a technique that determines a subset Ω_0 of Ω also containing the iterates $\{x_n\}$. Hence, the Hölder constants are at least as tight as the ones in Ω. This crucial

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modification leads to: weaker sufficient convergence criteria, the extension of the convergence domain, tighter error estimates on $\|x_* - x_n\|$, $\|x_{n+1} - x_n\|$ and a more precise information on x_*.

It is worth noticing that these advantages are obtained without additional conditions, since in practice the evolution of the old Hölderian constants require that of the new conditions as special cases.

2. Convergence

We introduce certain Hölder conditions crucial for the semi-local convergence. Let $p \in (0, 1]$. Suppose there exists $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in L(Y, X)$.

Definition 2.1. *Operator F' is center Hölderian on Ω if there exists $H_0 > 0$ such that*

$$\|F'(x_0)^{-1}F'(w) - F'(x_0)\| \leq H_0\|w - x_0\|^p \tag{2.1}$$

for all $w \in \Omega$.

Set

$$\Omega_0 = U(x_0, \frac{1}{H_0^{\frac{1}{p}}}) \cap \Omega. \tag{2.2}$$

Definition 2.2. *Operator F' is center Hölderian on Ω_0 if there exists $H > 0$ such that*

$$\|F'(x_0)^{-1}F'(w) - F'(u)\| \leq \tilde{H}\|w - u\|^p, \tag{2.3}$$

where $\tilde{H} = \begin{cases} H, & w = u - F'(u)^{-1}F(u), u \in D_0 \\ K, & w, u \in \Omega_0. \end{cases}$.

We present the results with H although K can be used too. But notice $H \leq K$.

Definition 2.3. *Operator F' is center Hölderian on Ω if there exists $H_1 > 0$ such that*

$$\|F'(x_0)^{-1}F'(w) - F'(u)\| \leq H_1\|w - u\|^p \tag{2.4}$$

for all $w, u \in \Omega$.

REMARK 2.4. *It follows from (2.2), that*

$$\Omega_0 \subseteq \Omega. \tag{2.5}$$

Then, by (2.1)-(2.5) the following items hold

$$H_0 \leq H_1 \tag{2.6}$$

and

$$H \leq H_1. \tag{2.7}$$

We shall assume that

$$H_0 \leq H. \tag{2.8}$$

Otherwise the results that follow hold with H_0 replacing H. Notice that $H_0 = H_0(x_0, \Omega)$, $H_1 = H_1(x_0, \Omega)$, $H = H(x_0, \Omega_0)$ and $\frac{H_0}{H_1}$ can be small (arbitrarily) [2–4]. In earlier studies [1, 5–14] the estimate

$$\|F'(z)^{-1}F'(x_0)\| \leq \frac{1}{1 - H_1\|z - x_0\|^{\frac{1}{p}}} \tag{2.9}$$

for all $z \in U(x_0, \frac{1}{H_1^{\frac{1}{p}}})$ was found using (2.4). But, if we use (2.1) to obtain the weaker and more precise estimate

$$\|F'(z)^{-1}F'(x_0)\| \leq \frac{1}{1 - H_0\|z - x_0\|^{\frac{1}{p}}} \tag{2.10}$$

for all $z \in U(x_0, \frac{1}{H_0^{\frac{1}{p}}})$. This modification in the proofs and exchanging H_1 by H leads to the advantages as already mentioned in the introduction. That is why we omit the proofs in our results that follow. Notice also that in practice the computation of H_1 require that of H_0 and H as special cases. Hence, the applicability of NM is extended without additional conditions.

Let $d \geq 0$ be such that

$$\|F'(x_0)^{-1}F(x_0)\| \leq d. \tag{2.11}$$

We assume that (2.1)-(2.3) hold from now on unless otherwise stated. First we extend the results by Keller [11] for NM. Similarly the results for the chord method can also be extended. We leave the details to the motivated reader. For brevity we skip the extensions on the radii of convergence balls, and only mention convergence criteria and error estimates.

THEOREM 2.5. *Assume:*

$$Hr^{\lambda} < \frac{1+\lambda}{2+\lambda},$$

$$d \leq \left[1 - \frac{2+\lambda}{1+\lambda}Hr^{\lambda}\right]\lambda$$

and

$$\bar{U}(x_0, r) \subset \Omega.$$

Then, $\lim_{n\longrightarrow\infty} x_n = x_* \in U(x_0, r_0)$ and $F(x_*) = 0$. Furthermore,

$$\|x_* - x_n\| \leq \left(\frac{\mu^{\frac{1}{\lambda}}}{2+\lambda}\right)^{(1+\lambda)^p} \frac{r}{\mu^{\frac{1}{\lambda}}},$$

where $\mu = \frac{Hr^{\lambda}}{1-H_0r^{\lambda}}\frac{1}{1+\lambda} < 1$.

Proof. See Theorem 2 in [11].

□

THEOREM 2.6. *Assume:*

$$Hd^{\lambda} < \frac{1}{2+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^{\lambda}$$

and

$$\bar{U}(x_0, r) \subset \Omega.$$

Then, $\lim_{n\longrightarrow\infty} x_n = x_* \in U(x_0, r_0)$, $F(x_*) = 0$ *and*

$$\|x_* - x_n\| \leq \left(\frac{\lambda^{\frac{1}{p}}}{1-\lambda}\right)^{(1+p)^n} \frac{d}{\mu^{\frac{1}{p}}},$$

where $\lambda = \frac{Hr_0^p}{1-H_0r_0^p}\left(\frac{d}{r_0}\right)^p \frac{1}{1+p} < 1$ *and* r_0 *is the minimal positive root of scalar equation*

$$(2+p)Ht^{1+p} - (1+p)(t-d) = 0$$

provided that $r \geq r_0$.

Proof. See Theorem 4 in [11].

□

THEOREM 2.7. *Assume:*

$$Hd^p \leq 1 - \left(\frac{p}{1+p}\right)^p,$$

$$R \geq \frac{1+p}{2+p-(1+p)^p}d$$

and

$$\bar{U}(x_0, r) \subset \Omega.$$

Then, $\lim_{n\longrightarrow\infty} x_n = x_* \in U(x_0, r), F(x_*) = 0$ *and*

$$\|x_* - x_n\| \leq \left(\frac{1}{1+p}\right)^n [(1+p)H^{\frac{1}{p}}d]^{(1+p)^n} H^{\frac{1}{p}}.$$

Proof. See Theorem 5 in [11].

□

Next, we extend a result given in [6] which in turn extended earlier ones [1,7–14]. It is convenient to define function on the interval $[0, \infty)$ by

$$g(t) = \frac{H}{1+p}t^{1+p} - t + d$$

$$g_{\beta}(t) = \frac{\beta H}{1+p}t^{1+p} - t + d \quad (\beta \geq 0)$$

$$h(t) = \frac{t^{1+p} + (1+p)t}{(1+p)^{1+p} - 1},$$

$$v(p) = \max_{t\geq 0} h(t),$$

$$\delta(p) = \min\{\beta \geq 1 : \max h(t) \leq \beta, 0 \leq t \leq t(\beta)\}$$

and scalar sequence $\{s_n\}$ by

$$s_0 = 0,\ s_n = s_{n-1} - \frac{g_d(s_{n-1})}{g'(s_{n-1})}.$$

Then, we can show:

THEOREM 2.8. *Assume:*

$$d \leq \frac{1}{v(p)} \left(\frac{p}{1+p} \right)^p$$

and

$$U(x_0, \bar{r}) \subseteq \Omega,$$

where $\bar{r}$ is the minimal solution of equation $g_v(p) = 0$,

$$g_v(t) = \frac{v(p)H}{1+p} t^{1+p} - t + d.$$

Proof. See Theorem 2.2 in [6].

□

Next, we present the extensions of the work by Rokne in [13] but for the Newton-like method (NLM)

$$x_{n+1} = x_n - L_n^{-1} F(x_n),$$

where L_n is a linear operator approximating $F'(x_n)$.

THEOREM 2.9. *Assume:*

$$\|L(x) - L(x_0)\| \leq M_0 \|x - x_0\|^p$$

for all $x \in \Omega$. Set $\Omega_0 = U(x_0, \frac{1}{(\gamma_2 M_0)^{\frac{1}{p}}})$.

$$\|F'(x) - F'(y)\| \leq \bar{M} \|x - y\|^p$$

for all $x, y \in \Omega_0$,

$$\|F'(x) - L(x)\| \leq \gamma_0 + \gamma_1 \|x - x_0\|^p$$

for all $x \in \Omega_0$, and some $\gamma_0 \geq 0, \gamma_1 \geq 0$. $L(x_0)^{-1} \in L(Y, X)$ with $\|L(x_0)^{-1}\| \leq \gamma_2$ and $\|L(x_0)^{-1} F(x_0)\| \leq \gamma_3$, function q defined by

$$q(t) = t^{1+p}(\gamma_2\gamma_0 + \gamma_2 M_0) + t(\frac{\gamma_2 \bar{M} d^p}{1+p} + \gamma_2\gamma_0 - 1) - \gamma_2 M_0 \gamma_3 t^p + \gamma_3$$

has a smallest positive zero $R > \gamma_3$,

$$\gamma_2 \bar{M} R^p < 1,$$

$$\rho = \frac{p}{1 - \gamma_2 \bar{M} R^p} \left[\frac{\gamma_2 \bar{M} d^p}{1+p} + \gamma_2\gamma_0 + \gamma_2\gamma_1 R^p \right] < 1,$$

$\bar{U}(x_0, R) \subset \Omega$. Then $\lim_{n \longrightarrow \infty} x_n = x_*$ and $F(x_*) = 0$.

Proof. See Theorem 1 in [13].

□

Many results on Newton's method were also reported in the elegant book in [9]. Next, we show how to extend one of them. The details of how to extend the result of them are left to the motivated reader.

THEOREM 2.10. *Suppose: conditions (2.1), (2.3), (2.8), and*
(C) $h_0 = Hd^p \in (0, \rho)$ *where* ρ *is the only solution of equation*

$$(1+p)^p(1-t)^{1+p} - t^p = 0,\ p \in (0,1]$$

in $(0, \frac{1}{2}]$ *and* $U(x_0, s) \subset \Omega$, *where* $s = \frac{(1+p)(1-h_0)}{(1+p)-(2+p)h_0}$ *hold. Then, sequence* $\{x_n\}$ *converges to a solution* x_* *of equation* $F(x) = 0$. *Moreover,* $\{x_n\}, x_* \in U[x_0, s]$ *and* x_* *is the only solution in* $\Omega \cap U(x_0, \frac{d}{h_0^{1/p}})$. *Moreover, the following error estimates hold*

$$\|x_n - x^*\| \leq e_n,$$

where $e_n = \delta^{\frac{(1+p)^n-1}{p^2}} \frac{A^n}{1-\delta^{\frac{(1+p)^n}{p}}A} d$, *with* $\delta = \frac{h_1}{h_0}$, $A = 1 - h_0$, $h_1 = h_0 f_1(h_0)^{1+p} f_2(h_0)^p$, $f_1(t) = \frac{1}{1-t}$ *and* $f_2(t) = \frac{t}{1+p}$.

Finally, we extend the results by F. Cianciaruso and E. De Pascale in [6] who in turn extended earlier ones [1,5,7,11,12,14]. Define scalar sequence $\{v_n\}$ for $h = d^p H$ by

$$\begin{aligned} v_0 &= 0, v_1 = h^{\frac{1}{p}}, \\ v_{n+1} &= v_n + \frac{(v_n - v_{n-1})^{1+p}}{(1+p)(1-v_n^p)}. \end{aligned} \tag{2.12}$$

Next, we extend Theorem 2.1 and Theorem 2.3 in [6], respectively.

THEOREM 2.11. *Let function* $f : [1, \infty) \longrightarrow [0, \infty)$, $R : [0, \infty) \longrightarrow [0, \infty)$ *be defined by*

$$f(t) = (1 - \frac{1}{t}) \frac{1+p}{((1+p)^{\frac{1}{1-p}} + (t(t-1)^p)^{\frac{1}{1-p}})^{1-p}}$$

and

$$R(t) = \frac{(1+p)^{\frac{1}{p}}}{((1+p)^{\frac{1}{1-p}} + (t(t-1)^p)^{\frac{1}{1-p}})^{1-p}}.$$

Suppose that

$$h \leq f(M), \tag{2.13}$$

where M *is a global maximum for function* f, *given explicitly by* $M = \frac{1+\sqrt{1+4(1+p)^p p^{1-p}}}{2}$. *Then, the following assertion hold*

$$v_n \leq R(M)(1 - \frac{1}{M^n}), \tag{2.14}$$

$$\frac{v_{n+1}}{v_n} \leq \frac{1 - \frac{1}{M^{n+1}}}{1 - \frac{1}{M^n}}, \tag{2.15}$$

$$v_n \leq v_{n+1} \leq R(M) < 1$$

and $\lim_{n\longrightarrow\infty} v_n = v^* \in [0, R(M)]$.

Simply use H for H_1 in [6].

□

THEOREM 2.12. *Under condition (2.13) further suppose that* $r^* = H^{-\frac{1}{p}}v^* \leq \rho$ *and* $U(x_0, \rho) \subseteq \Omega$. *Then, sequence* $\{x_n\}$ *generated by NM is well defined in* $U(x_0, v^*)$, *stays in* $U(x_0, v^*)$ *and converges to the unique solution* $x^* \in U[x_0, v^*]$ *of equation* $F(x) = 0$, *so that*

$$\|x_{n+1} - x_n\| \leq v_{n+1} - v_n$$

and

$$\|x^* - x_n\| \leq v^* - v_n.$$

Proof. Simply use H for H_1 used in [6].

REMARK 2.13. *(1) If* $K = H_1$ *the last two results coincide with the corresponding ones in [6]. But if* $K < H_1$ *then the new results constitute an improvement with benefits already stated in the introduction. Notice that the majorizing sequence* $\{w_n\}$ *in [6] was defined for* $h_1 = d^p H_1$ *by*

$$\begin{aligned} w_0 &= 0, w_1 = h_1^{\frac{1}{p}}, \\ w_{n+1} &= w_n + \frac{(w_n - w_{n-1})^{1+p}}{(1+p)(1-w_n^p)}, \end{aligned} \tag{2.16}$$

and the convergence criterion is

$$h_1 \leq f(M). \tag{2.17}$$

It then follows by (2.7), (2.12), (2.13), (2.16) and (2.17) that

$$h_1 \leq f(M) \Rightarrow h \leq f(M) \tag{2.18}$$

but not necessarily vice versa, unless if $H = H_1$,

$$v_n \leq w_n,$$

$$0 \leq v_{n+1} - v_n \leq w_{n+1} - w_n$$

and

$$0 \leq v^* \leq w^* = \lim_{n\longrightarrow\infty} w_n.$$

(2) In view of (2.9) and (2.10) sequence $\{u_n\}$ *defined for each* $n = 0, 1, 2, \ldots$ *by*

$$\begin{aligned} u_0 &= 0, u_1 = h_1^{\frac{1}{p}}, \\ u_2 &= u_1 + \frac{H_0(u_1-u_0)^{1+p}}{(1+p)(1-H_0u_1^p)}, \\ u_{n+1} &= u_n + \frac{H(u_n-u_{n-1})^{1+p}}{(1+p)(1-H_0u_n^p)} \end{aligned}$$

is a tighter majorizing sequence than $\{v_n\}$ *and can replace it in Theorem 2.11 and Theorem 2.12.*

Concerning the uniqueness of the solution x^* *we provide a result based only on (2.1).*

PROPOSITION 2.14. *Suppose:*

(1) The point $x^* \in U(x_0, a) \subset \Omega$ *is a simple solution of equation* $F(x) = 0$ *for some* $a > 0$.

(2) Condition (2.1) holds.

(3) There exist $b \geq a$ *such that*

$$H_0 \int_0^1 ((1-\tau)a + \tau b)^p d\tau < 1. \tag{2.19}$$

Let $G = U[x_0, b] \cap \Omega$. *Then, the point* x^* *is the only solution of equation* $F(x) = 0$ *in the set* G.

Proof. Let $z^* \in G$ with $F(z^*) = 0$. By (2.1) and (2.19), we obtain in turn for $Q = \int_0^1 F'(x^* + \tau(z^*-x^*))d\tau$

$$\begin{aligned} \|F'(x_0)^{-1}(Q - F'(x_0))\| &\leq H_0 \int_0^1 \|x^* + \tau(z^*-x^*) - x_0\|^p d\tau \\ &\leq H_0 \int_0^1 [(1-\tau)\|x^*-x_0\| + \tau\|z^*-x_0\|]^p d\tau \\ &\leq H_0 \int_0^1 ((1-\tau)a + \tau b)^p d\tau < 1, \end{aligned}$$

showing $z^* = x^*$ by the invertibility of Q and the approximation $Q(x^*-z^*) = F(x^*) - F(z^*) = 0$. □

Notice that if $K = H_1$ the results coincide to the ones of Theorem 3.4 in [9]. But, if $K < H_1$ then they constitute an extension.

REMARK 2.15. *(a) We gave the results in affine invariant form.*

(b)The results in this study can be extended more if we consider the set $S = U(x_1, \frac{1}{H^{1/p}} - d)$ *provided that* $H^{1/p}d < 1$. *Moreover, suppose* $S \subset \Omega$. *Then,* $S \subset \Omega_0$, *so the Hölderian constant corresponding to* S *is at least as small as* K, *and can replace it in all previous results.*

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