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Parameter Estimation for SPDEs Driven by Cylindrical Stable Processes

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Abstract. We consider infinite dimensional extension of affine models with heavy tails in finance. We study several estimators of the drift parameter in the stochastic partial differential equation driven by cylindrical stable processes. We consider several sampling schemes. We also consider random sampling scheme, e.g, when the solution process is observed at the arrival times of a Poisson process. We obtain the consistency and the asymptotic normality of the estimators.

1. Introduction

Parameter estimation in stochastic partial differential equations is a very young area of research in view of its applications in finance, physics, biology and oceanography. Loges [32] initiated the study of parameter estimation in infinite dimensional stochastic differential equations. When the length of the observation time becomes large, he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges [28] extended the work of Loges [32] to minimum contrast estimators. Koski and Loges [27] applied the work to a stochastic heat flow problem. Martingale estimation function for discretely observed diffusions was studied in Bibby and Srensen [2]. Bishwal [6] studied a new estimating function for discretely sampled diffusions by removing the stochastic integral in Girsanov likelihood. Bishwal [7] contains asymptotic theory on likelihood method and Bayesian method for drift estimation of finite and infinite dimensional stochastic differential equations. Bishwal [12] studied applications of Levy processes in stochastic volatility models in finance.

Huebner, Khasminskii and Rozovskii [23] started statistical investigation in SPDEs. They gave two contrast examples of parabolic SPDEs in one of which they obtained consistency, asymptotic normality and asymptotic efficiency of the MLE as noise intensity decreases to zero under the condition of absolute continuity of measures generated by the process for different parameters (the

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situation is similar to the classical finite dimensional case) and in the other they obtained these properties as the finite dimensional projection becomes large under the condition of singularity of the measures generated by the process for different parameters. The second example was extended by Huebner and Rozovskii [24] and the first example was extended by Huebner [22] to MLE for general parabolic SPDEs where the partial differential operators commute and satisfy different order conditions in the two cases.

Huebner [21] extended the problem to the ML estimation of multidimensional parameter. Lototsky and Rozovskii [33] studied the same problem without the commutativity condition. Small noise asymptotics of the nonparmetric estimation of the drift coefficient was studies by Ibragimov and Khasminskii [29].

Based on continuous observations, usually there can be two asymptotic settings in SPDE: 1) $T \to \infty$ 2) $n \to \infty$ where T is the length of the observations and n is the number of Fourier coefficients of the SPDE solution.

In a Bayesian approach, using the first setting, Bishwal [3] proved the Bernstein-von Mises theorem and asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding ergodic diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE.

Using the second setting, Bishwal [5] proved the Bernstein-von Mises theorem and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In this case, the measures generated by the process for different parameters are singular.

Bishwal [10] studied Bernstein-von Mises theorem and small noise Bayesian asymptotics for parabolic stochastic partial differential equations. Bishwal [9] studied hypothesis testing for fractional stochastic partial differential equations with applications to neurophysiology and finance.

In this paper we study the asymptotic properties of the quasi maximum likelihood estimator when we have observations of finite-dimensional projections at Poisson arrival time points. The asymptotic setting is only the large number of observations at random time points which are the arrivals of a Poisson process.

The rest of the paper is organized as follows: Section 2 contains model, assumptions and preliminaries. In Section 3 we prove estimation results with additive noise. Section 4 and 5, we provide estimation results with multiplicative noise. In section 6, we give several examples.

2. Model and Preliminaries

Let H be a real separable Hilbert space with inner product $\langle \cdot \rangle$ and norm $| \cdot |$. By $\mathcal{L}(H)$ we denote the Banach space of bounded linear operators from H into H endowded with the operator norm

$\|\cdot\|_{\mathcal{L}(H)}$. We fix an orthonormal basis (e_n) in H. Through the basis (e_n) we will often identify H in l^2. More generally, for a given sequence $\rho = (\rho_n)$ of real numbers we set

$$l^2_\rho = \{(x_n) \in \mathbb{R}^\infty : \sum_{n\geq 1} x_n^2 \rho_n^2 < \infty\}.$$

where $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{N}}$. The space l^2_ρ becomes a separable Hilbert space with the inner product: $\langle x, y\rangle = \sum_{n\geq 1} x_n y_n \rho_n^2$ for $x = (x_n), y = (y_n) \in l^2_\rho$. Let us fix θ_0, the unknown true value of the parameter θ. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $Z(t, x)$ be a process on this space with values in the Schwarz space of distributions $D'(G)$ such that for $\phi, \psi \in C_0^\infty(G)$, $\|\phi\|^{-1}_{L^2(G)}$ $\langle W(t,\cdot), \phi(\cdot)\rangle$ is a one dimensional stable process.

This process is usually referred to as the *cylindrical α-stable process* (C.S.P.), $\alpha \in (0, 2)$.

We assume that there exists a complete orthonormal system $\{h_i\}_{i=1}^\infty$ in $L_2(G))$ such that for every $i = 1, 2, \ldots, h_i \in Z_0^{m,2}(G) \cap C^\infty(\overline{G})$ and

$$\Lambda_\theta h_i = \beta_i(\theta) h_i, \text{ and } \mathcal{L}_\theta h_i = \mu_i(\theta) h_i \text{ for all } \theta \in \Theta$$

where $\mathcal{L}_\theta$ is a closed self adjoint extension of A^θ, $\Lambda_\theta := (k(\theta)I - \mathcal{L}_\theta)^{1/2m}$, $k(\theta)$ is a constant and and the spectrum of the operator Λ_θ consists of eigenvalues $\{\beta_i(\theta)\}_{i=1}^\infty$ of finite multiplicities and $\mu_i = -\beta_i^{2m} + k(\theta)$.

A Levy process (Z_t) with values in H is an H-valued process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ having stationary independent increments, cadlag trajectories such that $Z_0 = 0$, P-a.s. One has that

$$E[e^{i\langle Z_t, s\rangle}] = \exp(-t\psi(s)), \; s \in H$$

where $\psi : H \to \mathbb{C}$ is Sazonov continuous, negative definite function such that $\psi(0) = 0$. The function ψ is called the exponent of (Z_t).

The exponent ψ can be expressed by the infinite dimensional *Levy-Khintchine formula*

$$\psi(s) = \frac{1}{2}\langle Qs, s\rangle - i\langle a, s\rangle - \int_H \left(e^{i\langle s,y\rangle} - 1 - \frac{i\langle s, y\rangle}{1 + |y|^2}\right)\nu(dy), \; s \in H$$

where Q is the non-negative trace class operator on H, $a \in H$ and ν is the Levy measure or the jump intensity measure associated to (Z_t).

Cylindrical α-stable process (C.S.P.) is a Levy process taking values in the Hilbert space $U = l^2_\rho$, with a properly chosen weight ρ.

Consider the linear SPDE

$$dX_t = \theta A X_t dt + dZ_t, x \in H$$

C.S.P. $Z(t)$ is a cylindrical α-stable process, $\alpha \in (0, 2)$ which can be expanded in the series

$$Z(t) = \sum_{i=1}^\infty \gamma_i Z_i(t) h_i$$

where $\{Z_i(t)\}_{i=1}^\infty$ are independent, real valued, one dimensional, normalized, symmetric, α-stable processes and (γ_i) is a given sequence of, possibly unbounded, positive numbers, and h_i is a fixed orthonormal basis in H. The latter series converges P-a.s. in $H^{-\alpha}$ for $\alpha > d/2$. Indeed

$$\|Z(t)\|^2_{-\alpha} = \sum_{i=1}^{\infty} \gamma_i^2 Z_i^2(t)\|h_i\|^2_{-\alpha} = \sum_{i=1}^{\infty} Z_i^2(t)\beta_i^{-2\alpha}$$

and the later series converges P-a.s.

For any $j \in \mathbb{N}, t \geq 0$,

$$E[e^{iZ_j(t)h}] = e^{-t|h|^\alpha}.$$

Stable one-dimensional density :

A one-dimensional, normalized, symmetric α-stable distribution $\mu_\alpha,\ \alpha \in (0,2]$ has characteristic function

$$\hat{\mu}_\alpha(s) = e^{-|s|^\alpha}, s \in \mathbb{R}.$$

The density of μ_α with respect to Lebesgue measure will be denoted by p_α. This even function is known in closed form only if $\alpha = 1$ or 2. The precise asymptotic behavior of the density $p_\alpha, \alpha \in (0,2)$ is as follows:

For any $\alpha \in (0,2)$, there exists C_α such that

$$p_\alpha(x) \sim \frac{C_\alpha}{x^{\alpha+1}} \quad \text{as} \quad x \to \infty.$$

Stable measures on Hilbert space :

A random variable ξ on H is called α-stable ($\alpha \in (0,2]$) if for any n there exists a vector $a_n \in H$ such that for any independent copies $\xi_1, \xi_2, \ldots, \xi_n$ of ξ, the random variable $n^{-1/\alpha}(\xi_1 + \xi_2, \ldots + \xi_n) - a_n$ has the same distribution as ξ. A Borel probability measure μ on H is said to be α-stable if it is the distribution of a stable random variable with vales in H.

Stable OU Process:

$$dX_t = -\theta X_t dt + \sigma dZ_t,\ X_0 = x_0$$

The solution is

$$X_t = e^{-\theta t}x_0 + \int_0^t e^{-\theta(t-s)}\sigma dZ_s.$$

The stochastic integral can be defined as the limit in probability of Riemann sums.

Let

$$Y_t = \int_0^t e^{-\theta(t-s)}\sigma dZ_s.$$

Then

$$E[e^{ihY_t}] = \exp\left[-\sigma^\alpha |h|^\alpha \int_0^t e^{-\alpha\theta s}ds\right] = e^{-|h|^\alpha c^\alpha(t)}$$

where

$$c^\alpha(t) = \sigma\left(\frac{1 - e^{-\alpha\theta t}}{\alpha\theta}\right).$$

We show that the process X is stochastically continuous.

First we show that Y is stochastically continuous, i.e.,

$$\lim_{h\to 0+} \sup_{t\geq 0} P(|Y_{t+h} - Y_t| > \epsilon) = 0$$

Note that for any $t \geq 0$, $h \geq 0$,

$$Y_{t+h} - Y_t = \int_t^{t+h} e^{(t+h-s)A} dZ_s + e^{hA} \int_0^t e^{(t-s)A} dZ_s - \int_0^t e^{(t-s)A} dZ_s$$

$$= e^{hA} Y_t - Y_t + \int_t^{t+h} e^{(t+h-s)A} dZ_s$$

Let us choose $p \in (0, \alpha)$. We have

$$P(|Y_{t+h} - Y_t| > \epsilon) \leq P\left(|e^{hA} Y_t - Y_t| > \frac{\epsilon}{2}\right) + P\left(\left|\int_t^{t+h} e^{(t+h-s)A} dZ_s\right| > \frac{\epsilon}{2}\right)$$

$$\leq 2^p \frac{E|e^{hA} Y_t - Y_t|^p}{\epsilon^p} + 2^p \frac{E|\int_0^h e^{sA} dZ_s|^p}{\epsilon^p} = I_1(t, h) + I_2(h).$$

But

$$E|Y_t|^p \leq c_p \left(\sum_{n=1}^{\infty} \frac{1 - e^{-\alpha\theta t}}{\alpha\theta}\right)^{p/\alpha}$$

and so $[I_2(h)]^{\alpha/p} \to 0$ as $h \to 0$. Concerning I_1, by Khintchine inequality

$$|e^{hA} Y_t - Y_t| = \left(\sum_{n\geq 1} \left|(e^{-\theta h} - 1) Y_t^n\right|^2\right)^{1/2} \leq C_p \left(\tilde{E}\left|\sum_{n\geq 1} r_n (e^{-\theta h} - 1) Y_t^n\right|^p\right)^{1/p}.$$

where $\tilde{E}$ denotes expectation w.r.t. to the measure $\tilde{P}$ $\tilde{P}(r_n = 1) = \tilde{P}(r_n = 1) = 1/2$ where a Rademacher sequence (r_n) with $r_n : \tilde{\Omega} \to \{-1, 1\}$ is defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Hence

$$E|e^{hA} Y_t - Y_t|^p \leq C_p^p \tilde{E} E \left(\sum_{n\geq 1} \left|(e^{-\theta h} - 1) Y_t^n\right|^2\right)^{1/2}$$

$$\leq C_p \left(\sum_{n\geq 1} \left|(1 - e^{-\theta h t}) \beta_n\right|^\alpha \frac{(1 - e^{-\theta h t})}{\alpha\theta}\right)^{p/\alpha} \leq \frac{C_p}{\alpha^{p/\alpha}} \left(\sum_{n\geq 1} \frac{\left|(1 - e^{-\theta h t}) \beta_n\right|^\alpha}{\theta}\right)^{p/\alpha}$$

Since

$$\lim_{h\to 0+} \left(\sum_{n\geq 1} \frac{\left|(1 - e^{-\theta h t}) \beta_n\right|^\alpha}{\theta}\right)^{p/\alpha} = 0,$$

we get

$$\lim_{h\to 0+} \sup_{t\geq 0} 2^p \frac{E|e^{hA} Y_t - Y_t|^p}{\epsilon^p} = 0.$$

Since

$$E|Y_t|^p \leq C_p \left(\sum_{n\geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\theta h t})}{\alpha\theta}\right)^{p/\alpha},$$

hence

$$\lim_{h\to 0} \left(\sum_{n\geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\theta h t})}{\alpha\theta}\right)^{p/\alpha} = 0$$

hence

$$\lim_{h\to 0+} 2^p \frac{E|\int_0^h e^{sA}dZ_s|^p}{\epsilon^p} \to 0.$$

Thus

$$\lim_{h\to 0+} \sup_{t\geq 0} I_1(t,h) = 0$$

This proves stochastic continuity of Y_t.

Using the stochastic continuity and $\mathcal{F}_t$-adaptedness of X, we conclude that the process X has a predictable version.

Time Change:

Let L be a one dimensional α-stable process, $\alpha \in (0,2)$. Then there exists an α-stable process, $\alpha \in (0,2) \quad Z = (Z_t)$ such that

$$\int_0^t e^{-\theta s} dL_s = Z(u(t)) \;\text{ where }\; u(t) = \frac{1-e^{-\alpha\theta t}}{\alpha\theta}.$$

Recall that $u \in C^\infty([0,\infty])$ with $u'(t) \neq 0,\ t \geq 0$.

In the limiting Gaussian case of $\alpha = 2$, it becomes time change for Brownian motion.

Infinite Dimensional Stable OU Process

$$dX_t^n = -\theta X_t^n dt + \sigma dZ_t^n,\ X_0^n = x_n, n \in \mathbb{N}$$

with $x = (x_n) \in L^2 = H$. The solution is a stochastic process $X = X_t^x$ with values in $\mathbb{R}^\infty$ with components

$$X_t^x = e^{-\theta t} x_n + \int_0^t e^{-\theta(t-s)} \sigma dZ_s^n.$$

(The stochastic integral can be defined as the limit in probability of Riemann sums.)

$$X_t^x = \sum_{n=1}^\infty X_t^n e_n = e^{tA}x + Z_A(t)$$

where

$$Z_A(t) = \int_0^t e^{(t-s)A} dZ_s = \sum_{n=1}^\infty \left(\int_0^t e^{-\theta(t-s)} \sigma dZ_s^n \right) e_n.$$

The process X_t^x is an $\mathcal{F}_t$-adapted irreducible Markov process and its transition semigroup is strong Feller.

Let

$$Y_t^n = Z_A^n(t) = \int_0^t e^{-\theta(t-s)} \sigma dZ_s^n, n \in \mathbb{N},\ t \geq 0.$$

Then

$$E[e^{ihY_t^n}] = \exp\left[-\sigma^\alpha |h|^\alpha \int_0^t e^{-\alpha\theta s} ds \right] = e^{-|h|^\alpha c_n^\alpha(t)}$$

where

$$c_n(t) = \sigma \left(\frac{1 - e^{-\alpha\theta t}}{\alpha\theta} \right)^{1/\alpha}.$$

It follows that

$$E[e^{ihY_t^n}] = E[e^{ihc_n(t)L_n}], \;\; h \in \mathbb{R}$$

where (L_n) are independent α-stable random variables having the same law μ_α. Thus X_t^x is $\mathcal{F}_t$-adapted.

The Markov property easily follows from the identity

$$Z_A(t+h) - e^{hA} Z_A(t) = \int_t^{t+h} e^{(t+h-s)A} dZ_s, \;\; t, h \geq 0.$$

If the cylindrical Levy process Z takes values in Hilbert space H, the by the Kotelenez regularity results trajectories of the process X are cadlag with values in H.

Moments of the process

The OU process is stochastically continuous and trajectories in $L^p([0,T]; H)$ for any $0 < p < \alpha$ a.s. Set $Y_t := Z_A(t)$. Then we have

$$E|Y_t|^p \leq \tilde{c}_p \sigma^p \left(\sum_{n=1}^{\infty} \frac{1 - e^{-\alpha\theta t}}{\alpha\theta} \right)^{p/\alpha}$$

where $\tilde{c}_p$ depends on p.

Moments of the stochastic integral

Suppose (Z_t) is an α-stable Levy process with $0 \leq \alpha \leq 2$ and $y(t)$ is a predictable process satisfying $\int_0^T |y(t)|^\alpha dt < \infty$. Then for any $0 < r < \alpha$, there exists a constant C such that

$$E\left[\sup_{t \leq T} \left| \int_0^t y(s) dZ_s \right|^r \right] \leq E\left[\left(\int_0^t |y(t)|^\alpha dt \right)^{r/\alpha} \right].$$

Equivalence of Transition Probabilities

Assume

$$\sup_{n \geq 1} \frac{e^{-\gamma_n t} \gamma_n^{1/\alpha}}{\beta_n} = C_t < \infty, \quad E \int_0^T \left(\sum_{n \geq 1} |Y_t^n|^2 \right)^{p/2} dt < \infty.$$

Let p_α be the density of the one dimensional stable measure. Then the laws μ_t^x and μ_t^y of X_t^x and X_t^y respectively are equivalent for any $t > 0, \;\; x, y \in H, \alpha \in (0,2)$. Moreover, the density $\frac{d\mu_t^x}{d\mu_t^x}$ of

μ_t^x with respect to μ_t^y is given by

$$\frac{d\mu_t^x}{d\mu_t^y} = \lim_{n\to\infty}\prod_{k=1}^{n}\frac{p_\alpha\left(\frac{Z_k - e^{-\theta t}x_k}{c(t)}\right)}{p_\alpha\left(\frac{Z_k - e^{-\theta t}y_k}{c(t)}\right)}.$$

The corresponding MLE is denoted as $\hat{\theta}_n$.

Priola *et al.* [38] obtained exponential convergence to the invariant measure, in the total variation norm, for solutions to SDEs driven by α-stable noises in finite and infinite dimensions using two approaches: Lyapounov's function approach by Harris and Doeblin's coupling argument. In both approaches irreducibility and uniform strong Feller property play crucial role.

First we consider the method of moments estimation in modified tempered stable-Ornstein-Uhlenbeck model. Masuda and Uehara [36] studied two-step estimation in ergodic Levy driven SDE

$$dX_t = a(\theta, X_t)dt + b(\beta, X_{t-})dZ_t, \quad X_0 = x_0.$$

Masuda [35] studied multi-step estimation in stable OU Model:

$$dX_t = -\theta X_t dt + \sigma dZ_t,\ X_0 = x_0.$$

For the least squares estimator (LSE) $\tilde{\theta}_n$ of θ, Hu and Long [20] obtained

$$\left(\frac{T}{\log n}\right)^{1/\alpha}(\tilde{\theta}_n - \theta_0) \to^{\mathcal{D}} \frac{S'_\alpha}{S''^{+}_{\alpha/2}}$$

where S_α is stable distribution of order β.

While in Gaussian OU case, for different parts $\theta > 0,\ \theta < 0$ and $\theta = 0$, LAN, LAMN and LABF hold respectively (see Bishwal [11]), in stable case entirely different phenomena occur.

The solution of the SDE is given by

$$X_t = e^{-\theta(t-s)}X_s + \sigma\int_s^t e^{-\theta(t-s)}dZ_u, t \geq 0.$$

Due to the stable integral property,

$$\mathcal{L}\left(\int_s^t e^{-\theta(t-s)}dZ_u\right) = S_\alpha(\kappa_\Delta(\theta))$$

where

$$\kappa_\Delta(\theta) = \left\{\frac{1 - e^{-\theta\Delta}}{\theta\alpha}\right\}^{1/\alpha} \sim \Delta^{1/\alpha}.$$

For each $j \leq n$, the transition probability is given by

$$\mathcal{L}(X_{t_j}|X_{t_{j-1}} = x) = \delta_{x\exp(-\theta\Delta)} \star S_\alpha(\kappa_\Delta(\theta)).$$

LAMN holds for $\theta \in \mathbb{R}$ when T is fixed.

$$n^{1/\alpha - 1/2}(\hat{\theta}_n - \theta) \to^{D} MN(0, I_\theta(T)^{-1}).$$

where $I_\theta(T)$ is the Fisher information of the process.

We study estimation in MTS-OU SV model. The Inverse Gaussian-OU and Gamma-OU models are special cases.

An infinitely divisible distribution is said to be α-modified tampered stable distribution (α-MTS) distribution if its Levy triplet is given by

$$\sigma^2 = 0,$$
$$\nu(dx) = C\left(\frac{\lambda_+^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}\lambda_+ x}{x^{\alpha+\frac{1}{2}}} I_{x>0} + \frac{\lambda_+^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}\lambda_- x}{x^{\alpha+\frac{1}{2}}} I_{x<0}\right) dx,$$
$$\gamma = \mu + C\left(\frac{\Gamma(\frac{1}{2}-\alpha)}{2^{\alpha+\frac{1}{2}}}(\lambda_+^{2\alpha-1} - \lambda_-^{2\alpha-1}) - \lambda_+^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda_+) + \lambda_-^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\lambda_-)\right)$$

where $C > 0, \lambda_+, \lambda_- > 0,\ \mu \in \mathbb{R},\ \alpha \in (-\infty, 1)\backslash\{\frac{1}{2}\}$ and $K_p(x)$ is the modified Bessel function of second kind. We denote the MTS random variable by $X \sim MTS(\alpha, C, \lambda_+, \lambda_-, \mu)$. The Levy measure $\nu(dx)$ is called the MTS Levy measure with parameter $(\alpha, C, \lambda_+, \lambda_-)$.

The MTS distribution is obtained by taking a symmetric α-stable distribution with $\alpha \in (0, 1)$ and multiplying by a Levy measure with $\sqrt{|x|}\lambda^{\alpha+\frac{1}{2}} K_{\alpha+\frac{1}{2}}(\lambda|x|)$ on each half of the real axis. The measure can be extended to the case $\alpha \leq 0$. If $\alpha = \frac{1}{2}$, then γ may not be defined, so it is removed. The MTS distribution was introduced by Kim, Rachev and Chung [25].

The tails of the α-MTS distribution are thinner than those of the 2α-stable and fatter (heavier) than those of the 2α-TS distribution. At the zero neighborhood, all three have the same asymptotic behavior.

If $\lambda_+ > \lambda_-$, then the distribution is *skewed to the left.* If $\lambda_+ < \lambda_-$, then the distribution is *skewed to the right.* If $\lambda_+ = \lambda_-$, then the distribution is *symmetric.*

C controls the kurtosis of the distribution. If C increases, the peakedness of the distribution increases.

As α decreases, the distribution has fatter tails and increased peakedness. The Levy process corresponding to the MTS distribution has *finite activity* if $\alpha < 0$ and *infinite activity* if $\alpha > 0$. It has *finite variation* if $\alpha < \frac{1}{2}$ and *infinite variation* if $\alpha > \frac{1}{2}$.

With proper choice of C and μ, MTS distribution has zero mean and unit variance, and the distribution is called standard MTS distribution and denoted $X \sim stdMTS(\alpha, \lambda_+, \lambda_-)$.

CGMY process proposed in Carr *et al.* [14] is a tempered stable process. In order to obtain a closed form solution of the European option price, CGMY used the generalised Fourier transform of the distribution of the stock price under the assumption of Markov property.

The stochastic volatility model is given by

$$\begin{aligned} dY_t &= (\mu + \beta X_t)\ dt + \sqrt{X_t}\ dW_t + \rho\ dZ_t \\ dX_t &= -\theta\ X_t\ dt\ + dZ_t \end{aligned}$$

where μ is the drift parameter, β is the risk premium, $\theta > 0$ is the drift of the volatility and Z_t is a MTS process.

We estimate θ from the observations of $\{Y_t\}$ at the time points $t_k = k\Delta,\ k = 0, 1, 2, \ldots, n$, $\Delta > 0$.

$$c_m(Z) := \frac{d^m}{du^m} \log \phi_{TS}(u)|_{u=0}$$

For the tempered stable distribution TS(b, δ, γ) where $0 < b < 1, \delta > 0, \gamma \geq 0$, the m-th cumulant is given by

$$c_m(Z) = -\delta(-2)^m \gamma^{(b-m)/b} b(b-1)\ldots(b-(m-1))$$

for $\gamma > 0$. When $\gamma = 0$, it is positive b-stable distribution for which the moments of only order $k < b$ exist. For $b = 1/2$, TS distribution reduces to Inverse Gaussian (IG) distribution.

The infinite divisibility of this distribution allows one to construct the corresponding Levy process. A Levy process $Z = (Z_t)_{t\geq 0}$ is said to be a tempered stable process if Z_1 follows a tempered stable distribution. The tempered stable process is of *finite activity* if $\alpha < 0$ and *infinite activity* if $0 < \alpha < 2$. The tempered stable process is of *finite variation* if $0 < \alpha < 1$ and *infinite variation* if $1 < \alpha < 2$.

The MTS-GARCH model is given by

$$\log \frac{S_t}{S_{t-1}} = r_t - d_t + \lambda_t \sigma_t - g(\sigma_t; \alpha, \lambda_+, \lambda_-) + \sigma_t \epsilon_t$$

$$\sigma_t^2 = (\alpha_0 + \alpha_1 \sigma_{t-1}^2 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \wedge \rho,\ \ \epsilon_0 = 0$$

$\alpha_0, \alpha_1, \beta_1 \geq 0,\ \alpha_1 + \beta_1 < 1,\ 0 < \rho < \lambda_+^2,\ \ \epsilon_t \sim stdMTS(\alpha, \lambda_+, \lambda_-)$, r_t is the risk-free rate, d_t is the dividend rate, λ_t is the market price of risk, g is the characteristic exponent of the Laplace transform for the distribution $stdMTS(\alpha, \lambda_+, \lambda_-)$, i.e., $g(x; \alpha, \lambda_+, \lambda_-) = \log(E(\exp(x\epsilon_t)))$.

The characteristic function of Z is given by

$$\phi_Z(u) = \exp(iu\mu + G_R(u; \alpha, C, \lambda_+, \lambda_-) + G_I(u; \alpha, C, \lambda_+, \lambda_-))$$

where for $u \in \mathbb{R}$,

$$G_R(u; \alpha, C, \lambda_+, \lambda_-) = 2^{-\frac{\alpha+3}{2}} \sqrt{\pi} C \Gamma\left(1 - \frac{\alpha}{2}\right) \left[(\lambda_+^2 + u^2)^{\frac{\alpha}{2}} - \lambda_+^\alpha + (\lambda_-^2 + u^2)^{\frac{\alpha}{2}} - \lambda_-^\alpha\right],$$

$$\begin{aligned} & G_I(u; \alpha, C, \lambda_+, \lambda_-) \\ = \ & iuC2^{-\frac{\alpha+1}{2}} \Gamma\left(\frac{1-\alpha}{2}\right) \left[\lambda_+^{\alpha-1} F\left(1, \frac{1-\alpha}{2}; \frac{3}{2}, ; -\frac{u^2}{\lambda_+^2}\right) - \lambda_-^{\alpha-1} F\left(1, \frac{1-\alpha}{2}; \frac{3}{2}, ; -\frac{u^2}{\lambda_-^2}\right)\right] \end{aligned}$$

where F is the hyper-geometric function. The value of G_I for symmetric MTS distribution is always zero.

The m-th cumulant is given by

$$c_m(Z) = \mu \qquad if \quad m = 1,$$

$$c_m(Z) = 2^{m-\frac{\alpha+3}{2}} \left(\frac{m-1}{2}\right)! C\Gamma\left(\frac{m-\alpha}{2}\right)(\lambda_+^{\alpha-m} - \lambda_-^{\alpha-m}) \quad if \quad m = 3, 5, 7, \ldots$$

$$c_m(Z) = 2^{-\frac{\alpha+3}{2}} \sqrt{\pi} \left(\frac{m!}{\frac{m}{2}!}\right) C\Gamma\left(\frac{m-\alpha}{2}\right)(\lambda_+^{\alpha-m} + \lambda_-^{\alpha-m}) \quad if \quad m = 2, 4, 6, \ldots$$

The mean, variance, skewness and excess kurtosis are given by

$$E(Z) = c_1(Z) = \mu + 2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}),$$

$$V(Z) = c_2(Z) = 2^{-\frac{\alpha+1}{2}} \sqrt{\pi} C\Gamma\left(1-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}),$$

$$s(Z) = \frac{c_3(Z)}{c_2(Z)^{3/2}} = \frac{2^{\frac{\alpha+9}{4}}\Gamma\left(\frac{3-\alpha}{2}\right)(\lambda_+^{\alpha-3} - \lambda_-^{\alpha-3})}{\pi^{3/4}C^{1/2}(\Gamma(\frac{1-\alpha}{2})(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}))^{3/2}},$$

$$\kappa(Z) = \frac{c_4(Z)}{c_2(Z)^2} = \frac{3 \cdot 2^{\frac{\alpha+3}{2}} C\Gamma\left(2-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-4} + \lambda_-^{\alpha-4})}{\sqrt{\pi}C(\Gamma(\frac{1-\alpha}{2})(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}))^2}.$$

If $\alpha \in (0,2)\backslash\{1\}$, the Levy measure of α-stable, α-TS and α-MTS have the same asymptotic behavior at the zero neighborhood. However, the tails of the Levy measures for the α-MTS distribution are *thinner* than those of α-stable and *heavier* than those of α-TS distribution.

When Z is a IG process, the moment estimators of ρ and θ are given by

$$\hat{\theta}_n := \frac{\gamma\bar{y}}{\Delta\delta\hat{\rho}_n}, \quad \hat{\rho}_n := \frac{\gamma(\gamma s_y^2 - \Delta\delta)}{2\bar{y}}$$

where

$$\bar{y} := \frac{1}{n}\sum_{j=1}^{n} y_j, \quad y_j := Y_{j\Delta} - Y_{(j-1)\Delta},$$

$$s_y^2 := \frac{1}{n}\sum_{j=1}^{n}(y_j - \bar{y})^2 = \frac{1}{n}\sum_{j=1}^{n} y_j^2 - (\bar{y})^2.$$

When Z is a Gamma process, the moment estimators are given by

$$\hat{\theta}_n := \frac{\frac{1}{n^2}\left[\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})\right]^2}{\frac{1}{n^2}\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})^2 - \frac{\Delta}{n}\left[\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})\right]}\frac{2a^3(a+1)}{b^4\Delta},$$

$$\hat{\rho}_n := \frac{\frac{1}{n^2}\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})^2 - \frac{\Delta}{n}\left[\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})\right]}{\frac{1}{n^2}\left[\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})\right]}\frac{b^3\Delta}{2a^2(a+1)}.$$

For the MTS-OU model, the estimating functions are given by

$$c_1(y_1) = \lambda\rho\Delta c_1(Z),$$

$$c_2(y_1) = \Delta c_1(Z) + 2\lambda\rho^2\Delta c_1(Z),$$

$$c_3(y_1) = \Delta c_1(Z) + 2\lambda\rho^2\Delta c_2(Z),$$

$$c_4(y_1) = \Delta c_1(Z) + 2\lambda\rho^2\Delta c_3(Z)$$

which give

$$E(y_1) = c_1(y_1) = \mu + 2^{-\frac{\alpha+1}{2}} C\Gamma\left(\frac{1-\alpha}{2}\right)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}),$$

$$V(y_1) = c_2(y_1) = 2^{-\frac{\alpha+1}{2}} \sqrt{\pi} C\Gamma\left(1-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}).$$

This gives the moment estimators for the SOU model

$$\hat{\theta}_n := \frac{\frac{1}{n^2}\left[\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})\right]^2}{\frac{1}{n^2}\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})^2 - \frac{\Delta}{n}\left[\sum_{i=1}^{n}(Y_{i\Delta} - Y_{(i-1)\Delta})\right]}$$

$$\times\ [2^{-\frac{\alpha+1}{2}}C\Gamma\left(\frac{1-\alpha}{2}\right)(\lambda_+^{\alpha-1}-\lambda_-^{\alpha-1})]^2[2^{-\frac{\alpha+1}{2}}\sqrt{\pi}C\Gamma\left(1-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2})]2\Delta^{-1}.$$

$$\hat{\rho}_n := \frac{\frac{1}{n^2}\sum_{i=1}^n (Y_{i\Delta}-Y_{(i-1)\Delta})^2 - \frac{\Delta}{n}\left[\sum_{i=1}^n (Y_{i\Delta}-Y_{(i-1)\Delta})\right]}{\frac{1}{n^2}\left[\sum_{i=1}^n (Y_{i\Delta}-Y_{(i-1)\Delta})\right]}$$

$$\times\ [2^{-\frac{\alpha+1}{2}}C\Gamma\left(\frac{1-\alpha}{2}\right)(\lambda_+^{\alpha-1}-\lambda_-^{\alpha-1})2^{-\frac{\alpha+1}{2}}\sqrt{\pi}C\Gamma\left(1-\frac{\alpha}{2}\right)(\lambda_+^{\alpha-2}+\lambda_-^{\alpha-2})]^{-1}2^{-1}\Delta.$$

Let $\vartheta = (\rho, \theta)$ and $\hat{\vartheta}_n = (\hat{\rho}_n, \hat{\theta}_n)$. By using Theorem 2.2 in Masuda [34] (see also Theorem 4.1 Van der Vaart [41]), we obtain the strong consistency and asymptotic normality of the MM estimators:

Proposition 2.1 For fixed $\Delta > 0$ as $n \to \infty$,

$$(a) \quad \hat{\vartheta}_n \to \vartheta_0 \ \text{ a.s.} \quad \text{as } n \to \infty.$$

$$(b) \quad \sqrt{n}(\hat{\vartheta}_n - \vartheta_0) \to^{\mathcal{D}} \mathcal{N}_2(0, (J^{-1}(\vartheta_0)) \text{ as } n \to \infty.$$

where $J(\vartheta_0)$ is the Fisher information.

3. SPDEs with Additive Noise

Consider the parabolic SPDE

$$du^\theta(t,x) = \theta u^\theta(t,x) + \frac{\partial^2}{\partial x^2}u^\theta(t,x)dt + dZ(t,x), \ \ t \ge 0, \ \ x \in [0,1] \tag{3.1}$$

$$u(0,x) = u_0(x) \in L_2([0,1]), \tag{3.2}$$

$$u^\theta(t,0) = u^\theta(t,1), \ \ t \in \ [0,T]. \tag{3.3}$$

Here $\theta \in \Theta \subseteq \mathbb{R}$ is the unknown parameter to be estimated on the basis of the observations of the field $u^\theta(t,x), t \ge 0, \ x \in [0,1]$.

Let S_3 and S_4 be independent stable random variables, S_3 is positive $\alpha/2$-stable with distribution $S_{\alpha/2}(\sigma_1, 1, 0)$ and S_4 is symmetric α-stable random variable with distribution $S_\alpha(\sigma_2, 0, 0)$, $\sigma_1 = C_{\alpha/2}^{-2/\alpha}$, $\sigma_2 = C_\alpha^{-1/\alpha}$, $C_\alpha = (\int_0^\infty x^{-\alpha}\sin x dx)^{-1} = [\Gamma(1-\alpha)\cos(\pi\alpha/2)]^{-1}$.

In this case, in the limiting distribution, S_3 and S_4 are independent stable random variables with a rate faster than the cylindrical Brownian motion case.

For $x \in [0,1]$, we observe the process $\{u_t, t \ge 0\}$ at times $\{t_0, t_1, t_2, \ldots\}$. We assume that the sampling instants $\{t_i, i = 0,1,2,\ldots\}$ are generated by a Poisson process on $[0,\infty)$, i.e., $t_0 = 0, t_i = t_{i-1} + \alpha_i, i = 1,2,\ldots$ where α_i are i.i.d. positive random variables with a common exponential distribution $F(x) = 1 - \exp(-\lambda x)$. Note that intensity parameter $\lambda > 0$ is the average sampling rate which is assumed to be known. It is also assumed that the sampling process

t_i, $i = 0, 1, 2, \ldots$ is independent of the observation process $\{X_t, t \geq 0\}$. We note that the probability density function of $t_{k+i} - t_k$ is independent of k and is given by the gamma density

$$f_i(t) = \lambda(\lambda t)^{i-1} \exp(-\lambda t) I_t/(i-1)!, \;\; i = 0, 1, 2, \ldots. \tag{3.4}$$

where $I_t = 1$ if $t \geq 0$ and $I_t = 0$ if $t < 0$.

Consider the Fourier expansion of the process

$$u^\theta(t, x) = \sum_{t=1}^{\infty} u_i^\theta(t)\phi_i(x) \tag{3.5}$$

corresponding to some orthogonal basis $\{\phi_i(x)\}_{i=1}^\infty$. Note that $u_i^\theta(t), i \geq 1$ are independent one dimensional stable Ornstein-Uhlenbeck processes

$$du_i^\theta(t) = \mu_i^\theta u_i^\theta(t)dt + \beta_i^{-\alpha} dZ_i(t) \tag{3.6}$$

$$u_i^\theta(0) = u_{0i}^\theta,$$

Recall that $\mu_i(\theta) = k(\theta) - \beta_i^{2m}$. Thus

$$du_i^\theta(t) = (k(\theta) - \beta_i^{2m})u_i^\theta(t)dt + \beta_i^{-\alpha} dZ_i(t) \tag{3.7}$$

The random field $u(t, x)$ is observed at discrete times t and discrete positions x. Equivalently, the Fourier coefficients $u_i^\theta(t)$ are observed at discrete time points.

Define

$$\rho := \rho(\lambda, \theta) = \frac{\lambda}{\lambda - \kappa(\theta) + \beta_i^{2m}}.$$

The *quasi-likelihood estimator* is the solution of the estimating equation:

$$G_n^*(\theta) = 0 \tag{3.8}$$

where

$$G_n^*(\theta) = \frac{\beta_i^{2\alpha}\lambda(\rho(\lambda, \theta))^2}{\rho(\lambda, 2\theta)} \sum_{i=1}^{n} u_{t_{i-1}} \left((u_{t_{i-1}}\theta\rho(\lambda, \theta))^2 + \lambda\right)^{-1} (u_{t_i} - \rho(\lambda, \theta)u_{t_{i-1}}). \tag{3.9}$$

We call the solution of the estimating equation the *quasi-likelihood estimator*. There is no explicit solution for this equation.

The *optimal estimating function* for estimation of the unknown parameter θ is

$$G_n(\theta) = \beta_i^{2\alpha} \sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - \rho(\lambda, \theta)u_{t_{i-1}}]. \tag{3.10}$$

The *martingale estimation function (MEF) estimator* of ρ is the solution of

$$G_n(\theta) = 0 \tag{3.11}$$

and is given by

$$\hat{\rho}_n := \frac{\sum_{i=1}^n u_{t_{i-1}} u_{t_i}}{\sum_{i=1}^n u_{t_{i-1}}^2}.$$

We do the parameter estimation in two steps: The rate λ of the Poisson process can be estimated given the arrival times t_i, therefore it is done at a first step. Since we observe total number of arrivals n of the Poisson process over the T intervals of length one, the MLE of λ is given by

$$\hat{\lambda}_n := \frac{n}{T}.$$

Theorem 3.1 We have

$$\hat{\lambda}_n \to \lambda \text{ a.s.} \quad \text{as} \quad n \to \infty.$$

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \to^{\mathcal{D}} \mathcal{N}(0,\ e^{\lambda}(1 - e^{-\lambda})) \text{ as } \ n \to \infty.$$

Proof. Let V_i be the number of arrivals in the interval $(i-1, i]$. Then $V_i,\ i = 1, 2, \ldots, n$ are i.i.d. Poisson distributed with parameter λ. Since Φ is continuous, we have $I_{\{0\}}(V_i) = I_{\{0\}}(u(t_i))$ a.s. $i = 1, 2, \ldots, n$. Note that

$$\frac{1}{n}\sum_{i=1}^{n} I_{\{0\}}(u_{t_i}) \to^{a.s.} E(I_{\{0\}}V_1) = P(V_1 = 0) = e^{-\lambda} \text{ as } \ n \to \infty.$$

LLN and CLT and delta method applied to the sequence $I_{\{0\}}(u_{t_i}),\ i = 1, 2, \ldots, n$ give the results. $\square$

The CLT result above allows us to construct confidence interval for the jump rate λ.

Corollary 3.1 A $100(1-\alpha)\%$ confidence interval for λ is given by

$$\left[\frac{n}{T} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}},\ \ \frac{n}{T} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}}\right]$$

where $Z_{1-\frac{\alpha}{2}}$ is the $(1 - \frac{\alpha}{2})$-quantile of the standard normal distribution.

We obtain the strong consistency and asymptotic normality of the MEF estimator.

Theorem 3.2 When $\alpha = 2$, we have

$$\hat{\rho}_n \to^{a.s.} \rho \text{ as } \ n \to \infty$$

$$\sqrt{n}(\hat{\rho}_n - \rho) \to^{\mathcal{D}} \mathcal{N}(0,\ \lambda^{-i}(1 - e^{-\rho})) \text{ as } \ n \to \infty.$$

Proof: By using the fact that every stationary mixing process is ergodic, it is easy to show that if u_t is a stationary ergodic O-U Markov process and t_i is a process with nonnegative i.i.d. increments which is independent of u_t, then $\{u_{t_i}, i \geq 1\}$ is a stationary ergodic Markov process. Hence $\{u_{t_i}, i \geq 1\}$ is a stationary ergodic Markov process.

Observe that $u_i^\theta(t) := v_i$ is a stationary ergodic Markov chain and $v_i \sim \mathcal{N}(0, \sigma^2)$ where σ^2 is the variance of u_0. Thus by SLLN for zero mean square integrable martingales, we have

$$\frac{1}{n}\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i} \rightarrow^{a.s.} E(u_{t_0} u_{t_1}) = \rho E(u_{t_0}^2)$$

$$\frac{1}{n}\sum_{i=1}^{n} u_{t_{i-1}}^2 \rightarrow^{a.s.} E(u_{t_0}^2)$$

Thus

$$\frac{\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}}{\sum_{i=1}^{n} u_{t_{i-1}}^2} \rightarrow^{a.s.} \rho.$$

Further,

$$\sqrt{n}(\hat{\rho}_n - \rho) = \frac{n^{-1/2}\sum_{i=1}^{n} u_{t_{i-1}}(u_{t_i} - \theta u_{t_{i-1}})}{n^{-1}\sum_{i=1}^{n} u_{t_{i-1}}^2}.$$

Since

$$E(u_{t_1} u_{t_2} | u_{t_1}) = \theta u_{t_1}^2$$

it follows by Lemma 3.1 in Bibby and Srensen [2]

$$n^{-1/2}\sum_{i=1}^{n} u_{t_{i-1}}(u_{t_i} - \theta u_{t_{i-1}})$$

converges in distribution to normal distribution with mean zero and variance equal to

$$E[(u_{t_1} u_{t_2}) - E(u_{t_1} u_{t_2} | u_{t_1})]^2 = 1 - e^{2(\theta - \beta_i \delta)}\{2(\beta_i - \theta)(\beta_i + 1)\}^{-1}.$$

Applying delta method the result follows. □

In the next step, we use the estimator of λ to estimate θ.

Note that

$$\frac{1}{\hat{\rho}_n} = \frac{\sum_{i=1}^{n} u_{t_{i-1}}^2}{\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}}.$$

Thus

$$1 + \frac{\beta_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{i=1}^{n} u_{t_{i-1}}^2}{\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}}$$

which gives

$$\frac{\beta_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{i=1}^{n} u_{t_{i-1}}^2}{\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}} - 1 = -\frac{\sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - u_{t_{i-1}}]}{\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}}$$

Now replace λ by its estimator MLE $\hat{\lambda}_n$.

$$\beta_1^{2m} - \kappa(\theta) = -\frac{\sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - u_{t_{i-1}}]}{\frac{T}{n}\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}}$$

Thus

$$\hat{\theta}_n = \kappa^{-1}\left(\beta_i^{2m} + \frac{\sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - u_{t_{i-1}}]}{\frac{T}{n}\sum_{i=1}^{n} u_{t_{i-1}} u_{t_i}}\right).$$

Since the function $\kappa^{-1}(\cdot)$ is a continuous function, by application of delta method, the following result is a consequence of Theorem 3.2.

Theorem 3.3 When $\alpha = 2$,

$$a)\ \ \hat{\theta}_n \to \theta \text{ a.s. as } \ n \to \infty$$

$$b)\ \ \sqrt{n}(\hat{\theta}_n - \theta) \to^{\mathcal{D}} \mathcal{N}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\beta_1^{2m})})) \text{ as } \ n \to \infty.$$

In the second stage, we substitute λ by its estimator $\hat{\lambda}_n$.

Theorem 3.4 When $0 < \alpha < 2$,

$$a)\ \ \hat{\theta}_n \to^{a.s.} \theta \text{ as } \ n \to \infty$$

$$b)\ \ n^{(\alpha-1)/\alpha^2}(\hat{\theta}_n - \theta) \to^{\mathcal{D}} \left(\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\beta_1^{2m})}\right)^{1/\alpha} \frac{S_4}{S_3} \text{ as } \ n \to \infty.$$

where S_4 and S_3 are independent stable random variables.

In the second stage, we substitute λ by its estimator $\hat{\lambda}_n$. The limit distribution is normal only in the Gaussian case $\alpha = 2$.

4. SPDEs with Linear Multiplicative Noise

Consider the SPDE with multiplicative noise:

$$du^\theta(t,x) = (A_0 + \theta A_1)u^\theta(t,x)dt + Mu^\theta(t,x)dZ(t,x),\ \ t \geq 0,\ \ x \in [0,1] \tag{4.1}$$

where M is a known linear operator.

Equation (4.1) is called diagonalizable if A_0, A_1 and M have point spectrum and a common system of eigenfunction $\{h_j, j \geq 1\}$. Denote by ρ_k, ν_k and μ_k, the eigenvalues of the operators A_0, A_1 and M respectively.

Then

$$u^\theta(t,x) = \sum_{j\geq 1} u_{j,t} h_j.$$

The Fourier coefficients have the dynamics

$$du_k(t) = (\theta\nu_k + \rho_k)u_k(t)dt + \sigma_k u_k(t)dZ_k(t), \quad k \geq 1$$

which is the *Stable Black-Scholes Model* whose solution is geometric stable process.

Let

$$\theta\nu_k + \rho_k =: \mu_k(\theta), \quad \tilde{v}_{k,T} := \ln(u_{k,T}/u_{k,0}).$$

Conditional characteristic function (CCF) estimator is given by

$$\widehat{\mu_k(\theta)} = \frac{\tilde{v}_{k,T}}{T^{2(\alpha-1)/\alpha^2}} + \frac{\sigma_k^2}{b_1 T^{-((\alpha-1)^2+1)/\alpha^2}}.$$

Since $\mu_k(\theta)$ is strictly monotone function of θ, by invariance principle of CCFE, under invertible transformations, we can find the CCFE of the parameter θ

$$\hat{\theta}_{k,T} = \frac{\tilde{v}_{k,T}}{\nu_k T^{2(\alpha-1)/\alpha^2}} + \frac{\sigma_k^2}{b_1 \nu_k T^{-((\alpha-1)^2+1)/\alpha^2}} - \frac{\rho_k}{\nu_k}$$

which can be represented as

$$\hat{\theta}_{k,T} = \theta_0 + \frac{\sigma_k M_T}{\nu_k T^{2(\alpha-1)/\alpha^2}}$$

where M_T is a square-integrable martingale. Due to the LLN for martingales, we have strong consistency.

Note that in the standard Black-Scholes case where $\alpha = 2$, $\sigma_k = \sigma$, the MLE of the drift coefficient of the geometric Brownian motion is given by

$$\hat{\theta}_T = \frac{\ln(u_T/u_0)}{T} + \frac{\sigma^2}{2} = \theta_0 + \sigma \frac{W_t}{T}.$$

Due to the law of iterated logarithm for Brownian motion, the MLE is strongly consistent as $T \to \infty$.

Theorem 4.1 When $0 < \alpha < 2$,

a) $\hat{\theta}_{k,T}$ is an unbiased estimator of θ.

b)

$$\hat{\theta}_{k,T} \to \theta \quad \text{a.s.} \quad \text{as} \quad T \to \infty.$$

c)

$$T^{(\alpha-1)/\alpha^2}(\hat{\theta}_{k,T} - \theta) \to^{\mathcal{D}} \left(\frac{\sigma_k^2}{\nu_k^2}\right)^{1/\alpha} \frac{S_4}{S_3} \quad \text{as} \quad T \to \infty$$

where S_4 and S_3 are independent stable random variables.

d) If in addition,

$$\lim_{k\to\infty} \left|\frac{\sigma_k}{\nu_k}\right| = 0,$$

then for every fixed $T > 0$,

$$\hat{\theta}_{k,T} \to \theta \text{ a.s. as } \quad k \to \infty$$

and

$$\left|\frac{\nu_k}{\sigma_k}\right| (\hat{\theta}_{k,T} - \theta) \to^{\mathcal{D}} \left(T^{(\alpha-1)/\alpha^2}\right)^{1/\alpha} \frac{S_4}{S_3} \quad \text{as} \quad k \to \infty.$$

Remark: The parabolicity condition and the MLE consistency condition in general are not connected. In terms of operator's order, parabolicity states that the order of operator M from the diffusion term is smaller than half of the order of operators A_0 and A_1 from deterministic part. The consistency condition assumes that the order of the operator M from the diffusion part does not exceed the order of the operator A_1 from the deterministic part that contains the parameter of interest θ.

5. SPDEs with Nonlinear Multiplicative Noise

Consider the SPDE with multiplicative noise:

$$du^{\theta}(t,x) = (A_0 + \theta A_1)u^{\theta}(t,x)dt + Mu^{\theta}(t,x)dZ(t,x), \ t \geq 0, \ x \in [0,1] \tag{5.1}$$

where M is a known nonlinear operator.

Equation (5.1) is called diagonalizable if A_0, A_1 and M have point spectrum and a common system of eigenfunction $\{h_j, j \geq 1\}$. Denote by ρ_k, ν_k and μ_k, the eigenvalues of the operators A_0, A_1 and M respectively.

Then

$$u^{\theta}(t,x) = \sum_{j=1}^{\infty} u_{j,t} h_j.$$

We consider stable CIR model as example. Here S_1 and S_2 are dependent stable random variables unlike the linear case where S_3 and S_4 are independent stable random variables.

The existence and pathwise uniqueness of solutions to the SDEs with non-Lipschitz coefficient driven by spectrally positive Levy processes were studied in Fu and Li [17].

Consider the nonlinear SPDE

$$dX(t,x) = \frac{\theta}{2} X_{xx}(t,x)dt + \sqrt{X(t,x)}dW(t,x)$$

where $W(t,x)$ is a space-time white noise. Konno and Shiga [26] studied the existence and weak uniqueness of the above equation as a martingale problem for the associated super-Brownian motion. The pathwise uniqueness of nonnegative solution still remains open. The main difficulty comes from the unbounded drift coefficient and non-Lipschitz diffusion coefficient. Wang *et al.* [42] studied proved a comparison theorem and showed that the solution of the nonlinear SPDE is distribution function valued. They also established pathwise uniqueness. As application they obtained well-posedness of martingale problems for two classes of measure-valued diffusions: interacting super-Brownian motions and interacting Fleming-Viot processes. He *et al.* [18] obtained pathwise unique solution to nonlinear SPDE with super Levy process, which is a combination of space-time Gaussian white noises and Poisson random measures which is a generalization of work of Xiong [43] where the result for a super-Brownian motion with binary branching mechanism was obtained. Using an extended Yamada-Watanabe argument, Xiong [43] established strong existence and uniqueness of the solution to the SPDE. Super-Brownian motion (SMB), also called the Dawson-Watanabe process introduced by Sawson and Watanabe is a measure valued process arising as the limit of empirical measure process of a branching particle system. SBM satisfies a martingale problem. When the state space is $\mathbb{R}$, SBM has a density w.r.t. Lebesgue measure and this density valued process $X(t,x)$ satisfies the above SPDE. When the space $\mathbb{R}$ is s single point, the SPDE becomes an SDE which is CIR diffusion $dX_t = \sqrt{X_t}dW_t$ whose uniqueness is established using the Yamada-Watanabe argument. Xiong and Yang (2019) studied existence and pathwise uniqueness to an SPDE with Hölder continuous coefficient driven by α-stable colored noise. The existence of the solution is shown by considering the weak limit of a sequence of SDE system which is obtained by replacing the laplacian operator in the SPDE by its discrete version. The pathwise uniqueness is shown by using a backward doubly stochastic differential equation to take care of the Laplacian. In the case of $d = 1$, the pathwise uniqueness of a nonnegative solution to the corresponding equation was established by Yang and Zhou [45] for $1 < \alpha < \sqrt{5} - 1$ and pathwise uniqueness for $\sqrt{5} - 1 < \alpha < 2$ is still open.

Consider SPDE model with multiplicative noise and mean reversion, where the j-th Fourier coefficient is the stable Cox-Ingersoll-Ross (SCIR) model:

$$du_{j,t} = (a - \theta u_{j,t})dt + \sigma u_{j,t-}^{1/\alpha} dZ_{j,t}, \quad j \geq 1 \tag{5.2}$$

where a is the mean reverting level and θ is mean reverting speed. Recall that for $\alpha = 2$, for every $j \geq 1$, the process $Z_{j,t}$ is a standard Brownian motion, this is the famous Cox-Ingersoll-Ross (CIR) model used for modeling interest rate, which is also used a stochastic volatility process in Heston model. Note that there are Brownian CIR models with additive compound Poisson type jumps. When $1 < \alpha < 2$, $Z_{j,t}$ is stable process with Levy measure

$$\nu_\alpha(dz) = \frac{1_{\{z>0\}}dz}{\alpha\Gamma(-\alpha)z^{\alpha+1}}. \tag{5.3}$$

The discontinuous SCIR model captures the heavy tailed property in the sense of infinite variance. There is empirical evidence from high frequency data available in support of application of pure jump models in financial modeling.

The SCIR model has the unique stationary distribution μ with Laplace transform given by

$$L_\mu(\lambda) = \int_0^\infty e^{-\lambda x}\mu(dx) = \exp\left\{-\int_0^\lambda \frac{\alpha a}{\alpha\theta + \sigma^\alpha z^{\alpha-1}}dz\right\}, \quad \lambda \geq 0. \tag{5.4}$$

Applying Itô's formula, for $t \geq r \geq 0$, we obtain

$$u_{j,t} = e^{-\theta(t-r)}u_{j,r} + a\int_r^t e^{-\theta(t-s)}ds + \sigma\int_r^t e^{-\theta(t-s)}u_{j,s-}^{1/\alpha}dZ_{j,s}, \ j \geq 1. \tag{5.5}$$

Let the process be observed at $\{kh, k = 0, 1, \ldots, n\}$ from a single realization $\{u_{j,t}, t \geq 0\}$ for fixed h. For simplicity, we take $h = 1$. This equation can be considered as a first order autoregressive (AR(1)) equation

$$u_{j,k} = \rho + \gamma u_{j,k-1} + \epsilon_{j,k}, \quad j \geq 1 \tag{5.6}$$

where $\gamma = e^{-\theta}, \quad \rho = a\theta^{-1}(1-\gamma)$ and

$$\epsilon_{j,k} = \sigma\int_{k-1}^k e^{-\theta(k-s)}u_{j,s-}^{1/\alpha}dZ_{j,s}, \quad k \geq 1, \ j \geq 1. \tag{5.7}$$

For $B \in \mathcal{B}(R^+)$, let

$$S_{2,j,n}(B) = \sum_{k=1}^{n} u_{j,k-1}\epsilon_k I_B(|u_{j,k-1}\epsilon_{j,k}|), \quad S_{1,j,n}(B) = \sum_{k=1}^{n} u_{j,k-1}^2 I_B(u_{j,k-1}), \quad j \geq 1. \tag{5.8}$$

It is easy to see that

$$\epsilon_{j,k} = u_{j,k} - E(u_{j,k}|\mathcal{F}_{k-1}), \; k \geq 1, \; j \geq 1. \tag{5.9}$$

is a sequence of martingale differences for every fixed j.

Let $S_{1,j,n} := S_{1,j,n}(0,\infty)$, $\quad S_{2,j,n} := S_{2,j,n}(0,\infty)$ and recall that $\gamma = e^{-\theta}$.

Then

$$\hat{\theta}_{j,n} - \theta = \frac{S_{2,j,n}}{S_{1,j,n}} \tag{5.10}$$

where $\hat{\theta}_n$ is the conditional least squares estimator (CLSE) which minimizes

$$\sum_{k=1}^{n} \epsilon_{j,k}^2 = \sum_{k=1}^{n} [u_{j,k} - E(u_{j,k}|\mathcal{F}_{k-1})]^2 = \sum_{k=1}^{n} [u_{j,k} - \rho - \gamma u_{j,k-1}]^2 \tag{5.11}$$

and are given by

$$\hat{\gamma}_{j,n} = \frac{\sum_{k=1}^{n} u_{j,k-1} \sum_{k=1}^{n} u_{j,k} - n \sum_{k=1}^{n} u_{j,k-1} u_{j,k}}{(\sum_{k=1}^{n} u_{j,k-1})^2 - n \sum_{k=1}^{n} u_{j,k-1}^2},$$

$$\hat{\rho}_{j,n} = \frac{1}{n} \sum_{k=1}^{n} u_{j,k} - \hat{\gamma}_n \frac{1}{n} \sum_{k=1}^{n} u_{j,k-1},$$

$$\hat{\theta}_{j,n} = -\log \hat{\gamma}_{j,n}, \quad \hat{a}_{j,n} = \frac{\hat{\rho}_n \hat{\theta}_n}{1 - \hat{\gamma}_n}.$$

Let (S_1, S_2) have the characteristic function given by

$$\begin{aligned} & E[\exp\{i\lambda_1 S_1 + i\lambda_2 S_2\}] \\ := & \exp\left\{ -\frac{\sigma^\alpha}{\theta^2 \Gamma(-\alpha)} \int_0^\infty E\left(1 - \exp\{i\lambda_1 y^2 + i\lambda_2 y^{(\alpha+1)/\alpha} V_{j,1}\}\right) \right. \\ \times & \left. E\left(\exp\left\{ \frac{ie^{-2\theta}\lambda_1 y^2}{1 - e^{-2\theta}} + \frac{ie^{-\theta(\alpha+1)/\alpha}\lambda_2 y^{(\alpha+1)/\alpha} V_{j,2}}{(1 - e^{\theta(\alpha+1)})^{1/\alpha}} \right\}\right) \frac{dy}{y^{\alpha+1}} \right\} \end{aligned} \tag{5.12}$$

and

$$V_{j,k} := \sigma \int_{k-1}^{k} e^{-\theta(k-s)} e^{-\theta(s-k+1)/\alpha} dZ_{j,s}, \quad k = 1, 2, \; j \geq 1 \tag{5.13}$$

which are i.i.d. with the same distribution as

$$\sigma \left(\frac{e^{-\theta} - 1}{(\alpha - 1)\theta} \right)^{1/\alpha} Z_{j,1}$$

which is regularly varying with index α. The limit distribution is normal only in the Gaussian case $\alpha = 2$.

Following Li and Ma [31] it can be shown that for every fixed j, if we have $1 < \alpha < (1 + \sqrt{5})/2$, then we have as $n \to \infty$

$$(d_n^{-2} S_{1,j,n}, c_n^{-1} S_{2,j,n}) \xrightarrow{\mathcal{D}} (S_1, S_2) \quad \text{on} \quad \mathbb{R}^2$$

where $d_n = n^{1/\alpha}$ and $c_n = n^{(\alpha+1)/\alpha^2} = d_n^{(\alpha+1)/\alpha}$.

For the stable SPDE model, we have the following result on the consistency and the limit distribution of the CLSE:

Theorem 5.1 If we have $1 < \alpha < (1+\sqrt{5})/2$, then for every fixed $j \geq 1$

a)

$$\hat{\theta}_{j,n} \to^{P} \theta \text{ as } \ n \to \infty.$$

b)

$$n^{(\alpha-1)/\alpha^2}(\hat{\theta}_{j,n} - \theta) \to^{\mathcal{D}} \left(\frac{\sigma^2}{\nu_j^2}\right)^{1/\alpha} \frac{S_2}{S_1} \text{ as } \ n \to \infty.$$

c) If in addition, $\lim_{j\to\infty} |\nu_j| = \infty$, then for every fixed $n \geq 1$,

$$\hat{\theta}_{j,T} \to^{P} \theta \ \text{ as } \ j \to \infty$$

and

$$|\nu_j| (\hat{\theta}_{j,n} - \theta) \to^{\mathcal{D}} \sigma \left(n^{-(\alpha-1)/\alpha^2}\right)^{1/\alpha} \frac{S_2}{S_1} \text{ as } \ j \to \infty.$$

where S_2 and S_1 are defined in (5.12).

Remarks

1) The limit distribution in the case $(1+\sqrt{5})/2 < \alpha < 2$ is still open.

2) The process (X_j) is exponentially ergodic and hence strongly mixing.

3) For the Gaussian case $(\alpha = 2)$, the limit results are based on ergodic theory and martingale convergence theorem. For the non-Gaussian case $(1 < \alpha < 2)$, limit results are obtained by the theory of regular variation and convergence of point processes.

4) Let $0 < \alpha < 2$ and let Z_t be a one dimensional α-stable process with Levy measure $\nu(dz)$. Then as $n \to \infty$, $nP(n^{-1/\alpha}Z_t \in \cdot) \to^{v} t\nu(\cdot)$.

6. Examples

(a) Consider the *linear stochastic heat equation with additive noise*

$$du(t,x) = \theta u_{xx}(t,x)dt + dZ(t,x)$$

for $0 \leq t \leq T$ and $x \in (0,1)$ and $\theta > 0$ with periodic boundary conditions.

Here $2m = m_1 = 2$ and $\mu_j = -\theta\pi^2 j^2, \gamma > 1/2$. The eigenfunctions are $h_j(x_1,\ldots,x_n) = (\sqrt{2/\pi})^d(\sin(n_1x_1),\ldots,\sin(n_dx_d))$, $x = (x_1,\ldots,x_n) \in \mathbb{R}^d, j = (n_1,\ldots,n_d) \in \mathbb{N}^d$. The corresponding eigenvalues are $-\nu_j$ where $\nu_j = (n_1^2 + \ldots + n_d^2)$.

As $n \to \infty, h \to 0$, $nh^{1+\alpha}/\log n \to 0$, $nh^{2\alpha-1}\log n \to \infty$, $nh^{2-\alpha/2+\rho} \to \infty$ for some $\rho > 0$,

$$\left(\frac{n}{\log n}\right)^{1/\alpha} h^{1/\alpha}(\hat{\theta}_n - \theta_0) \to^{\mathcal{D}} 2\theta_0(\alpha\theta_0)^{-1/\alpha}\frac{S_4}{S_3}$$

where S_3 and S_4 are independent stable random variables, S_3 is positive $\alpha/2$-stable with distribution $S_{\alpha/2}(\sigma_1, 1, 0)$ and S_4 is symmetric α-stable random variable with distribution $S_\alpha(\sigma_2, 0, 0)$, $\sigma_1 = C_{\alpha/2}^{-2/\alpha}$, $\sigma_2 = C_\alpha^{-1/\alpha}$, $C_\alpha = (\int_0^\infty x^{-\alpha} \sin x dx)^{-1} = [\Gamma(1-\alpha)\cos(\pi\alpha/2)]^{-1}$.

Observe the rate of convergence $(nh)^{1/\alpha}(\log n)^{-1/\alpha} = (T)^{1/\alpha}(\log n)^{-1/\alpha}$. For $\alpha = 2$, this rate is $T^{1/2}(\log n)^{-1/2}$.

(b)

Consider the *linear stochastic heat equation with multiplicative noise*

$$du(t,x) = \theta u_{xx}(t,x)dt + u(t,x)dZ(t,x)$$

for $0 \le t \le T$ and $x \in (0,1)$ and $\theta > 0$ with zero boundary conditions and nonzero initial value $u(0) \in L_2(0,1)$. Here A_1 is the Laplace operator on $(0,1)$ with zero boundary conditions that has the eigenfunctions $h_k(x) = \sqrt{2/\pi}\sin(kx)$, $k > 0$ and the eigenvalues $\nu_k = -k^2, \rho_k = 0$, $\sigma_k = 1, k > 0$.

$$u_k(t) = \int_0^1 h_k(x)u(t,x)dx,$$

$$du_k(t) = (\theta\nu_k + \rho_k)u_k(t)dt + \sigma_k u_k(t)dZ_k(t).$$

Recall that

$$\tilde{v}_{k,T} := \ln(u_{k,T}/u_{k,0}).$$

The CCFE has the form

$$\hat{\theta}_{k,T} = \frac{\tilde{v}_{k,T} - 1}{k^2}.$$

(c) Consider the following SPDE

$$du(t,x) = [\Delta u(t,x) + \theta u(t,x)]dt + (1-\Delta)^r u(t,x)dZ(t,x).$$

In this case $A_0 = \Delta, A_1 = I, M = (1-\Delta)^r$ with the eigenvalues $\nu_k = 1, \rho_k = \sigma_k, \mu_k = (1+\sigma_k)^r$. It has a unique solution for any $r \le 1/2$.

The CCFE has the form

$$\hat{\theta}_{k,T} = \frac{\tilde{v}_{k,T}}{k^2 T^{2(\alpha-1)/\alpha^2}} - \frac{(1-\sigma_k)^{2r}}{k^2 T^{-((\alpha-1)^2+1)/\alpha^2}} - \frac{1}{\sigma_k}$$

(d) *Stable Cox-Ingersoll-Ross Model*

Xiong and Yang [44] studied existence and strong uniqueness of the following SPDE:

$$du_k(t) = (\theta\nu_k + \rho_k)u_k(t)dt + \sigma_k(u_k(t))^{1/\alpha}dZ_k(t), \ k \ge 1.$$

The existence of the solution in the case of space-time white noise is shown by considering the weak limit of a sequence of SDE systems which is obtained by replacing the Laplacian operator in the SPDE by its discrete version. The weak uniqueness follows from the uniqueness of solution to the martingale problem for the associated super-Brownian motion. In the case of α-stable noise the existence and pathwise uniqueness of the solution is studied in Xiong and Yang [44].

Concluding Remark We considered Levy process driving term in this paper. Using fractional Levy process as the driving term, maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [8]. Recently, sub-fractional Brownian (sub-FBM) motion which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0$$

for $0 < H < 1$ introduced by Bojdecki, Gorostiza and Talarczyk [13] has received some attention recently in finite dimensional models. The interesting feature of this process is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. It would be interesting to see extension of this paper to sub-FBM case. We generalize sub-fBM to Sub-fractional Levy process (sub-FLP).

Sub-fractional Levy process (SFLP) is defined as

$$S_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dM_s, \quad t \in \mathbb{R}$$

where $M_t, t \in \mathbb{R}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0$, $E(M_1^2) < \infty$ and without Brownian component. SFLP has the following properties:

1) The covariance of the process is given by

$$\text{Cov}(S_{H,t}, S_{H,s}) = s^{2H} + t^{2H} + \frac{E[L(1)^2]}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) S_H is not a martingale. For a large class of Levy processes, S_H is neither a semimartingale nor a Markov process. 3) S_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) S_H has nonstationary increments. 5) S_H is symmetric. 6) S_H is self similar. 7) S_H has infinite total variation on compacts.

It would be interesting to investigate QML estimation in SPDE driven by subfractional Levy processes which incorporate both jumps and long memory apart from nonstationarity.

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