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Moment Properties of Generalized Order Statistics Based on Modified Weighted Rayleigh Distribution

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Abstract. The generalized order statistics is a unified model of several ordered random schemes such as order statistics, record values, progressive type II right censored order statistics, etc. Order statistics and record values appear in many statistical applications and are widely used in statistical modeling and inference. Progressive Type-II censored sampling scheme is a versatile censoring scheme because it allows the experimenter to save time and cost of the life-testing experiment and is quite useful in reliability and lifetime studies. Rayleigh distribution and its extended version such as modified weighted Rayleigh distribution are important distributions for modeling lifetime data in reliability analysis and engineering science. In this paper, we studied moment properties of generalized order statistics from modified weighted Rayleigh distribution. Further, characterization results based on moment properties and conditional expectation are presented. The simulation studies based on order statistics, upper record values, and progressive type II right censored order statistics are also carried out.

1. Introduction

1.1. **Definition:** Let $X_1, X_2, \dots, X_n$ be a sequence of independent and identically distributed (iid) random variables (RVs) with cumulative distribution function (CDF) $F(x)$ and probability density function (PDF) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, \dots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \dots, n$ are called generalized order statistics (GOS) if their joint (PDF) is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}\, f(x_i)\right)[1-F(x_n)]^{k-1}\, f(x_n) \tag{1}$$

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on the cone $F^{-1}(0) \leq x_1 \ldots \leq x_n \leq F^{-1}(1)$ of $\mathbb{R}^n$

The variants of GOS are defined in Table 1.1 by choosing the parameters appropriately (See Cramer [17]):

Table 1.1: Variants of the GOS

		$\gamma_n = k$	γ_r	m_r
i)	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$\gamma_r - \gamma_{r+1} - 1$
ii)	Order statistics (OS)	1	$(n-r+1)$	0
iii)	Record values	1	1	-1
iv)	Progressively type II right censored OS	$R_n + 1$	$n - r + 1 + \sum_{j=r}^{n} R_j$	R_γ
v)	Pfeifer's record values	β_n	β_r	$\beta_r - \beta_{r+1} - 1$

1.2. **Marginal and Joint Distributions:** This subsection discusses the marginal distribution of single GOS and joint distribution of two GOS. Here we consider two cases:

Case I. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1, i \neq j$.

In view of (1), the PDF of r^{th} GOS $X(r, n, \tilde{m}, k)$ is given as (Kamps and Cramer [21])

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} \sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1} f(x), \tag{2}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0,$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \ 1 \leq i \leq r \leq n.$$

The joint PDF of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$, is given as (Kamps and Cramer [21])

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x, y) = C_{s-1} \sum_{j=r+1}^{s} a_j{}^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_j} \times \left[\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \ x < y, \tag{3}$$

where

$$a_j^{(r)}(s) = \prod_{\substack{l=r+1 \\ l \neq j}}^{s} \frac{1}{(\gamma_l - \gamma_j)}, \ r+1 \leq j \leq s \leq n.$$

Case II : $m_i = m$, $i = 1, 2, \ldots, n-1$.

The PDF of r^{th} GOS $X(r, n, m, k)$ is given as (Kamps [22])

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x)), \tag{4}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i \quad , \gamma_i = k+(n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1} & , \ m \neq -1 \\ \log\left(\dfrac{1}{1-x}\right) & , \ m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \ x \in [0,1).$$

The joint PDF of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given as (Pawlas and Szynal [34])

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!\,(s-r-1)!}[\bar{F}(x)]^m\, g_m^{r-1}(F(x))$$
$$\times\, [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(x)\, f(y), \ -\infty \le x < y \le \infty. \tag{5}$$

1.3. **Modified Weighted Rayleigh Distribution:** A RV X is said to follow modified weighted Rayleigh distribution (MWRD), if its PDF is given by (Badmus et at. [15])

$$f(x) = 2\alpha(\beta\theta^2+1)x\, e^{-\alpha(\beta\theta^2+1)x^2}, \ x>0; \ \alpha, \beta, \theta > 0. \tag{6}$$

and the corresponding survival function (SF) is

$$\bar{F}(x) = e^{-\alpha(\beta\theta^2+1)x^2}, \ x>0; \ \alpha, \beta, \theta > 0. \tag{7}$$

In view of (6) and (7), it can be seen that

$$\bar{F}(x) = \frac{1}{2\alpha(\beta\theta^2+1)} x^{-1} f(x). \tag{8}$$

1.4. **Literature Review:** The moment properties of GOS for various distributions and recurrence relations between them have received considerable attention in recent years. Recurrence relations are interesting in their own right and used in minimizing the time and labour of direct computation. Many authors have studied the moment of GOS for several distributions. For example, see Kamps and Cramer [21], Pawlas and Szynal [34], Keseling [23], Cramer and Kamps [16], Saran and Pandey [35], Ahmad and Fawzy [1], Athar and Islam [7], Al-Hussaini et al. [4], Anwar et al. [5], Khan et al. [29], Khwaja et al. [31], Khan and Zia [30], Athar and Nayabuddin [11], Khan and Khan [28], Nayabuddin and Athar [33], Singh et al. [36], Zarrin et al. [38], Athar et al. [9], [10], [12], [14] etc.

The characterization of probability distribution has significant applications in statistical studies. There are several approaches to characterize a probability distribution. In recent years there has been a great interest in the characterization of probability distributions through recurrence relations, conditional expectations and truncation moment. For more detail survey on characterization, one may refer to Galambos and Kotz [18], Kotz and Shanbhag [32], Su and Huang [37], Khan and Abouammoh [24], Khan and Alzaid [25], Khan and Athar [26], Khan et al. [27], Ahsanullah [3], Athar and Noor [13], Athar and Akhter [6], Ahsanullah et al. [2], Athar and Aty [8] and references therein.

The paper is organized as follows. In the section 2, exact expression for single moment of GOS from MWRD and recurrence relation between them are obtained. The particular cases of the main result are also shown here while in section 3, the results based on product moment of GOS from MWRD are given. The section 4 is related to the characterization of MWRD through recurrence relations and conditional expectation. A simulation study is presented in Section 5 and whole study is concluded in Section 6.

2. Single Moments

In this section, we find the exact expression as well as the relation between single moments of GOS from MWRD. For the sake of convenience during the calculation and discussion, we shall consider

$$E[X^p(r,n,\tilde{m},k)] = \mu^{(p)}_{r,n,\tilde{m},k}.$$

Theorem 2.1. *Let Case I be satisfied. For MWRD as given in (6) and for $n \in N, \tilde{m} \in \mathbb{R}, k > 0,\ 1 \le r \le n,\ p = 1, 2, \ldots$*

$$\mu^{(p)}_{r,n,\tilde{m},k} = \frac{C_{r-1}\Gamma\left(\frac{p}{2}+1\right)}{\alpha^{p/2}(\beta\theta^2+1)^{p/2}} \sum_{i=1}^{r} a_i(r)\,\frac{1}{\gamma_i^{\frac{p}{2}+1}}. \tag{9}$$

Proof. In view of (2), we have

$$\mu^{(p)}_{r,n,\tilde{m},k} = C_{r-1}\int_0^\infty x^p \sum_{i=1}^{r} a_i(r)\,\left[\bar{F}(x)\right]^{\gamma_i-1} f(x)\,dx.$$

$$= C_{r-1}\int_0^\infty x^p \sum_{i=1}^{r} a_i(r)\,2\alpha(\beta\theta^2+1)x\,e^{-\alpha\gamma_i(\beta\theta^2+1)x^2}\,dx. \tag{10}$$

Let $\alpha\gamma_i(\beta\theta^2+1)x^2 = t$ in (10), this implies $2\alpha\gamma_i(\beta\theta^2+1)x dx = dt$. Thus, we have

$$\begin{aligned}\mu^{(p)}_{r,n,\tilde{m},k} &= C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^\infty \frac{t^{\frac{p}{2}}}{[\alpha\gamma_i(\beta\theta^2+1)]^{\frac{p}{2}}}\ e^{-t}\,\frac{1}{\gamma_i}\,dt.\\ &= C_{r-1}\sum_{i=1}^{r} a_i(r)\,\frac{1}{\gamma_i\,[\alpha\gamma_i(\beta\theta^2+1)]^{\frac{p}{2}}}\int_0^\infty t^{\frac{p}{2}}\,e^{-t}\,dt.\end{aligned}$$

$$= \quad C_{r-1}\sum_{i=1}^{r} a_i(r)\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{\gamma_i\,[\alpha\,\gamma_i(\beta\theta^2+1)]^{\frac{p}{2}}}.$$

Hence, Theorem 2.1 is proved.

Corollary 2.1. *Let case II is satisfied. For the condition as stated in Theorem 2.1*

$$\mu^{(p)}_{r,n,m,k} = \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\frac{\Gamma\left(\frac{p}{2}+1\right)}{\alpha^{p/2}(\beta\theta^2+1)^{p/2}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{\gamma_{r-i}^{\frac{p}{2}+1}}. \tag{11}$$

Proof. For $\gamma_i \neq \gamma_j$ but at $m_i = m_j = m$, $i, j = 1, 2, \ldots, n-1$, we have

$$a_i(r) = \frac{1}{(m+1)^{r-1}}\,(-1)^{r-i}\,\frac{1}{(i-1)!(r-i)!} \qquad \text{(see Khan et al. [27])}$$

Therefore, in view of (9), we get

$$\mu^{(p)}_{r,n,m,k} = \frac{C_{r-1}}{(m+1)^{r-1}}\frac{\Gamma\left(\frac{p}{2}+1\right)}{\alpha^{p/2}(\beta\theta^2+1)^{p/2}}\sum_{i=1}^{r}(-1)^{r-i}\,\frac{1}{(i-1)!(r-i)!}\,\frac{1}{\gamma_i^{\frac{p}{2}+1}}.$$

Hence, the expression (11) holds. □

Remark 2.1. *By putting $m_i = 0$; $i = 1, 2, \ldots, n-1$ and $k = 1$ in (11), we obtain the expression for single moment of order statistics as*

$$\mu^{(p)}_{r:n} = C_{r:n}\,\frac{\Gamma\left(\frac{p}{2}+1\right)}{[\alpha\,(\beta\theta^2+1)]^{\frac{p}{2}}}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{1}{(n-r+i+1)^{\frac{p}{2}+1}},$$

where $C_{r:n} = \frac{\Gamma(n+1)}{\Gamma r\,\Gamma(n-r+1)}$ and $\mu^{p}_{r:n} = E(X^{(p)}_{r:n})$, which represents the p^{th} moment of r^{th} order statistics.

Remark 2.2. *Let $m_i \to -1$; $i = 1, 2, \ldots, n-1$ in (11), then we get the expression for single moment of k^{th} upper record values as*

$$\mu^{(p)}_{U_n^{(k)}} = \frac{\Gamma\left(\frac{p}{2}+n\right)}{\Gamma n\,k^{\frac{p}{2}}}\,\frac{1}{[\alpha\,(\beta\theta^2+1)]^{\frac{p}{2}}},$$

where $\mu^{(p)}_{U_n^{(k)}} = E(X^{p}_{U_n^{(k)}})$ denotes the p^{th} moment of sequence of k^{th} upper record values.

Remark 2.3. *If $X^{\tilde{R}}_{1:m:n} < \ldots < X^{\tilde{R}}_{r:m:n} < \ldots < X^{\tilde{R}}_{m:m:n}$ be the m ordered progressive type II censored data, for $1 \leq m \leq n$. Then in view of (9), we get the expression for single moment of progressive type II right censored order statistics.*

$$E(X^{\tilde{R}}_{r:m:n})^p = \mu^{\tilde{R}(p)}_{r:m:n} = \frac{C_{r-1}\Gamma\left(\frac{p}{2}+1\right)}{\alpha^{p/2}(\beta\theta^2+1)^{p/2}}\sum_{i=1}^{r}a_i(r)\,\frac{1}{\gamma_i^{\frac{p}{2}+1}},$$

where $\gamma_i = \sum_{j=i}^{m}(R_j+1)$, $\tilde{R} = (R_1, R_2, \ldots, R_m)$ and $r = 1, 2, \ldots, m$.

Theorem 2.2. *Under the similar conditions as stated in Theorem 2.1, the recurrence relation for moment of GOS from MWRD is given as*

$$\mu^{(p)}_{r,n,\tilde{m},k} - \mu^{(p)}_{r-1,n,\tilde{m},k} = \frac{p}{2\,\alpha\,(\beta\theta^2+1)\gamma_r}\,\mu^{(p-2)}_{r,n,\tilde{m},k}. \tag{12}$$

Proof. From Athar and Islam [7], we have

$$E\big[\xi\{X(r,n,\tilde{m},k)\}\big] - E\big[\xi\{X(r-1,n,\tilde{m},k)\}\big] = C_{r-2}\int_{-\infty}^{\infty}\xi'(x)\sum_{i=1}^{r}a_i(r)\,\big[\bar{F}(x)\big]^{\gamma_i}\,dx. \tag{13}$$

By putting $\xi(x) = x^p$ in (13), we get

$$\mu^{(p)}_{r,n,\tilde{m},k} - \mu^{(p)}_{r-1,n,\tilde{m},k} = p\,C_{r-2}\int_{-\infty}^{\infty}x^{p-1}\sum_{i=1}^{r}a_i(r)\,\big[\bar{F}(x)\big]^{\gamma_i}\,dx. \tag{14}$$

Now, using (8) in (14), we get the required result. □

Corollary 2.2. *For Case II and under the conditions as stated in Theorem 2.1, the recurrence relations for the single moment of GOS is given as*

$$\mu^{(p)}_{r,n,m,k} - \mu^{(p)}_{r-1,n,m,k} = \frac{p}{2\,\alpha\,(\beta\theta^2+1)\gamma_r}\,\mu^{(p-2)}_{r,n,m,k}. \tag{15}$$

Proof. It may be noted that for $\gamma_i \neq \gamma_j$ but at $m_i = m_j = m$, $i, j = 1, 2, \ldots, n-1$, the PDF of $X(r,n,\tilde{m},k)$ given in (2) reduces to the PDF of $X(r,n,m,k)$ given in (4). Thus, Corrlary 2.2 is the particular case of Theorem 2.1 and obtained by replacing $\tilde{m}$ with m in (12) □

Remark 2.4. *By putting $m_i = 0$; $i = 1, 2, \ldots, n-1$ and $k = 1$, we obtain the recurrence relation for single moments of order statistics as*

$$\mu^{(p)}_{r:n} - \mu^{(p)}_{r-1:n} = \frac{p}{2(n-r+1)\,\alpha(\beta\theta^2+1)}\,\mu^{(p-2)}_{r:n}.$$

Remark 2.5. *Let $m_i = m \to -1$; $i = 1, 2, \ldots, n-1$, then we get the recurrence relation for single moments of k^{th} upper record values as*

$$\mu^{(p)}_{U_n^{(k)}} - \mu^{(p)}_{U_{n-1}^{(k)}} = \frac{p}{2k\,\alpha(\beta\theta^2+1)}\,\mu^{(p-2)}_{U_n^{(k)}}.$$

Remark 2.6. *If $X^{\tilde{R}}_{1:m:n} < \ldots < X^{\tilde{R}}_{r:m:n} < \ldots < X^{\tilde{R}}_{m:m:n}$ be the m ordered progressive type II censored data, then for $1 \le m \le n$ and $r = 1, 2, \ldots, m$*

$$\mu^{\tilde{R}(p)}_{r:m:n} - \mu^{\tilde{R}(p)}_{r-1:m:n} = \frac{p}{2\,\alpha\,(\beta\theta^2+1)\gamma_r}\,\mu^{\tilde{R}(p-2)}_{r:m:n},$$

where $\gamma_r = m - r + 1 + \sum_{l=r}^{m} R_l$.

3. Product Moments

The product moments of GOS for MWRD is studied in this section. For the sake of convenience throughout the calculations and discussion, we shall assume

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)] = \mu^{(p,q)}_{r,s,n,\tilde{m},k}$$

Theorem 3.1. *Let Case I be satisfied. For MWRD as given in (6) and $n \in N, \tilde{m} \in \mathbb{R}$, $k > 0$, $1 \leq r < s \leq n$, $p, q = 1, 2, ...,$*

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} = C_{s-1} \frac{\Gamma\left(\frac{p+q}{2}+2\right)}{\left(\frac{p}{2}+1\right)[\alpha(\beta\theta^2+1)]^{\frac{p+q}{2}}\gamma_i^{\frac{p+q}{2}+2}} \times \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s)\, {}_2F_1\left(1, \frac{p+q}{2}+2, \frac{p}{2}+2; \frac{\gamma_i-\gamma_j}{\gamma_i}\right), \tag{16}$$

where

$${}_2F_1(a,b;c;z) = \sum_{u=0}^{\infty} \frac{(a)_u\,(b)_u}{(c)_u} \frac{z^u}{u!}, \quad |z|<1,$$

and $(a)_n$ is the Pochammer symbol and defined by $(a)_n = \frac{\Gamma(a+n)}{\Gamma a}$.

Proof. We know that

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} = C_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^\infty \int_x^\infty x^p y^q \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j} [\bar{F}(x)]^{\gamma_i} \times \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}\, dy\, dx.$$

$$= C_{s-1} \sum_{i=1}^{r} a_i(r) \sum_{j=r+1}^{s} a_j^{(r)}(s) \int_0^\infty x^p [\bar{F}(x)]^{\gamma_i-\gamma_j-1} f(x) I(x) dx, \tag{17}$$

where

$$I(x) = \int_x^\infty y^q [\bar{F}(y)]^{\gamma_j-1} f(y)\, dy = \int_x^\infty y^q e^{-\alpha\gamma_j(\beta\theta^2+1)y^2}\, 2\alpha(\beta\theta^2+1) y\, dy.$$

Let $\alpha\gamma_j(\beta\theta^2+1)y^2 = t \Rightarrow 2\alpha\gamma_j(\beta\theta^2+1)y dy = dt$ and assume that $u(x) = \alpha\gamma_j(\beta\theta^2+1)x^2$. Thus, we have

$$I(x) = \frac{1}{\gamma_j}\int_{u(x)}^\infty \frac{t^{\frac{q}{2}}}{[\alpha\gamma_j(\beta\theta^2+1)]^{\frac{q}{2}}} e^{-t}\, dt = \frac{1}{\gamma_j[\alpha\gamma_j(\beta\theta^2+1)]^{\frac{q}{2}}} \int_{u(x)}^\infty t^{\frac{q}{2}} e^{-t}\, dt = \frac{1}{\gamma_j^{\frac{q}{2}+1}\alpha^{\frac{q}{2}}(\beta\theta^2+1)^{\frac{q}{2}}} \Gamma\left(\frac{q}{2}+1, u(x)\right). \tag{18}$$

Now, using (18) in (17), we get

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} = \frac{C_{s-1}}{\gamma_j^{\frac{q}{2}+1}\alpha^{\frac{q}{2}}(\beta\theta^2+1)^{\frac{q}{2}}}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a_j^{(r)}(s) \times \int_0^\infty x^p\,[\bar{F}(x)]^{\gamma_i-\gamma_j-1}f(x)\,\Gamma\Big(\frac{q}{2}+1,\alpha\gamma_j(\beta\theta^2+1)x^2\Big)\,dx. \tag{19}$$

Again using (6) and (7) in (19), we get

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} = \frac{2\alpha(\beta\theta^2+1)\,C_{s-1}}{\gamma_j^{\frac{q}{2}+1}\alpha^{\frac{q}{2}}(\beta\theta^2+1)^{\frac{q}{2}}}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a_j^{(r)}(s) \times \int_0^\infty x^{p+1}\,(e^{-\alpha(\beta\theta^2+1)x^2})^{\gamma_i-\gamma_j}\,\Gamma\Big(\frac{q}{2}+1,\alpha\gamma_j(\beta\theta^2+1)x^2\Big)\,dx. \tag{20}$$

Let $\alpha(\beta\theta^2+1)x^2 = z \Rightarrow 2\alpha(\beta\theta^2+1)x\,dx = dz$. Thus, we have

$$\begin{aligned}\mu^{(p,q)}_{r,s,n,\tilde{m},k} &= \frac{C_{s-1}}{\gamma_j^{\frac{q}{2}+1}[\alpha(\beta\theta^2+1)]^{\frac{q}{2}}}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a_j^{(r)}(s)\\ &\quad\times\int_0^\infty\left[\frac{z}{\alpha(\beta\theta^2+1)}\right]^{\frac{p}{2}}e^{-(\gamma_i-\gamma_j)z}\,\Gamma\Big(\frac{q}{2}+1,\gamma_j z\Big)\,dz\\ &= \frac{C_{s-1}}{\gamma_j^{\frac{q}{2}+1}[\alpha(\beta\theta^2+1)]^{\frac{p+q}{2}}}\sum_{i=1}^{r}a_i(r)\sum_{j=r+1}^{s}a_j^{(r)}(s)\\ &\quad\times\int_0^\infty z^{\frac{p}{2}}\,e^{-(\gamma_i-\gamma_j)z}\,\Gamma\Big(\frac{q}{2}+1,\gamma_j z\Big)\,dz.\end{aligned} \tag{21}$$

From Gradshteyn and Ryzhik [19] $p.$ 657, we have

$$\int_0^\infty x^{\mu-1}\,e^{-\beta x}\,\Gamma(v,\alpha x)\,dx = \frac{\alpha^v\,\Gamma\mu+v}{\mu\,(\alpha+\beta)^{\mu+v}}\;{}_2F_1\left(1,\mu+v;\mu+1;\frac{\beta}{\alpha+\beta}\right). \tag{22}$$

Now by using (22) in (21), we obtain (16).

Thus, the proof of Theorem 3.1 is completed. □

Corollary 3.1. *Suppose Case II is satisfied. For the condition as stated under Theorem 3.1, the expression for product moments of GOS for MWRD is given as*

$$\begin{aligned}\mu^{(p,q)}_{r,s,n,m,k} &= \frac{C_{s-1}\,\Gamma\big(\frac{p+q}{2}+2\big)}{(r-1)!\,(s-r-1)!\,(m+1)^{s-2}\,\big(\frac{p}{2}+1\big)\,[\alpha(\beta\theta^2+1)]^{\frac{p+q}{2}}}\\ &\quad\times\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\frac{1}{\gamma_{r-i}^{\frac{p+q}{2}+2}}\\ &\quad\times{}_2F_1\left(1,\frac{p+q}{2}+2;\frac{p}{2}+2;\frac{\gamma_{r-i}-\gamma_{s-j}}{\gamma_{r-i}}\right),\end{aligned} \tag{23}$$

where $\gamma_i = k+(n-i)(m+1) > 0$.

Proof. In view of Khan et al. [27] for $\gamma_i \neq \gamma_j$ but at $m_i = m_j = m$, we have

$$a_i(r) = \frac{1}{(m+1)^{r-1}} (-1)^{r-i} \frac{1}{(i-1)!(r-i)!}$$

$$a_j^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}} (-1)^{s-j} \frac{1}{(j-r-1)!(s-j)!}$$

Therefore, from (16), we have

$$\mu_{r,s,n,m,k}^{(p,q)} = \frac{C_{s-1}\Gamma\left(\frac{p+q}{2}+2\right)}{\left(\frac{p}{2}+1\right)\left[\alpha(\beta\theta^2+1)\right]^{\frac{p+q}{2}}} \left\{ \sum_{i=1}^{r} \frac{1}{(m+1)^{r-1}} (-1)^{r-i} \frac{1}{(i-1)!(r-i)!} \frac{1}{\gamma_{r-i}^{\frac{p+q}{2}+2}} \right\}$$
$$\times \left\{ \sum_{j=r+1}^{s} \frac{1}{(m+1)^{s-r-1}} (-1)^{s-j} \frac{1}{(j-r-1)!(s-j)!} \right\}$$
$$\times {}_2F_1\left(1, \frac{p+q}{2}+2; \frac{p}{2}+2; \frac{\gamma_i - \gamma_j}{\gamma_i}\right),$$

which can be expressed as (23). Hence, the corollary 3.1 is proved. □

Remark 3.1. *If $m_i = 0,\ i = 1, 2, ..., n-1$ and $k = 1$, then we get the expression for product moment of order statistics from MWRD*

$$\mu_{r,s:n}^{(p,q)} = C_{r,s:n} \frac{\Gamma\left(\frac{p+q}{2}+2\right)}{\left(\frac{p}{2}+1\right)\left[\alpha(\beta\theta^2+1)\right]^{\frac{p+q}{2}}}$$
$$\times \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \frac{1}{(n-r+i+1)^{\frac{p+q}{2}+2}}$$
$$\times {}_2F_1\left(1, \frac{p+q}{2}+2; \frac{p}{2}+2; \frac{s-r+i-j}{n-r+i+1}\right),$$

where $\mu_{r,s:n}^{(p,q)} = E\left(X_{r:n}^p . X_{s:n}^q\right)$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Remark 3.2. *Let $m_i \to -1;\ i = 1, 2, \ldots, n-1$, then we get the expression for product moment of k^{th} upper record values as*

$$\mu_{U_{r,s}^{(k)}}^{(p,q)} = \frac{1}{\Gamma(r)\Gamma(s-r)} \frac{1}{[k\alpha(\beta\theta^2+1)]^{(p+q)/2}} \sum_{t=0}^{s-r-1} (-)^{s-r-t-1} \binom{s-r-1}{t}$$
$$\times \frac{\Gamma(s+(p+q)/2)}{(s+(p/2)-t-1)},$$

where $\mu_{U_{r,s}^{(k)}}^{(p,q)} = E\left(X_{U_r^{(k)}}^p . X_{U_s^{(k)}}^q\right)$.

Remark 3.3. *If $\gamma_i = \sum_{j=i}^{m}(R_i+1)$ and $\tilde{R} = (R_1, R_2, ..., R_m)$ in (16), we get the expression for product moment of progressive type II right censored order statistics.*

Theorem 3.2. *Under the conditions as stated in Theorem 3.1, the recurrence relations of product moments of GOS for MWRD is given as*

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,q)}_{r,s-1,n,\tilde{m},k} = \frac{q}{2\gamma_s\,\alpha\,(\beta\theta^2+1)}\,\mu^{(p,q-2)}_{r,s,n,\tilde{m},k}. \tag{24}$$

Proof. We have by Athar and Islam [7],

$$E[\xi\,\{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)\}] - E[\xi\,\{X(r,n,\tilde{m},k),X(s-1,n,\tilde{m},k)\}]$$
$$= C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}\frac{\partial}{\partial y}\xi(x,y)\sum_{j=r+1}^{s} a_j^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\frac{f(x)}{\bar{F}(x)}dy\;dx. \tag{25}$$

Let $\xi(x,y)=\xi_1(x)\xi_2(y)=x^p y^q$ in (25). Thus, we have

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} - \mu^{(p,q)}_{r,s-1,n,\tilde{m},k} = q\,C_{s-2}\int_{0}^{\infty}\int_{x}^{\infty}x^p\,y^{q-1}\left[\sum_{j=r+1}^{s}a_j^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j}\right]$$
$$\times\left[\sum_{i=1}^{r}a_i\,(r)\,[\bar{F}(x)]^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\,dy\,dx. \tag{26}$$

Now, the relations (24) can be proved by using (8) in (26). □

Corollary 3.2. *Under the conditions as stated in Theorem 3.1 and for Case II, the recurrence relations of the product moments is*

$$\mu^{(p,q)}_{r,s,n,m,k} - \mu^{(p,q)}_{r,s-1,n,m,k} = \frac{q}{2\gamma_s\,\alpha\,(\beta\theta^2+1)}\,\mu^{(p,q-2)}_{r,s,n,\tilde{m},k}. \tag{27}$$

Remark 3.4. *If $m_i=0$; $i=1,2,\ldots,n-1$ and $k=1$, then the relation for product moment of order statistics for MWRD is given by*

$$\mu^{(p,q)}_{r,s:n} - \mu^{(p,q)}_{r,s-1:n} = \frac{q}{2\,\alpha\,(\beta\theta^2+1)(n-s+1)}\,\mu^{(p,q-2)}_{r,s:n}$$

where, $\mu^{(p,q)}_{r,s:n}(x,y)=E[X^p_{r:n}.X^q_{s:n}]$.

Remark 3.5. *Let $m_i\to -1$; $i=1,2,\ldots,n-1$, then relation for product moment of k^{th} upper record values is given as*

$$\mu^{(p,q)}_{U^{(k)}_{(n,m)}} - \mu^{(p,q)}_{U^{(k)}_{(n,m-1)}} = \frac{q}{2\,k\alpha\,(\beta\theta^2+1)}\,\mu^{(p,q-2)}_{U^{(k)}_{(n,m)}},$$

where $E\big[X^p_{U^{(k)}_{(n)}}X^q_{U^{(k)}_{(m)}}\big]=\mu^{(p,q)}_{U^{(k)}_{(n,m)}}$.

Remark 3.6. *For $1\le m\le n$ and $\gamma_i=\sum_{j=i}^{m}(R_i+1)$, $\tilde{R}=(R_1,R_2,\ldots,R_m)$ in (24), we get the relation for product moment of progressive type II right censored order statistics.*

$$\mu^{\tilde{R}(p)}_{r,s:m:n} - \mu^{\tilde{R}(p)}_{r,s-1:m:n} = \frac{p}{2\,\alpha\,(\beta\theta^2+1)\gamma_r}\,\mu^{\tilde{R}(p-2)}_{r,s:m:n}.$$

4. Characterization

This section describes the characterization of underlying distribution using recurrence relations between moments of GOS and through conditional expectation.

Theorem 4.1. *Fix a positive integer k and let p a non-negative integer. A necessary and sufficient condition for a RV X to be distributed with PDF as stated in (6) is that*

$$\mu^{(p)}_{r,n,\tilde{m},k} = \mu^{(p)}_{r-1,n,\tilde{m},k} + \frac{p}{2\alpha\,(\beta\theta^2+1)\,\gamma_r}\mu^{(p-2)}_{r-1,n,\tilde{m},k}. \tag{28}$$

Proof. The necessary part follows from (12). For the sufficiency part, suppose that the relation in (28) is satisfied.

Now, from Athar and Islam [7] for $\xi(x) = x^p$, we have

$$p\,C_{r-2}\int_{-\infty}^{\infty} x^{p-1}\sum_{i=1}^{r} a_i(r)\,\left[\bar{F}(x)\right]^{\gamma_i} dx$$
$$= \frac{p\,C_{r-1}}{2\alpha\,(\beta\theta^2+1)\,\gamma_r}\int_{-\infty}^{\infty} x^{p-1}(x^{-1})\sum_{i=1}^{r} a_i(r)\left[\bar{F}(x)\right]^{\gamma_{i-1}} f(x)\;dx.$$

This implies

$$\frac{pC_{r-1}}{2\alpha\,(\beta\theta^2+1)\,\gamma_r}\int_{-\infty}^{\infty} x^{p-1}\sum_{i=1}^{r} a_i(r)\left[\bar{F}(x)\right]^{\gamma_{i-1}}\left\{2\alpha\left(\beta\theta^2+1\right)\bar{F}(x) - x^{-1}f(x)\right\}dx. \tag{29}$$

Applying the extension of Müntz-Szász theorem, which is given in Hwang and Lin [20] to (29), we obtain

$$\frac{\bar{F}(x)}{f(x)} = \frac{x^{-1}}{2\alpha\,(\beta\theta^2+1)},$$

which implies

$$f(x) = 2\alpha(\beta\,\theta^2+1)x\,e^{-\alpha(\beta\,\theta^2+1)x^2},\quad x>0;\quad \alpha,\,\beta,\,\theta>0.$$

Hence, Theorem 4.1 is proved. □

Theorem 4.2. *Fix a positive integer k and let p, q are non-negative integer. A necessary and sufficient condition for a RV x to be distributed with PDF as stated in (6) is that*

$$\mu^{(p,q)}_{r,s,n,\tilde{m},k} = \mu^{(p,q)}_{r,s-1,n,\tilde{m},k} + \frac{q}{2\alpha\,(\beta\theta^2+1)\,\gamma_s}\mu^{(p,q-2)}_{r,s,n,\tilde{m},k}. \tag{30}$$

Proof. Necessary part follows from (24).

Now, suppose that the relation in (30) is satisfied. From Athar and Islam [7] for $\xi(x,y) = x^p y^q$, we have

$$q\,C_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^{p}y^{q-1}\sum_{j=r+1}^{s}a_{i}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_{j}}$$
$$\times\sum_{i=1}^{r}a_{i}(r)\left[\bar{F}(x)\right]^{\gamma_{i}}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx$$
$$=\frac{q\,C_{s-1}}{2\alpha\left(\beta\theta^{2}+1\right)\gamma_{s}}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^{p}y^{q-2}\sum_{j=r+1}^{s}a_{i}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_{j}}$$
$$\times\sum_{i=1}^{r}a_{i}(r)\left[\bar{F}(x)\right]^{\gamma_{i}}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx.$$

This implies

$$\frac{q\,C_{s-1}}{2\alpha\left(\beta\theta^{2}+1\right)\gamma_{s}}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^{p}y^{q-1}\sum_{j=r+1}^{s}a_{i}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_{j}}\sum_{i=1}^{r}a_{i}(r)\left[\bar{F}(x)\right]^{\gamma_{i}}\frac{f(x)}{\bar{F}(x)}$$
$$\times\left\{2\alpha\left(\beta\theta^{2}+1\right)-y^{-1}\frac{f(y)}{\bar{F}(y)}\right\}dydx=0. \tag{31}$$

Applying the extension of Müntz-Szász theorem [Hwang and Lin [20]] to (31), we get

$$\frac{\bar{F}(y)}{f(y)}=\frac{y^{-1}}{2\alpha\left(\beta\theta^{2}+1\right)},$$

which gives the *pdf* $f(y)$ as given in (6).

Hence, Theorem 4.2 is proved. □

Theorem 4.3. *Let X be a RV with CDF $F(x)$ and PDF $f(x)$ over the support $(0,\infty)$, and $w(x)$ be a monotonic, continuous and differentiable function of x, then for two consecutive values r and $r+1$, $2\le r+1\le s\le n$,*

$$E\left[w\left\{X\left(s,n,m,k\right)\right\}|X\left(t,n,m,k\right)=x\right]=h_{s|t}\left(x\right)$$
$$=w\left(x\right)+\sum_{i=t}^{s-1}\frac{1}{\gamma_{i+1}},\quad i=r,r+1,\ \gamma_{i+1}\neq0 \tag{32}$$

if and only if

$$\bar{F}(x)=e^{-w(x)},\ w(x)=\alpha(\beta\theta^{2}+1)x^{2},\ x>0;\ \alpha,\theta,\beta>0. \tag{33}$$

Proof. For $s\ge r+1$

$$E\left[w\left\{X\left(s,n,m,k\right)\right\}|X\left(t,n,m,k\right)=x\right]=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}$$

$$\times\int_{x}^{\infty}w(y)\left[1-\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1}\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_{s}-1}\frac{f(y)}{\bar{F}(y)}dy. \tag{34}$$

Let

$$\psi=\frac{\bar{F}(y)}{\bar{F}(x)}=\frac{e^{-w(x)}}{e^{-w(y)}}, \text{implies } w(y)=w(x)-\log\psi.$$

Then, the right hand side of (34) reduces to

$$=\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_0^1 (w(x)-\log\psi)\left[1-\psi^{m+1}\right]^{s-r-1}[\psi]^{\gamma_s-1}\,d\psi.$$

Let $\psi^{m+1}=u$, then

$$E\left[w\left\{X(s,n,m,k)\right\}|X(t,n,m,k)\right]=w(x)-\frac{\prod_{i=r+1}^{s-1}\gamma_i}{(s-r-1)!(m+1)^{s-r-1}}$$

$$\times\int_0^1 u^{\frac{\gamma_s}{m+1}-1}(1-u)^{s-r-1}\log u\,du.$$

$$=w(x)-\frac{\prod_{i=r+1}^{s-1}\gamma_i}{(s-r-1)!(m+1)^{s-r-1}}B\left(\frac{\gamma_s}{m+1},s-r\right)\left[\varphi\left(\frac{\gamma_s}{m+1}\right)-\varphi\left(\frac{\gamma_r}{m+1}\right)\right],$$

where

$$\varphi(x)\frac{d}{dx}\ln\Gamma(x),\quad \varphi(x-n)-\varphi(x)=\sum_{i=1}^{n}\frac{1}{x-i}. \tag{35}$$

[c.f. Gradshteyn and Ryzhik [19], pp-540, 905] and $B(.,.)$ is the complete beta function.

Therefore,

$$E\left[w\left\{X(s,n,m,k)\right\}|X(t,n,m,k)=x\right]=h_{s|t}(x)=w(x)+\sum_{i=t}^{s-1}\frac{1}{\gamma_{i+1}}$$

To prove sufficiency part, let

$$E\left[w\left\{X(s,n,m,k)\right\}|X(r,n,m,k)=x\right]=h_{s|r}(x)$$

Therefore,

$$\frac{C_{s-1}}{C_{r-1}(s-r-1)!(m+1)^{s-r-1}}\int_x^b w(y)\left[\left(\bar{F}(x)\right)^{m+1}-\left(\bar{F}(y)\right)^{m+1}\right]^{s-r-1}$$

$$\times\left[\bar{F}(y)\right]^{\gamma_s-1}f(y)dy=h_{s|r}(x)\left[\bar{F}(x)\right]^{\gamma_{r+1}}.$$

Differentiating both sides with respect to x and adjusting the terms, we get

$$\frac{f(x)}{\bar{F}(x)}=-\frac{1}{\gamma_{r+1}}\frac{h'_{s|r}(x)}{\left[h_{s|r+1}(x)-h_{s|r}(x)\right]}\qquad \text{[Khan et al. [27]]}$$

$$= \frac{1}{\gamma_{r+1}} \frac{w'(x)}{\left[w(x) + \sum_{i=r+1}^{s-1} \frac{1}{\gamma_{i+1}} - w(x) - \sum_{i=r}^{s-1} \frac{1}{\gamma_{i+1}} \right]} = w'(x).$$

Implying that $\bar{F}(x) = e^{-w(x)} = e^{-\alpha(\beta\theta^2+1)x^2}$.

Thus, the proof of theorem is completed. □

5. Simulation Study

In this section, a simulation study is conducted to compute first four moments of order statistics and record values for arbitrary chosen values of the parameters of MWRD. Moreover, the first four moments of progressive type II censored order statistics for different sample sizes, different effective sample sizes and for different censoring schemes are computed. These first four moments can be used to study the statistical properties like mean, variance, skewness and kurtosis of order statistics, record values and progressive type II right censored order statistics from MWRD.

First four moments of order statistics

Table 5.1

		$\alpha = 0.3$							
		$\beta = 0.3,\ \theta = 1.0$				$\beta = 0.5\ \theta = 2.0$			
n	r	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$\mu_{r:n}^{(3)}$	$\mu_{r:n}^{(4)}$	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$\mu_{r:n}^{(3)}$	$\mu_{r:n}^{(4)}$
1	1	1.419099	2.564103	5.458072	13.14924	0.934165	1.111111	1.556942	2.499136
2	1	1.003454	1.282051	1.929720	3.287311	0.660555	0.555556	0.550462	0.617284
	2	1.834743	3.846154	8.986425	23.01118	1.207776	1.666667	2.563422	4.320988
3	1	0.819317	0.854701	1.050407	1.461027	0.539341	0.370370	0.299634	0.274348
	2	1.371729	2.136752	3.688347	6.939879	0.902983	0.925926	1.052119	1.303155
	3	2.06625	4.700855	11.63546	31.04683	1.360172	2.037037	3.319073	5.829904
4	1	0.709549	0.641026	0.682259	0.821828	0.467083	0.277778	0.194618	0.154321
	2	1.14862	1.495726	2.154849	3.378625	0.756114	0.648148	0.614681	0.634431
	3	1.594838	2.777778	5.221845	10.50113	1.049851	1.203704	1.489557	1.971879
	4	2.223388	5.34188	13.77334	37.89539	1.463613	2.314815	3.928912	7.115912
5	1	0.634640	0.512821	0.488185	0.525970	0.417771	0.222222	0.139257	0.098765
	2	1.009186	1.153846	1.458556	2.00526	0.664328	0.500000	0.416060	0.376543
	3	1.357771	2.008547	3.199288	5.438673	0.893795	0.870370	0.912613	1.021262
	4	1.752882	3.290598	6.570216	13.8761	1.153888	1.425926	1.874186	2.605624
	5	2.341014	5.854701	15.57412	43.90021	1.541044	2.537037	4.442593	8.243484

Table 5.2

		$\alpha = 0.5$							
		$\beta = 1.0,\ \theta = 2.0$				$\beta = 2.0\ \theta = 5.0$			
n	r	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$\mu_{r:n}^{(3)}$	$\mu_{r:n}^{(4)}$	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$\mu_{r:n}^{(3)}$	$\mu_{r:n}^{(4)}$
1	1	0.560499	0.400000	0.336300	0.320000	0.175499	0.039216	0.010324	0.003076
2	1	0.396333	0.200000	0.118900	0.080000	0.124097	0.019608	0.003650	0.000769
	2	0.724666	0.600000	0.553699	0.560000	0.226902	0.058824	0.016997	0.005383
3	1	0.323604	0.133333	0.064721	0.035555	0.101324	0.013072	0.001987	0.000342
	2	0.541790	0.333333	0.227258	0.168889	0.169641	0.032680	0.006976	0.001623
	3	0.816104	0.733333	0.716920	0.755556	0.255532	0.071895	0.022007	0.007262
4	1	0.280250	0.100000	0.042037	0.020000	0.087750	0.009804	0.001290	0.000192
	2	0.453669	0.233333	0.132771	0.082222	0.142049	0.022876	0.004076	0.000790
	3	0.629911	0.433333	0.321744	0.255556	0.197233	0.042484	0.009877	0.002456
	4	0.878168	0.833333	0.848645	0.922222	0.274965	0.081699	0.026051	0.008864
5	1	0.250663	0.080000	0.030080	0.012800	0.078486	0.007843	0.000923	0.000123
	2	0.398597	0.180000	0.089869	0.048800	0.124805	0.017647	0.002759	0.000469
	3	0.536277	0.313333	0.197124	0.132356	0.167915	0.030719	0.006051	0.001272
	4	0.692333	0.513333	0.404824	0.337689	0.216778	0.050327	0.012427	0.003246
	5	0.924627	0.913333	0.959600	1.068356	0.289512	0.089542	0.029457	0.010269

Table 5.3

		$\alpha = 1.0$							
		$\beta = 0.5,\ \theta = 1.0$				$\beta = 5.0,\ \theta = 10.0$			
n	r	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$\mu_{r:n}^{(3)}$	$\mu_{r:n}^{(4)}$	$\mu_{r:n}^{(1)}$	$\mu_{r:n}^{(2)}$	$\mu_{r:n}^{(3)}$	$\mu_{r:n}^{(4)}$
1	1	0.723601	0.666667	0.723601	0.888889	0.039594	0.001996	0.000119	0.000008
2	1	0.511663	0.333333	0.255832	0.222222	0.027997	0.000998	0.000042	0.000002
	2	0.935539	1.000000	1.191371	1.555556	0.051190	0.002994	0.000195	0.000014
3	1	0.417771	0.222222	0.139257	0.098765	0.022859	0.000665	0.000023	0.000001
	2	0.699447	0.555556	0.488981	0.469136	0.038272	0.001663	0.000080	0.000004
	3	1.053585	1.222222	1.542566	2.098765	0.057650	0.003659	0.000253	0.000019
4	1	0.361801	0.166667	0.090450	0.055556	0.019797	0.000499	0.000015	0.000000
	2	0.585684	0.388889	0.285678	0.228395	0.032047	0.001164	0.000047	0.000002
	3	0.813211	0.722222	0.692284	0.709877	0.044497	0.002162	0.000113	0.000006
	4	1.133710	1.388889	1.825993	2.561728	0.062034	0.004158	0.000299	0.000023
5	1	0.323604	0.133333	0.064721	0.035556	0.017707	0.000399	0.000011	0.000000
	2	0.514586	0.300000	0.193367	0.135556	0.028157	0.000898	0.000032	0.000001
	3	0.692330	0.522222	0.424144	0.367654	0.037883	0.001564	0.000069	0.000003
	4	0.893798	0.855556	0.871043	0.938025	0.048906	0.002562	0.000143	0.000008
	5	1.193688	1.522222	2.064731	2.967654	0.065316	0.004558	0.000338	0.000027

Table 5.4

n	$k=1,\ \alpha=0.3$							
	$\beta=0.3,\ \theta=0.8$				$\beta=0.5,\ \theta=2.0$			
	$\mu^{(1)}_{U_n^{(1)}}$	$\mu^{(2)}_{U_n^{(1)}}$	$\mu^{(3)}_{U_n^{(1)}}$	$\mu^{(4)}_{U_n^{(1)}}$	$\mu^{(1)}_{U_n^{(1)}}$	$\mu^{(2)}_{U_n^{(1)}}$	$\mu^{(3)}_{U_n^{(1)}}$	$\mu^{(4)}_{U_n^{(1)}}$
1	1.481993	2.796421	6.216414	15.63994	0.934165	1.111111	1.556942	2.469136
2	2.222990	5.592841	15.54104	46.91981	1.401248	2.222222	3.892355	7.407407
3	2.778737	8.389262	27.19681	93.83962	1.751560	3.333333	6.811621	14.81481
4	3.241860	11.18568	40.79522	156.3994	2.043486	4.444444	10.21743	24.69136
5	3.647092	13.9821	56.09342	234.599	2.298922	5.555556	14.04897	37.03704
6	4.011802	16.77852	72.92145	328.4387	2.528814	6.666667	18.26366	51.85185
7	4.346118	19.57494	91.15181	437.9182	2.739549	7.777778	22.82957	69.13580
8	4.656556	22.37137	110.6843	563.0377	2.935231	8.888889	27.72163	88.88889
9	4.947590	25.16778	131.4377	703.7971	3.118683	10.00000	32.91943	111.1111
10	5.222456	27.96421	153.3439	860.1965	3.291943	11.11111	38.40600	135.8025

Table 5.5

n	$k=2,\ \alpha=0.5$							
	$\beta=2.0,\ \theta=5.0$				$\beta=3.0,\ \theta=10.0$			
	$\mu^{(1)}_{U_n^{(2)}}$	$\mu^{(2)}_{U_n^{(2)}}$	$\mu^{(3)}_{U_n^{(2)}}$	$\mu^{(4)}_{U_n^{(2)}}$	$\mu^{(1)}_{U_n^{(2)}}$	$\mu^{(2)}_{U_n^{(2)}}$	$\mu^{(3)}_{U_n^{(2)}}$	$\mu^{(4)}_{U_n^{(2)}}$
1	0.124097	0.019608	0.003650	0.000769	0.051081	0.003322	0.000255	0.000022
2	0.186145	0.039216	0.009125	0.002307	0.076622	0.006645	0.000636	0.000066
3	0.232681	0.058824	0.015968	0.004614	0.095777	0.009967	0.001114	0.000132
4	0.271461	0.078431	0.023952	0.007689	0.111740	0.013289	0.001671	0.000221
5	0.305394	0.098039	0.032935	0.011534	0.125708	0.016611	0.002297	0.000331
6	0.335933	0.117647	0.042815	0.016148	0.138279	0.019934	0.002986	0.000464
7	0.363928	0.137255	0.053519	0.021530	0.149802	0.023256	0.003733	0.000618
8	0.389923	0.156863	0.064987	0.027682	0.160502	0.026578	0.004532	0.000795
9	0.414293	0.176471	0.077172	0.034602	0.170533	0.029900	0.005382	0.000993
10	0.437309	0.196078	0.090034	0.042291	0.180007	0.033223	0.006279	0.001214

Table 5.6

n	$k=3,\ \alpha=5.0$							
	$\beta=0.5,\ \theta=3.0$				$\beta=0.8,\ \theta=8.0$			
	$\mu^{(1)}_{U^{(3)}_n}$	$\mu^{(2)}_{U^{(3)}_n}$	$\mu^{(3)}_{U^{(3)}_n}$	$\mu^{(4)}_{U^{(3)}_n}$	$\mu^{(1)}_{U^{(3)}_n}$	$\mu^{(2)}_{U^{(3)}_n}$	$\mu^{(3)}_{U^{(3)}_n}$	$\mu^{(4)}_{U^{(3)}_n}$
1	0.097570	0.012121	0.001774	0.000294	0.031671	0.001277	0.000061	0.000003
2	0.146356	0.024242	0.004435	0.000882	0.047507	0.002554	0.000152	0.000010
3	0.182944	0.036364	0.007761	0.001763	0.059383	0.003831	0.000265	0.000020
4	0.213435	0.048485	0.011642	0.002938	0.069281	0.005109	0.000398	0.000033
5	0.240115	0.060606	0.016008	0.004408	0.077941	0.006386	0.000547	0.000049
6	0.264126	0.072727	0.020810	0.006171	0.085735	0.007663	0.000712	0.000069
7	0.286137	0.084848	0.026012	0.008228	0.092879	0.008940	0.000890	0.000091
8	0.306575	0.096970	0.031587	0.010579	0.099514	0.010217	0.001080	0.000117
9	0.325736	0.109091	0.037509	0.013223	0.105733	0.011494	0.001283	0.000147
10	0.343832	0.121212	0.043760	0.016162	0.111607	0.012771	0.001497	0.000179

Table 5.7

$\alpha = 0.3,\ \beta = 0.5,\ \theta = 5.0$

n	m	$\tilde{R} = R_1, R_2, \ldots, R_m$	r	$\mu_{r:m:n}^{\tilde{R}(1)}$	$\mu_{r:m:n}^{\tilde{R}(2)}$	$\mu_{r:m:n}^{\tilde{R}(3)}$	$\mu_{r:m:n}^{\tilde{R}(4)}$
5	2	$(0, 3)$	1	0.196939	0.049383	0.014588	0.004877
			2	0.313167	0.111111	0.043585	0.018595
10	5	$(0, 0, 1, 1, 3)$	1	0.139257	0.024691	0.005158	0.001219
			2	0.214585	0.052126	0.013988	0.004079
			3	0.276304	0.082990	0.026780	0.009202
			4	0.341187	0.124143	0.047841	0.019420
			5	0.419383	0.185871	0.086673	0.042367
15	6	$(0, 0, 1, 1, 2, 5)$	1	0.113703	0.016461	0.002807	0.000542
			2	0.173565	0.034098	0.007399	0.001745
			3	0.221029	0.053091	0.013696	0.003761
			4	0.266313	0.075538	0.022663	0.007153
			5	0.312745	0.102972	0.035533	0.012803
			6	0.371022	0.144125	0.058436	0.024665
20	8	$(0, 0, 2, 2, 2, 1, 3, 2)$	1	0.098470	0.012346	0.001824	0.000305
			2	0.149631	0.025341	0.004740	0.000963
			3	0.189591	0.039059	0.008641	0.002035
			4	0.228323	0.055519	0.014279	0.003863
			5	0.268837	0.076096	0.022576	0.006994
			6	0.314612	0.103530	0.035523	0.012675
			7	0.365064	0.138804	0.054839	0.022467
			8	0.459175	0.221108	0.111527	0.058864

Table 5.8

$\alpha = 0.5,\ \beta = 0.8,\ \theta = 8.0$

n	m	$\tilde{R} = R_1, R_2, \ldots, R_m$	r	$\mu_{r:m:n}^{\tilde{R}(1)}$	$\mu_{r:m:n}^{\tilde{R}(2)}$	$\mu_{r:m:n}^{\tilde{R}(3)}$	$\mu_{r:m:n}^{\tilde{R}(4)}$
10	3	(1, 2, 4)	1	0.054856	0.003831	0.000315	0.000029
			2	0.087230	0.008621	0.000942	0.000112
			3	0.122083	0.016284	0.002345	0.000361
20	5	(3, 2, 1, 0, 9)	1	0.038789	0.001916	0.000111	0.000007
			2	0.061681	0.004310	0.000333	0.000028
			3	0.081652	0.007258	0.000694	0.000071
			4	0.100336	0.010741	0.001218	0.000146
			5	0.117597	0.014572	0.001894	0.000257
25	7	(2, 0, 3, 1, 2, 3, 7)	1	0.034694	0.001533	0.000080	0.000006
			2	0.053777	0.003274	0.000220	0.000016
			3	0.068492	0.005099	0.000408	0.000035
			4	0.083073	0.007352	0.000689	0.000068
			5	0.097008	0.009907	0.001060	0.000118
			6	0.111964	0.013100	0.001596	0.000202
			7	0.131048	0.017889	0.002538	0.000373
30	8	(4, 1, 1, 2, 1, 1, 0, 12)	1	0.031671	0.001277	0.000061	0.000003
			2	0.049808	0.002810	0.000175	0.000012
			3	0.064158	0.004476	0.000335	0.000027
			4	0.076905	0.006300	0.000546	0.000005
			5	0.089491	0.008429	0.000832	0.000086
			6	0.101815	0.010823	0.001197	0.000137
			7	0.114281	0.013560	0.001667	0.000212
			8	0.126363	0.016507	0.002225	0.000309

6. Conclusions

The order statistics, record values and progressive type II right censored order statistics, which are the sub-models of GOS, have wide application in statistical modeling and inference. Moreover, Rayleigh distribution and extended forms have significant role in reliability theory and engineering. Thus, it is important to study the moment properties of these models. The results presented in this paper and simulation study based on it will be useful in the field of reliability analysis and engineering sciences. Further, the characterization of probability distributions is a essential procedure to check the proposed model fits the requirement of given distribution or not. The proposed characterization result in this paper may be useful for the researchers of applied sciences.

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