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Hypotheses Testing in Nonergodic Fractional Ornstein-Uhlenbeck Models

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Abstract. We obtain explicit form of fine large deviation theorems for the log-likelihood ratio in testing models with fractional nonergodic Ornstein-Uhlenbeck processes with Hurst parameter more than half and get explicit rates of decrease of the error probabilities of Neyman-Pearson, Bayes and minimax tests.

1. Introduction and Preliminaries

Parameter estimation for directly observed stochastic differential equations is studied in Bishwal (2008). Parameter estimation in partially observed stochastic volatility models is studied in Bishwal (2022a). Hypothesis testing for stochastic differential equations is studied in Linkov (1993). Parameter estimation and hypotheses testing in ergodic diffusion processes is studied in Kutoyants (1984). Parameter estimation for SPDEs driven by cylindrical stable processes is studied in Bishwal (2023).

Long memory or long range dependent processes have received recent attention in finance, engineering and physics. The simplest continuous time long memory process is the fractional Brownian motion discovered by Kolmogorov (1940) and later on studied by Levy (1948) and Mandelbrot and van Ness (1968).

A normalized fractional Brownian motion $\{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s, t \geq 0.$$

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a Dirichlet process. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for $H > \frac{1}{2}$ and

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in this case they display long-range dependence. The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence. The ARIMA(p,d,q) with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0,\ 0.5)$ process converge in Donsker sense to fBm.

As a generalization of fractional Brownian motion we get the Hermite process of order k with Hurst parameter $H \in (\frac{1}{2}, 1)$ which is defined as a multiple Wiener-Itô integral of order k with respect to standard Brownian motion $(B(t))_{t\in\mathbb{R}}$

$$Z_t^{H,k} := c(H,k)\int_{\mathbb{R}}\int_0^t \Pi_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{H-1}{2})} ds\ dB(y_1)dB(y_2)\cdots dB(y_k)$$

where $x_+ = \max(x, 0)$.

For $k = 1$ the process is fractional Brownian motion W_t^H with Hurst parameter $H \in (0, 1)$. For $k = 2$ the process is Rosenblatt process. For $k \geq 2$, the process is non-Gaussian.

The Rosenblatt process is not a semimartingale and for $H > 1/2$, the quadratic variation is 0. The distribution of the process is infinitely divisible. It is unknown yet whether the process is Markov or not.

The covariance kernel $R(t, s)$ is given by

$$R(t,s) := E[Z_t^{H,k} Z_s^{H,k}] = c(H,k)^2 \int_0^t \int_0^s \left[(u-s)_+^{-(\frac{1}{2}+\frac{H-1}{2})} ds (v-y)_+^{-(\frac{1}{2}+\frac{H-1}{2})} dy\right]^k dudv.$$

Let

$$\beta(p,q) := \int_0^1 z^{p-1}(1-z)^{q-1}dz,\ \ p,q > 0$$

be the beta function.

Using the identity

$$\int_0^1 \int_{\mathbb{R}} (u-s)_+^{a-1} ds (v-y)_+^{a-1} dy\ = \beta(a, 2a-1)|u-v|^{2a-1},$$

we have

$$\begin{aligned} R(t,s) &= c(H,k)^2 \beta\left(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k}\right)^k \int_0^t \int_0^s \left(|u-v|^{\frac{2H-2}{k}}\right)^k dvdu \\ &= c(H,k)^2 \frac{\beta(\frac{1}{2}-\frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)} \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}). \end{aligned}$$

In order to obtain $E(Z_t^{(H,k)})^2 = 1$, choose

$$c(H,k)^2 = \left(\frac{\beta(\frac{1}{2}-\frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)}\right)^{-1}$$

and we have

$$R(t,s) = \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

Thus the covariance structure of the Hermite process and fractional Brownian motion are the same. The process $Z_t^{(H,k)}$ is H-self similar with stationary increments and all moments are finite.

For any $p \geq 1$,

$$E|Z_t^{(H,k)} - Z_s^{(H,k)}|^p \leq c(p, H, k)|t-s|^{pH}.$$

Thus the Hermite process has Hölder continuous paths of order $\delta < H$.

A weighted fBm (wfBm) ξ_t has the covariance function

$$Q(s,t) = \int_0^{s\wedge t} u^a[(t-u)^b + (s-u)^b]du, \quad s,t \geq 0$$

where $a > -1, -1 < b \leq 1, |b| \leq 1+a$. When $a = 0$, it is the usual fBm with Hurst parameter $(b+1)/2$ up to a multiplicative constant. For $b = 0$ it is a time-inhomogeneous Bm.

The function u^a is called the weight function of wfBm. For $a = 0$, this process is usual fBm with Hurst parameter $(b+1)/2$. For the case $b = 1$, this process has the covariance of the process $\int_0^t W_{r^a} dr$ where W is standard Brownian motion. For $b = 0$, this process is time-inhomogeneous Bm. The finite dimensional distributions of the process $(T^{-a/2}(\xi_{t+T} - \xi_T)), t \geq 0$ converge as $T \to \infty$ to those of fBm with Hurst parameter $(1+b)/2$ multiplied by $(2/(1+b)))^{1/2}$. The process has asymptotically stationary increments for long time intervals, but not for short time intervals. For $b \neq 0$, the process is neither a semimartingale nor a Markov process.

This process occurs as the limit of occupation time fluctuations of a particle system of independent particles moving in $\mathbb{R}^d$ with symmetric α-stable Levy process, $0 < \alpha \leq 2$, started from an inhomogeneous Poisson configuration with intensity measure $dx/(1+|x|^\gamma), 0 < \gamma \leq d = 1 < \alpha, a = -\gamma/\alpha, \; b = 1 - 1/\alpha, -1 < a < 0, 0 < b \leq 1+a$. The homogeneous case $\gamma = 0$ gives fBm.

A bi-fractional Brownian motion (bfBm) has covariance

$$\frac{1}{2}(s^{2H} + t^{2H})^k - |t-s|^{2Hk}), \quad s,t \geq 0, \quad 0 < k \leq 1.$$

For $k = 1$, it reduces to fBm. For $H = 1/2$, bfBm can be extended for $1 < k < 2$.

Consider the Gaussian process with the covariance function

$$K_H(s,t) = (2-2H)\left(s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right]\right), \quad s,t > 0$$

for $1 < H \leq 2$. The case $H = 1/2$ corresponds to Bm.

This process occurs as the limit of occupation time fluctuations of a particle system undergoing a critical branching, i.e., each particle independently, at an exponentially distributed lifetime, disappears with probability $1/2$ or is replaced with two particles at the same site with probability $1/2$. For $\alpha = 2$, one reaches superprocesses.

Recently, sub-fractional Brownian (sub-FBM) motion which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0$$

for $0 < H < 1$ introduced by Bojdecki, Gorostiza and Talarczyk (2004) has received some attention in finite dimensional models. The interesting feature of this process is that this process has some of the main properties of fBm, but the increments of the process are nonstationarity, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. However, in this paper we will focus only on the fractional noise.

Sub-fBm is intermediate between Bm and fBm in the sense that its increments over non-overlapping time intervals are more weakly correlated and their covariance decays faster than fBm, see Bojdecki *et al.* (2007).

A negative sub-fBm (nsfBm) has the covariance function

$$K_H(s,t) = (2-2H)\left(s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right]\right), \quad s,t > 0$$

for $1 < H < 2$. The nsfBm is a semimartingale where as sfbM is not. The nsfBm is not a Markov process so as the sfBm.

High-density occupation time fluctuation limits of the branching particle system for the case $d = \alpha$ leads to the centered Gaussian process with covariance function

$$R(s,t) = -\left(s^2 \log s + t^2 \log t - \frac{1}{2}\left[(s+t)^2 \log(s+t) + (s-t)^2 \log|s-t|\right]\right), \ s,t > 0.$$

Though it has similar interpretation to that of nsfBm, this process is not a semimartingale.

Sub-fBm can be embedded into a larger family of long range dependent self similar process with covariance function

$$C_H(s,t) = (s^{2H} + t^{2H})^k - \frac{1}{2^k}\left[(s+t)^{2H} + |s-t|^{2H}\right]^k, \quad s,t > 0, 0 < k \leq 1.$$

The case $k = 1$ corresponds to sub-fBm. For $H = 1/2$, this yields a family of such processes with covariance function

$$(s+t)^k - (s \wedge t)^k, \quad k \geq 1$$

which corresponds to Bm for $k = 1$.

Another family of such processes has covariance function

$$C_H(s,t) = (s^{2H} + t^{2H})^k - \frac{1}{2}\left[(s+t)^{2Hk} + |s-t|^{2Hk}\right], \quad s,t > 0, \quad 0 < k \leq 1.$$

This corresponds to the process $\{\frac{1}{2}(\eta_t + \eta_{-t}), t \geq 0\}$ where η is a bi-fBm. Using the kernel for the sfBm, one can define sub-Rosenblatt process, a non-Gaussian process, see Bojdecki *et al.* (2006).

The fractional Ornstein-Uhlenbeck (fOU) process, is an extension of Ornstein-Uhlenbeck process with fractional Brownian motion (fBm) driving term, see Bishwal (2011). In finance, it is known as fractional Vasicek model and has been as one-factor short-term interest rate model which takes into account the long memory effect of the interest rate, see Bishwal (2022b). This process has been modeled as a telecom process by Wolpert and Taqqu (2005). Using suitable transformation of the process, one can obtain a nonlinear stationary process satisfying a fractional SDE, see Buchmann and Kluppelberg (2006). Benth (2003) used the process as temperature and obtained weather derivative arbitrage free pricing formulas for European and average type options. Cheridito *et al.* (2003) obtained the fOU process as a Lamperti transformation of the fBM. The model parameter is usually unknown and must be tested from data.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis on which is defined the Ornstein-Uhlenbeck process X_t satisfying the Itô stochastic differential equation

$$dX_t = \theta X_t dt + dW_t^H, \quad t \geq 0, \quad X_0 = 0 \tag{1.1}$$

where $\{W_t^H\}$ is a fractional Brownian motion with $H > 1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta > 0$ is the unknown parameter to be tested on the basis of continuous observation of the process $\{X_t\}$ on the time interval $[0, T]$.

In the stationary case $\theta < 0$ one can construct fOU process in another way by time change. Lamperti transform provided one-to-one correspondence between a stationary process and a H-self similar process. Recall that an ordinary OU process is a Lamperti transform of Brownian motion. If (B_t) is a standard Brownian motion, then $\frac{e^{-\theta t}}{\sqrt{2\theta}} B_{e^{2\theta t}}$ is an OU process. Similarly one can represent fOU process as a time changed fBM:

$$X_t = \frac{e^{-2H\theta t}}{\sqrt{4H\theta}} W^H_{e^{2\theta t}}.$$

It is a zero mean Gaussian process with covariance kernel

$$\begin{aligned} C_t = E(X_s X_{t+s}) &= \frac{1}{8H\theta}\left[e^{2H\theta t} + e^{-2H\theta t} - |e^{\theta t} - e^{-\theta t}|^{2H}\right] \\ &= \frac{1}{8H\theta}\left[\cosh 2H\theta t - 2^{2H-1}(\sinh \theta t)^{2H}\right]. \end{aligned}$$

For $H = 1/2$, this reduces to the covariance of the ordinary OU process $\frac{1}{2\theta} e^{-\theta t}$. The covariance structure shows that the fOU process is locally asymptotically stationary for small time lag and is a sum of mutually independent Gaussian Markov processes for large time lag.

We study the problem of testing hypotheses

$$\mathcal{H}_0 : \theta = \theta_0 \quad \text{against the alternative} \quad \mathcal{H}_1 : \theta = \theta_1. \tag{1.2}$$

Bishwal (2008b) studied hypothesis testing in the ergodic case $\theta < 0$. We study the nonergodic case $\theta > 0$ in this paper. Define

$$\begin{aligned} \kappa_H &:= 2H\Gamma(3/2 - H)\Gamma(H + 1/2), \\ k_H(t, s) &:= \kappa_H^{-1}(s(t - s))^{\frac{1}{2} - H}, \quad \lambda_H = \frac{2H\Gamma(3 - 2H)\Gamma(H + \frac{1}{2})}{\Gamma(3/2 - H)} \\ v_t \equiv v_t^H &:= \lambda_H^{-1} t^{2-2H}, \quad M_t^H = \int_0^t k_H(t, s) dW_s^H. \end{aligned}$$

From Norros *et al.* (1999) it is well known that M_t^H is a Gaussian martingale, called the *fundamental martingale* whose variance function $\langle M^H \rangle_t$ is v_t^H. The natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H since

$$W_t^H := \int_0^t K(t, s) dM_s^H$$

holds for $H \in (1/2, 1)$ where

$$K_H(t,s) := H(2H-1)\int_s^t r^{h-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}, \quad 0 \le s \le t$$

and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define

$$Q_t := \frac{d}{dv_t}\int_0^t k_H(t,s)X_s ds.$$

It is easy to see that

$$Q_t = \frac{\lambda_H}{2(2-2H)}\left\{t^{2H-1}Z_t + \int_0^t r^{2H-1}dZ_s\right\}.$$

X admits the representation

$$X_t = \int_0^t K_H(t,s)dZ_s.$$

The natural filtration generated by the fundamental semimartingale process

$$Z_t = \theta\int_0^t Q_s dv_s + M_t^H$$

and the process X coincide, see Kleptsyna and Le Breton (2002). The available information for X and Z are strictly equivalent.

Let the realization $\{X_t, 0 \le t \le T\}$ or equivalently $\{Z_t, 0 \le t \le T\}$ be denoted by Z_0^T. Let P_θ^T be the measure generated on the space (C_T, B_T) of continuous functions on $[0,T]$ with the associated Borel σ-algebra B_T generated under the supremum norm by the process X_0^T and P_0^T be the standard Wiener measure. Applying fractional Girsanov formula, when θ is the true value of the parameter, P_θ^T is absolutely continuous with respect to P_0^T and the Radon-Nikodym derivative (likelihood) of P_θ^T with respect to P_0^T based on Z_0^T is given by

$$L_T(\theta) := \frac{dP_\theta^T}{dP_0^T}(Z_0^T) = \exp\left\{\theta\int_0^T Q_t dZ_t - \frac{\theta^2}{2}\int_0^T Q_t^2 dv_t\right\}. \tag{1.3}$$

Consider the score function, the derivative of the log-likelihood function, which is given by

$$Y_T(\theta) := \int_0^T Q_t dZ_t - \theta\int_0^T Q_t^2 dv_t. \tag{1.4}$$

Under the hypothesis $\mathcal{H}_0$, the log-likelihood ratio process admits the representation

$$\Lambda_T := \log\frac{dP_{\theta_1}^T}{dP_{\theta_0}^T}(Z_0^T) = \left\{(\theta_1-\theta_0)\int_0^T Q_t dM_t - \frac{(\theta_1-\theta_0)^2}{2}\int_0^T Q_t^2 dv_t\right\}. \tag{1.5}$$

The *Hellinger integral* of order $\varepsilon \in (-\infty,\ \infty)$ is defined as

$$h_T(\varepsilon) = E_{\theta_0}[\exp(\varepsilon\Lambda_t)].$$

Note that $h_T(\varepsilon) := h_T(\varepsilon; P_{\theta_1}^T, P_{\theta_0}^T) = h_T(1-\varepsilon; P_{\theta_0}^T, P_{\theta_1}^T)$.

Under $\mathcal{H}_0$ we have

$$\int_0^T Q_t dM_t = \frac{1}{2}\left(U_T + 2\theta_0\int_0^T Q_s^2 dv_s - T\right). \tag{1.6}$$

where $U_T := \frac{\lambda_H}{2-2H}Z_T\int_0^T t^{2H-1}dZ_s$.

Recall that if $X \sim \mathcal{N}(m_1, \sigma_1)$ and $Y \sim \mathcal{N}(m_2, \sigma_2)$ are two independent random variables, then X/Y follows Cauchy distribution. The pdf of the standard Cauchy distribution is given by

$$g(x) = \frac{1}{\pi(1+x^2)}, \ x \in \mathbb{R}.$$

The cdf of standard Cauchy distribution is denoted by $G(x)$ which is given by

$$G(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x, \ x \in \mathbb{R}$$

and whose characteristic function is given by

$$\int_{-\infty}^{\infty} e^{i\lambda x} dG(x) = e^{-|\lambda|}, \lambda \in \mathbb{R}.$$

The least squares estimator $\hat{\theta}_T = \int_0^T X_t dX_t / \int_0^T X_t^2 dt$ of the parameter θ is strongly consistent and has the limiting Cauchy distribution with rate $e^{\theta T}$ as $T \to \infty$, see Belfadli *et al.* (2011), El Machkouri *et al.* (2016), Belfadli *et al.* (2020) and Es-Sebaiy *et al.* (2021). More specifically, $\hat{\theta}_T \to \theta$ almost surely as $T \to \infty$ and $e^{\theta T}(\hat{\theta}_T - \theta) \to^{\mathcal{D}} 2\theta G(1)$ as $T \to \infty$.

Now we introduce the hypothesis testing method. Luschgy (1993, 1994a, 1994b, 1994c, 1995) and Linkov (1993) studied hypothesis testing for semimartingales. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_0, P_1), t \geq 0$ be the family of statistical experiments generated by the observations $X_0^t = (X_s, 0 \leq s \leq t)$. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be the statistical hypotheses consisting of the distributions of the observations X_t defined by the measures P_0 and P_1 respectively. Denote $\delta_t = (\delta_t(x))_{x \in \Omega}$ measurable map of the space $(\Omega, \mathcal{F})$ to the space $([0,1], \mathcal{B}[0,1])$. The map δ_t is called the test for distinguishing between $\mathcal{H}_0$ and $\mathcal{H}_1$ from the observations X_t, $\delta_t(x)$ being the conditional probability to reject the hypothesis $\mathcal{H}_0$ under the condition that $X_o = x$. Denote by Σ_t the totality of all the tests δ_t for distinguishing between the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$. The probability of *error of the first kind* (Type I Error) is defined as $\alpha(\delta_t) = E_0(\delta_t)$ and probability of the *error of the second kind* (Type II Error) is defined as $\beta(\delta_t) = E_1(1-\delta_t)$ where E_0 is expectation under the P_0 and E_1 is expectation under P_1. The probability of error of the first kind $\alpha(\delta_t)$ is the probability to accept the hypothesis $\mathcal{H}_1$ with the help of the test δ_t under the condition that the hypothesis $\mathcal{H}_0$ is true. The quantity $1 - \beta(\delta_t)$ is called the *power* of a test δ_t. For any $\alpha \in [0,1]$, we denote by Σ_t^α the totality of all tests in δ_t in Σ_t that satisfy the condition $\alpha(\delta_t) \leq \alpha$.

A family of hypotheses $(\mathcal{H}_1^t)$ is said to be *contiguous* to a family of hypotheses $(\mathcal{H}_0^t)$ if for any test $\delta_t \in \Sigma_t$ such that $\alpha(\delta_t) \to 0, t \to \infty$, we have $\beta(\delta_t) \to 1, t \to \infty$.

To test the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$, introduce the test

$$\delta_t^{c,\varepsilon} = I(X_t > c) + \epsilon I(X_t = c)$$

where $c \in (0, \infty)$ and $\varepsilon \in [0,1]$ are parameters of the test.

The Kakutani-Hellinger distance between the measures P_0 and P_1 is defined as

$$\rho^2(P_0, P_1) = \frac{1}{2} E_Q |\Lambda_{t,1}^{1/2} - \Lambda_{t,0}^{1/2}|^2$$

where $Q = (P_0 + P_1)/2$ and $\Lambda_{t,0} = \frac{dP_0}{dQ}(X_0^t)$ and $\Lambda_{t,1} = \frac{dP_1}{dQ}(X_0^t)$.

The Hellinger integral $h(\epsilon; P_1, P_0)$ of order ε of P_1 and P_0 is defined as

$$h(\epsilon; P_1, P_0) = E_Q \Lambda_{t,1}^{\varepsilon} \Lambda_{t,0}^{1-\varepsilon}.$$

The quantity $h(\frac{1}{2}; P_1, P_0)$ simply called the Hellinger integral for measures P_1 and P_0.

Note that

$$\rho^2(P_0, P_1) = 1 - h(\frac{1}{2}; P_1, P_0).$$

The distance in variation between measures P_0 and P_1 is the total variation of the measures, namely

$$\|P_1 - P_0\| = E_Q|\Lambda_{t,1} - \Lambda_{t,0}|.$$

$$\|P_0 \wedge P_1\| = 1 - \frac{1}{2}\|P_1 - P_0\| = E_Q((\Lambda_{t,0} \wedge \Lambda_{t,1}).$$

The quantity

$$I(P_0|P_1) = E_Q \Lambda_{t,0} \log(\Lambda_{t,0}/\Lambda_{t,1}).$$

is called the *entropy* of the measure P_0 with respect to P_1. The *relative entropy* $I(P_0|P_1)$ called the *Kullback-Leibler divergence* or Kullback-Liibler information. Observe that $E\Lambda_t = -I(P_0|P_1)$.

The Neyman-Pearson fundamental lemma is the following:

For any $\alpha \in [0, 1]$, there exists a test $\delta_t^{c(\alpha),\varepsilon(\alpha)}$ of level α, where $(c(\alpha), \varepsilon(\alpha))$ is some solution to the equation $\delta_t^{c(\alpha),\varepsilon(\alpha)} = \alpha$ with respect to (c, α). For any $\alpha \in [0, 1]$, the test $\delta_t^{c(\alpha),\varepsilon(\alpha)}$ with $\varepsilon(0) = 1$ is the *most powerful test*.

A test $\delta_t^{c(\alpha),\varepsilon(\alpha)}$ with $\varepsilon(0) = 1$ is called the *Neyman-Pearson test*.

Next let us introduce *Bayes test*. Consider the nonrandomized test

$$\delta_t^{*,c} = \delta_t^{c,1} = I(X_t \geq c), \;\; c \geq 0.$$

For any $c \in [0, \infty)$,

$$\inf\{c\alpha(\delta_t) + \beta_t(\delta_t) : \delta_t \in \Sigma_t\} = c\alpha(\bar{\delta}_t) + \beta_t(\bar{\delta}_t)$$

for any $\bar{\delta}_t \in \Sigma_t$ coinciding with the test $\delta_t^{*,c}$ on the set $\{X_t \neq c\}$ and arbitrarily defined on the set $\{X_t = c\}$. From this it follows that for every $c \in [0, \infty)$ and $\varepsilon \in [0, 1]$,

$$\inf\{c\alpha(\delta_t) + \beta_t(\delta_t) : \delta_t \in \Sigma_t\} = c\alpha(\delta_t^{*,c}) + \beta_t(\delta_t^{*,c}) = c\alpha(\delta_t^{c,\varepsilon}) + \beta_t(\delta_t^{c,\varepsilon}).$$

This test is called *Bayes test* with respect to an a priori distribution $(c/(c + 1), 1/(c + 1))$.

For $c = 1$, in the Bayes risk,

$$\|P_0 \wedge P_1\| = \inf\{\alpha(\delta_t) + \beta_t(\delta_t) : \delta_t \in \Sigma_t\}.$$

Thus $\|P_0 \wedge P_1\|/2$ is the probability of error of the Bayes test in the case of equiprobable hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$.

For the definitions of Neyman-Pearson test and Bayes test, see Linkov (1993); Chapter II, Section 2.1. For the definition of *minimax test*, see Borovkov (1984), Chapter III, Section 41.3.

No single decision rule minimizes the weighted average, i.e., Bayes risk for every prior distribution. The conservative approach is to minimize the worst case risk over all possible prior state distributions. Intuitively, the minimax decision rule is the Bayesian decision rule with constant Bayesian risk over the priors. The minimax rule allows one to guarantee a worst-case (maximum) risk over all priors. Minimax test is a Bayes test with respect to a least favorable prior distribution.

We assume the following *regularity conditions*: if for some function ψ_T such that $\psi_T \to \infty$ as $T \to \infty$, the (possibly infinite) limit:

$$\lim_{T\to\infty} \psi_T^{-1} \log h_T(\varepsilon) = \chi(\varepsilon) \tag{1.7}$$

exists for all $\varepsilon \in (-\infty, \infty)$, and $\chi(\varepsilon)$ is a convex differentiable function on $(\varepsilon_-, \varepsilon_+)$ with

$$\gamma_- := \lim_{\varepsilon\downarrow\varepsilon_-} \chi'(\varepsilon) \le \gamma_+ := \lim_{\varepsilon\uparrow\varepsilon_+} \chi'(\varepsilon), \quad \varepsilon_- := \inf\{\varepsilon : \chi(\varepsilon) < \infty\} < \varepsilon_+ := \sup\{\varepsilon : \chi(\varepsilon) < \infty\}.$$

Obviously, $\varepsilon_- \le 0$ and $\varepsilon_+ \ge 1$. If $\varepsilon_- < 0$, then the derivative $\chi'(0) =: \gamma_0$ is well defined. If $\varepsilon_+ > 1$, then the derivative $\chi'(1) =: \gamma_1$ is well defined too.

Let us introduce the *Legendre-Fenchel transform* of the function $\chi(\varepsilon)$, i.e.,

$$F(\gamma) := \sup_{\varepsilon}(\varepsilon\gamma - \chi(\varepsilon))$$

and the quantities

$$\Gamma_0 := \gamma_0 I(\varepsilon_- < 0) + \gamma_- I(\varepsilon_- = 0), \quad \Gamma_1 := \gamma_1 I(\varepsilon_- > 1) + \gamma_+ I(\varepsilon_- = 1)$$

where $I(\cdot)$ is the indicator function.

Substituting (1.6) into (1.5), we obtain the *Hellinger integral*

$$h_T(\varepsilon) = \exp\left(\frac{\varepsilon(\theta_1-\theta_0)}{2}T\right) E_{\theta_0}\left[\exp\left(\frac{\varepsilon(\theta_1-\theta_0)}{2}Z_T^2 - \frac{\varepsilon(\theta_1^2-\theta_0^2)}{2}\int_0^T Q_s^2 dv_s\right)\right]. \tag{1.8}$$

Kleptsyna and Le Breton (2002) obtained the following *Cameron-Martin type formula*:

Denote $V_T := \int_0^T Q_s^2 dv_s$. Let $\phi_T^H(u) := E\exp(-uV_T)$, $u > 0$. Then $\phi_T^H(u)$ is given by

$$\phi_T^H(u) = \left\{\frac{4\sin\pi H\sqrt{\theta^2+2u}e^{-\theta T}}{\pi T D_T^H(\theta;\sqrt{\theta^2+2u})}\right\}^{1/2} \tag{1.9}$$

where

$$\begin{aligned} D_T^H(\theta;\alpha) \quad := \quad & [\alpha\cosh(\frac{\alpha}{2}T) - \theta\sinh(\frac{\alpha}{2}T)]^2 J_{-H}(\frac{\alpha}{2}T)J_{H-1}(\frac{\alpha}{2}T) \\ & -[\alpha\sinh(\frac{\alpha}{2}T) - \theta\cosh(\frac{\alpha}{2}T)]^2 J_{1-H}(\frac{\alpha}{2}T)J_{H}(\frac{\alpha}{2}T) \end{aligned}$$

for $\alpha > 0$ and J_ν is the modified Bessel function of first kind of order ν.

Bishwal (2008b) generalized the above Cameron-Martin type formula to obtain joint generating function of U_T and V_T.

Lemma 1.1 Let $\Psi_T^H(z_1, z_2) := E \exp(z_1 V_T + z_2 U_T), z_1, z_2 \in \mathbb{C}$. Then $\Psi_T^H(z_1, z_2)$ exists for $|z_i| \leq \delta$, $1 = 1{,}2$ for some $\delta > 0$ and is given by

$$\Psi_T^H(z_1, z_2) = \exp\left(\frac{-\theta T}{2}\right)\left[\frac{4(\sin \pi H)\gamma}{\pi T \mathcal{D}_T^H(\gamma; z_2)}\right]^{1/2} \tag{1.10}$$

where $\gamma := (\theta^2 + 2z_1)^{1/2}$ and we choose the principal branch of the square root and

$$\begin{aligned}\mathcal{D}_T^H(\gamma; z_2) \quad := \quad & [\gamma \cosh(\frac{\gamma}{2}T) - (\theta - 2z_2)\sinh(\frac{\gamma}{2}T)]^2 J_{-H}(\frac{\gamma}{2}T) J_{H-1}(\frac{\gamma}{2}T) \\ & -[\gamma \sinh(\frac{\gamma}{2}T) - (\theta - 2z_2)\cosh(\frac{\gamma}{2}T)]^2 J_{1-H}(\frac{\gamma}{2}T) J_H(\frac{\gamma}{2}T)\end{aligned}$$

and J_ν is the modified Bessel function of first kind of order ν.

2. Main Results

Let ψ_t be a nonrandom positive function such that $\psi_t \to \infty$ as $t \to \infty$. Then the following is the law of large numbers for the log-likelihood ratio process Λ_t:

(LLN) $\lim_{t\to\infty} P_0[|\psi_t^{-1}\Lambda_t + 1| > a] = 0$ for any $a > 0$.
(L1) $\lim_{t\to\infty} P_0[\psi_t^{-1}\Lambda_t > \gamma] = 0$ for any $\gamma > -1$.
(L2) $\lim_{t\to\infty} P_0[\psi_t^{-1}\Lambda_t < \gamma] = 0$ for any $\gamma < -1$.

Note that (L1) and L2) hold if and only if (LLN) holds.

The following assertion is a large deviation theorem of Chernoff type for the log-likelihood ratio process Λ_T. The assertion was proved by means of large deviations theorems for extended random variables, see Linkov (1999).

Proposition 2.1 Let the regularity condition (1.7) be satisfied. Then the following conclusions hold:

(i) if $\Gamma_0 < \gamma_+$, then for all $\gamma \in (\Gamma_0, \gamma_+)$, we have

$$\lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t > \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t \geq \gamma] = -F(\gamma) \in (-\infty, 0);$$

(ii) if $\varepsilon_- < 0$ and $\gamma_- < \gamma_0$, then for all $\gamma \in (\Gamma_-, \gamma_0)$, we have

$$\lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t < \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t \leq \gamma] = -F(\gamma) \in (-\infty, 0);$$

(iii) if $\gamma_- < \Gamma_1$, then for all $\gamma \in (\gamma_-, \Gamma_1)$, we have

$$\lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t < \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t \leq \gamma] = \gamma - F(\gamma) \in (-\infty, 0);$$

(iv) if $\varepsilon_+ > 1$ and $\gamma_1 < \gamma_+$, then for all $\gamma \in (\Gamma_1, \gamma_+)$, we have

$$\lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t > \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t \geq \gamma] = \gamma - F(\gamma) \in (-\infty, 0).$$

The following assertions give the rate of decrease of error probabilities (Stein's Lemma) for Neyman-Pearson, Bayes and minimax tests.

Let $\delta_t = \delta_t(\alpha_t)$ be a Neyman-Pearson test of level $\alpha_t \in (0,1)$ for testing hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ under the observations $\{Z_s, 0 \le s \le t\}$. The following proposition gives the rate of decrease of the error probabilities of the first kind α_t and second kind $\beta(\alpha_t)$ for the test $\delta_t(\alpha_t)$.

Proposition 2.2

$$\text{(i)} \quad \lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t) = -a \quad \text{if and only if} \quad \lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t) = -b(a)$$

where

$$b(a) := a - \gamma(a) \in [F(\Gamma_1) - \Gamma_1, \quad F(\Gamma_0) - \Gamma_0]$$

and $\gamma(a)$ is the unique solution of the equation $F(\gamma) = a$ with respect to $\gamma \in [\Gamma_0,\ \Gamma_1]$.

(ii) for all $a \in [0, F(\gamma_0)]$,

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t) = -a \quad \text{implies} \quad \limsup_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t) \le \Gamma_0 - F(\Gamma_0)$$

and for all $a \in [F(\gamma_1), \infty]$,

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t) = -a \quad \text{implies} \quad \liminf_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t) \ge \Gamma_1 - F(\Gamma_1).$$

(iii) for all $b \in [0, F(\Gamma_1) - \Gamma_1]$,

$$\lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t) = -b \quad \text{implies} \quad \limsup_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t) \le -F(\Gamma_1)$$

and for all $b \in [F(\Gamma_0) - \Gamma_0, \infty]$,

$$\lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t) = -b \quad \text{implies} \quad \liminf_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t) \ge -F(\Gamma_0).$$

These results under more restrictive conditions were proved in Linkov (1993). The only of part of (i) for the sequence of i.i.d. random variables was proved in Birge (1981).

Let δ_t^π be a Bayes test for testing hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ under the observations $\{X_s, 0 \le s \le t\}$, where π and $1-\pi$, $\pi \in (0,1)$ are the a priori probabilities of testing $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

The following assertion gives the rate of decrease of the error probabilities of the first kind and second kind $\alpha(\delta_t^\pi)$ and $\beta(\delta_t^\pi)$, and the risk $e(\delta_t^\pi)$ under the regularity condition (1.7).

Proposition 2.3

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \log e(\delta_t^\pi) = -F(0).$$

This assertion was proved by Chernoff (1952) for the i.i.d. case.

Let $\delta_t^*(\alpha_t)$ be a minimax test. The following theorem gives the rate of decrease of the error probabilities of the first kind and second kind $\alpha(\delta_t^*)$ and $\beta(\delta_t^*)$, and the minimax risk $e(\delta_t^*)$ of the test δ_t^* under the regularity condition (1.7).

Proposition 2.4

$$\lim_{t\to\infty}\psi_t^{-1}\log\alpha(\delta_t^*)=\lim_{t\to\infty}\psi_t^{-1}\log\beta(\delta_t^*)=\lim_{t\to\infty}\psi_t^{-1}\log e(\delta_t^*)=-F(0).$$

The following two theorems are the main results of the paper.

Theorem 2.5 *In the model (1.1) for testing (1.2) with $\theta_1>\theta_0=0$, the following conclusions hold:*

(a) In Proposition 2.1: (i) holds for all $\gamma\in(\infty,1/2)$, (iii) holds for all $\gamma\in(\infty,1/4)$ and (iv) holds for all $\gamma\in(1/4,1/2)$.

(b) In Proposition 2.2: (i) holds for all $a\in(0,1/4)$ with $b(a)=(4a-1)^2/16a$, $F(0)=1/2$. First part of (ii) holds with $a=0$. Second part of (ii) holds for all $a\in(1/4,\infty)$. First part of (iii) holds with $b=0$. Second part of (iii) holds with $b=\infty$.

(c) Proposition 2.3 and Proposition 2.4 hold with $F(0)=1/2$.

Proof. We consider $\theta_1>\theta_0=0$. We have

$$\log h_T(\varepsilon)\quad=\quad\frac{\varepsilon\theta_1 T}{2}+\log E_{\theta_0}\left[\exp\left(\frac{\varepsilon\theta_1}{2}U_T-\frac{\varepsilon\theta_1^2}{2}\int_0^T Q_s^2\,dv_s\right)\right].$$

Then assuming that $\varepsilon>0$ and denoting $\varphi:=\frac{\varepsilon\theta_1}{2}$ and $\xi:=\pm\sqrt{\varepsilon}\theta_1$, by means of Lemma 1.1, we obtain that the logarithm of the Hellinger integral admits the representation

$$\log h_T(\varepsilon)\quad=\quad\varphi T+\log E_{\theta_0}\left[\exp\left(\varphi U_T-\varphi\theta_1\int_0^T Q_s^2\,dv_s\right)\right]=\varphi T+\log\left\{\frac{4(\sin\pi H)\varrho e^{-\theta_0 T}}{\pi T\mathcal{D}_T^H(\varrho,\varphi)}\right\}^{1/2}$$

where $\varrho:=(\theta_0^2+2\varphi\theta_1)^{1/2}$.

For $\varepsilon<0$ and sufficiently large $T>0$, we have $h_T(\varepsilon)=\infty$. Hence taking $\psi_T=2(1-H)C_0^2B_1^2(\theta_1-\theta_0)^2e^{2\theta_1 T}$, where $B_1=B(3/2-H,\ 3/2-H)$, $B_3=B(H-1/2,\ 3/2-H)$, $C_0=\frac{1}{2}(H-1/2)H(1-H)B_1B_3)^{-1/2}$ with $B(m,n)$ is the beta-function and letting T go to ∞ and applying asymptotic properties of the modified Bessel functions, we obtain $\chi(\varepsilon)=-\frac{1}{2}$ for all $\varepsilon\in(0,1)$, $\chi(0)=\chi(1)=0$ and $\chi(\varepsilon)=\infty$ for all $\varepsilon\notin[0,1]$.

The function $\chi(\varepsilon)$ satisfies the condition in (1.7) with,

$$\chi'(\varepsilon)=0,\quad\gamma_-=0,\quad\gamma_+=0,\quad\varepsilon_-=-\frac{1}{2},\quad\varepsilon_+=-\frac{1}{2},$$

$$\gamma_0=0,\quad\gamma_1=0,\quad\Gamma_0=0,\quad\Gamma_1=0,\quad F(\Gamma_0)=\frac{1}{2},\quad F(\Gamma_1)=\frac{1}{2}.$$

Since $\Gamma_0=0=\Gamma_1$, we obtain Theorem 2.5. □

Next suppose that $\theta_1>\theta_0>0$. Assuming that $\varepsilon>-\theta_0/[2(\theta_1^2-\theta_0^2)]$ and $\varphi:=\varepsilon(\theta_1-\theta_0)/2$ and $\xi:=\pm\sqrt{2\varepsilon(\theta_1^2-\theta_0^2)/\theta_0+1}$ which implies that $(\xi^2-1)\theta_0/4=\varepsilon(\theta_1^2-\theta_0^2)/2$, by means of Lemma

1.1, we obtain that the logarithm of the Hellinger integral admits the representation:

$$\begin{aligned}\log h_T(\varepsilon) &= \frac{\varepsilon(\theta_1-\theta_0)T}{2} + \log E_{\theta_0}\left[\exp \frac{\varepsilon(\theta_1-\theta_0)}{2}U_T - \frac{\varepsilon(\theta_1^2-\theta_0^2)}{2}\int_0^T Q_s^2 dv_s\right] \\ &= \varphi T + \log E_{\theta_0}\left[\exp\left(\varphi U_T - \frac{(\xi^2-1)\theta_0}{4}\int_0^T Q_s^2 dv_s\right)\right] \\ &= \varphi T + \log\left\{\frac{4(\sin\pi H)\zeta e^{-\theta T}}{\pi T \mathcal{D}_T^H(\zeta,\phi)}\right\}^{1/2}\end{aligned}$$

where $\zeta := \left(\theta_0^2 - \frac{(\xi^2-1)\theta_0}{2}\right)^{1/2}$.

For $\varepsilon < -\theta_0/[2(\theta_1^2-\theta_0^2)]$ and sufficiently large $T>0$, we have $h_T(\varepsilon)=\infty$. Hence substituting the above expression into (1.7), taking $\psi_T := 2(1-H)C_0^2B_1^2(\theta_1-\theta_0)^2e^{2\theta_1 T}$ by applying the asymptotic properties of the modified Bessel functions, and letting T go to ∞, we have

$$\chi(\varepsilon) = \frac{\varepsilon}{2} - \frac{\sqrt{2\varepsilon\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}}{2(\theta_1-\theta_0)} + \frac{\theta_0}{2(\theta_1-\theta_0)}$$

which is a strictly convex function on $(\varepsilon_-,\varepsilon_+)$ with:

$$\varepsilon_- = \inf\{\varepsilon : \chi(\varepsilon)<\infty\} = -\frac{\theta_0}{2(\theta_1^2-\theta_0^2)}, \quad \varepsilon_+ = \sup\{\varepsilon : \chi(\varepsilon)<\infty\} = \infty,$$

$$\chi'(\varepsilon) = \frac{1}{2} - \frac{\theta_0(\theta_1+\theta_0)}{2\sqrt{2\varepsilon\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}}, \quad \gamma_- = -\infty, \quad \gamma_+ = \frac{1}{2}, \quad \gamma_0 := \chi'(0) = \frac{1-\theta_0-\theta_1}{2}$$

$$\gamma_1 := \chi'(1) = \frac{1}{2} - \frac{\theta_0(\theta_1+\theta_0)}{2\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}}, \quad F(\gamma) := \sup_{\varepsilon>\varepsilon_-}(\varepsilon\gamma - \chi(\varepsilon)) = \frac{\theta_0(1-2\gamma-\theta_0-\theta_1)^2}{4(\theta_1^2-\theta_0^2)(1-2\gamma)}$$

$$\Gamma_0 = \gamma_0 = \frac{1-\theta_0-\theta_1}{2}, \quad \Gamma_1 = \gamma_1 = \frac{1}{2} - \frac{\theta_0(\theta_1+\theta_0)}{2\sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}}$$

$$F(\Gamma_0) = 0, \quad F(\Gamma_1) = \frac{(\theta_0 - \sqrt{2\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2})^2}{4(\theta_1-\theta_0)\sqrt{\theta_0(\theta_1^2-\theta_0^2)+\theta_0^2}}$$

Since $\Gamma_0 < \Gamma_1$, from Propositions 2.1-2.4, we obtain the following theorem.

Theorem 2.6 *In the model (1.1) for testing (1.2) with $\theta_1 > \theta_0 > 0$, the following conclusions hold:*

(a) In Proposition 2.1: (i) holds for all $\gamma \in (\Gamma_0, \gamma_+)$. (ii) holds for all $\gamma \in (\gamma_-, \gamma_0)$. (iii) holds for all $\gamma \in (\gamma_-, \Gamma_1)$. (iv) holds for all $\gamma \in (\gamma_1, \gamma_+)$.

(b) In Proposition 2.2: (i) holds for all $a \in (0, F(\Gamma_1))$ with

$$b(a) = \frac{1-\theta_0-\theta_1}{2} - \frac{\theta_0+\theta_1}{\theta_0}\left(a(\theta_1-\theta_0) - \sqrt{a\theta_0(\theta_1-\theta_0)+a^2(\theta_1-\theta_0)^2}\right).$$

First part of (ii) holds with $a=0$. Second part of (ii) holds for all $a \in (F(\Gamma_1), \infty]$. First part of (iii) holds with $b \in [0, F(\Gamma_1) - \Gamma_1]$. Second part of (iii) holds with $b \in [F(\Gamma_0) - \Gamma_0, \infty]$.

(c) Proposition 2.3 and Proposition 2.4 hold with

$$F(0) = \frac{\theta_0(1-\theta_0-\theta_1)^2}{4(\theta_1^2-\theta_0^2)}.$$

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