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A Variable Step Size Multi-Block Backward Differentiation Formula for Solving Stiff Initial Value Problem of Ordinary Differential Equations

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Abstract. A variable step size multi-block backward differentiation formula for solving stiff initial value problems of ordinary differential equations with a variable step size strategy was derived. The proposed method (VSSMBBDF) computes two approximate solution values at a time per integration step. The stability properties are achieved by varying the step size ratio in the formula to generate more zero stable schemes. The proposed method is also found to be an A-Stable scheme across different choices of the step size. The method is capable of solving stiff IVPs of ODEs. Approximates result from the system of stiff ODE problems considered are found to favorably validate the performance of the new method in terms of accuracy of the scale error and less executional time in respect to the two methods compared in the study. Hence, the proposed method can be an alternative solver for stiff IVPs of ODEs.

Introduction

Backward differentiation formula came to existence from the work of (Cutis & Hirschfield, 1952), then extended backward differentiation formulae by (Cash, 1980); Implicit r-point block backward differentiation formula for solving first- order stiff ODEs by (Ibrahim *et al.,*2007); Super class aspect of block backward differentiation formula by (Sulaiman*et al.,* 2013a); diagonally implicit super class of block backward differentiation formula for solving Stiff IVPs by (Musa *et al,* 2016) and the work of Sagir (2012, 2014, 2013) possesses good error when compared with some existing methods. Due to the preferences of seeking numerical approximate solutions to most of the modern problems, numerical methods are been developed with various capacities to handle current realities of initial vaue problem of ODEs and stiff ODEs, some of the recent method with very good stability properties, at one point or the other are found with following, an A-stable block integrator scheme for the

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solution of first order system of IVPs of ordinary differential equations by (Abdullahi *et al*, 2022); Enhanced 3-Point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems; Order and Convergence of the enhanced 3 point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems by (Abdullahi & Musa, 2021a; Abdullahi & Musa, 2021b); An Order Five Implicit 3-StepBlock Method for Solving Ordinary Differential Equations by (Yahaya & Sagir, 2013); Diagonally Implicit Super Class of Block Backward Differentiation Formula with Off-Step Points for Solving Stiff Initial Value Problems by (Babangida & Musa, 2016); Development of an improved numerical integration method via the transcendental function of exponential form by (Fadugba, 2020); Implicit five-step block method with generalized equidistant points for solving fourth order linear and non-linear initial value problems by (Adeyeye & Omar, 2019); Variable step block backward differentiation formula for solving first order stiff odes by (Zarina *et al.,* 2007); Predictor-Corrector Block Iteration Method for Solving Ordinary Differential Equations by (Majid & Suleiman, 2011); An Accurate Block Solver for Stiff Initial Value Problems and A new fifth order implicit block method for solving first order stiff ordinary differential equations by (Musa *et al*., 2014a, 2014b). Suleiman *et al.,* (2013b) developed the super class form of variable step size block backward differentiation formula formally known as a new variable step size block backward differentiation formula for solving stiff initial value problems. Most of the methods stated are zero stable, A- stable or both, and displays different degree of accuracy of the scale error and executional time.

This study considers deriving a non – super class aspect of the variable step size block backward differentiation developed by Sulaiman *et al* (2013b) of the form

$$\sum_{j=0}^{4} \alpha_{j,i,r} y_{n+j-2} = h\beta_{k,i,r}(f_{n+k} - \rho f_{n+k-1}) \qquad k = 1,2,3 \tag{1}$$

The propose scheme is archive by reducing the order of (1) by one and consider the back value of y_{n-1} and y_n. The feature point, f_{n+k-1} replaced $-\rho f_{n+k-1}$ in (1), to came-up with the new non–super class formula of the form

$$\sum_{j=0}^{3} \alpha_{j,i,r} y_{n+j-1} = h\beta_{k,i,r}(f_{n+k} + f_{n+k-1}) \;\; k = 1, 2 \tag{2}$$

The proposed formula (2) is A variable step size multi-block backward differentiation formula for solving stiff initial value problems of ordinary differential equations of the form

$$y' = f(x,\ \hat{Y}), \qquad \hat{Y}(a) = \varphi\eta, \qquad a \le x \le b \tag{3}$$

where $\hat{Y} = (y_1, y_2, y_3, \dots \dots \dots y_n)$, $\eta\bar{\varphi} = (\varphi\eta_1, \varphi\eta_2, \varphi\eta_3, \dots, \varphi\eta_n)$

The system of ordinary differential equations (1) and can be solves by analytical or numerical methods. However, when the system is stiff, the analytical solution seems difficult in most cases, the preferences is geared toward obtaining the numerical aspect of solutions, that is what makes the block backward differentiation formulae more suitable in handling any sort of stiff IVPs of ODEs.

Methodology

Formulation of the Proposed Method (VSSMBBDF)

In this section, two approximate solution values y_{n+1} and y_{n+2} with step size h, are formulated in a block simultaneously. The formula is computed using two back values y_n and y_{n-1} with step size h. The linear difference operator L_i associated with (2) defined by

$$L\{y(x), h\} = \sum_{j=0}^{k}\left[\alpha_j y(x + jh) - h\beta_j y'(x + jh)\right]. \tag{3}$$

where$y(x)$is an arbitrary test function and it is continuously differential on [a, b].

The method (2) is constructed using a linear operator L_i. To derive the first and second points, define the linear operator L_1 and L_2 associated with (2) as

$$L_1[y(x_n), h)]: \alpha_{0,i}\, y_{n-1+}\alpha_{1,i}\, y_n + \alpha_{2,i}\, y_{n+1} + \alpha_{3,i}\, y_{n+2} - h\beta_{k,i}[f_{n+k} + f_{n+k-1}] = 0 \tag{4}$$

$$L_2[y(x_n), h)]: \alpha_{0,i}\, y_{n-1+}\alpha_{1,i}\, y_n + \alpha_{2,i}\, y_{n+1} + \alpha_{3,i}\, y_{n+2} - h\beta_{k,i}[f_{n+k} + f_{n+k-1}] = 0 \tag{5}$$

Case 1&2: $k = i = 1$&$k = i = 2$

The associated relationship for (4) and (5) are

$$\alpha_{0,1}y(x_n - rh) + \alpha_{1,1}y(x_n) + \alpha_{2,1}y(x_n + h) + \alpha_{3,1}y(x_n + 2h) - h\beta_{1,1}[f(x_n + h) + f(x_n)] = 0 \tag{6}$$

$$\alpha_{0,2}y(x_n - rh) + \alpha_{1,2}y(x_n) + \alpha_{2,2}y(x_n + h) + \alpha_{3,2}y(x_n + 2h)) - h\beta_{2,2}[f(x_n + 2h) + f(x_n + h)] = 0 \tag{7}$$

Expanding $y(x_n - rh)$, $y(x_n), y(x_n + h)$, $y(x_n + 2h)$, $f(x_n + h)$, $f(x_n + 2h)$ and $f(x_n)$ from (6) and (7) using Taylor's series expansion about x_n. We obtained the following coefficients for the two cases respectively

$$\left.\begin{aligned} C_{0,1} &= \alpha_{0,1} + \alpha_{1,1} + \alpha_{2,1} + \alpha_{3,1} = 0 \\ C_{1,1} &= -r\alpha_{0,1} + \alpha_{1,1} + 2\alpha_{2,1} - 2\beta_{1,1} = 0 \\ C_{2,1} &= \tfrac{1}{2}\alpha_{0,1} + \tfrac{1}{2}\alpha_{1,1} + 2\alpha_{2,1} - 3\beta_{1,1} = 0 \\ C_{3,1} &= -\tfrac{1}{6}\alpha_{0,1} + \tfrac{1}{6}\alpha_{1,1} + \tfrac{4}{3}\alpha_{2,1} - \tfrac{5}{2}\beta_{1,1} = 0 \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} C_{0,2} &= \alpha_{0,2}+\alpha_{1,2}+\alpha_{2,2}+\alpha_{3,2}=0\\ C_{1,2} &= -r\alpha_{0,2}+\alpha_{2,2}+2\alpha_{3,2}-2\beta_{2,2}=0\\ C_{2,2} &= \tfrac{1}{2}\alpha_{0,2}+\tfrac{1}{2}\alpha_{2,2}+2\alpha_{3,1}-3\beta_{1,1}=0\\ C_{3,2} &= -\tfrac{1}{6}\alpha_{0,2}+\tfrac{1}{6}\alpha_{2,2}+\tfrac{4}{3}\alpha_{3,2}-\tfrac{5}{2}\beta_{1,1}=0 \end{aligned}\right\} \qquad (9)$$

In deriving the first pointy_{n+1} and second point y_{n+2} the coefficient $\alpha_{2,1}$ and $\alpha_{3,2}$ are normalized to 1 solving the simultaneous equation (8) and (9) for the values of $\alpha_{j,i's}$, $\beta_{j,i's}$. Re-arranging and substituting the values in (4) and (5), to get the first and second points as

$$y_{n+1}=\frac{1}{2r+7}y_{n-1}+\frac{5r+13}{2(2r+7)}y_n-\frac{r+1}{2(2r+7)}+\frac{2(r+2)}{2r+7}hf_{n+1}+\frac{2(r+2)}{2r+7}hf_n \qquad (10)$$

$$y_{n+2}=\frac{2}{9r+17}y_{n-1}-\frac{3(r+1)}{9r+17}y_n+\frac{6(2r+3)}{9r+17}y_{n+1}+\frac{4(r+2)}{9r+17}hf_{n+2}+\frac{4(r+2)}{9r+17}hf_{n+1} \qquad (11)$$

Hence, (10) - (11) is called a variable step size multi-block backward differentiation formula (VSSMBBDF) for solving stiff initial value problems of ordinary differential equations.

From the proposed scheme different stable methods can be obtain by carefully varying the value of the step size ratio r.

Table 1: Variable step size ratios with the stable methods obtained

Step Size Ratio (r)	Approximate Points	Formulae (VSSMBBDF)
$r=1$	y_{n+1} y_{n+2}	$y_{n+1}=\frac{1}{9}y_{n-1}+y_n-\frac{1}{9}y_{n+2}+\frac{2}{3}hf_{n+1}+\frac{2}{3}hf_n$ $y_{n+2}=\frac{1}{13}y_{n-1}-\frac{3}{13}y_n+\frac{15}{13}y_{n+1}+\frac{6}{13}hf_{n+2}+\frac{6}{13}hf_{n+1}$
$r=2$	y_{n+1} y_{n+2}	$y_{n+1}=\frac{1}{11}y_{n-1}+\frac{22}{22}y_n-\frac{3}{22}y_{n+2}+\frac{8}{11}hf_{n+1}+\frac{8}{11}hf_n$ $y_{n+2}=\frac{2}{35}y_{n-1}-\frac{9}{35}y_n+\frac{6}{5}y_{n+1}+\frac{16}{35}hf_{n+2}+\frac{16}{35}hf_{n+1}$
$r=\frac{1}{2}$	y_{n+1} y_{n+2}	$y_{n+1}=\frac{1}{8}y_{n-1}+\frac{31}{32}y_n-\frac{3}{32}y_{n+2}+\frac{5}{8}hf_{n+1}+\frac{5}{8}hf_n$ $y_{n+2}=\frac{4}{43}y_{n-1}-\frac{9}{43}y_n+\frac{48}{43}y_{n+1}+\frac{20}{43}hf_{n+2}+\frac{20}{43}hf_{n+1}$
$r=\frac{5}{6}$	y_{n+1} y_{n+2}	$y_{n+1}=\frac{3}{26}y_{n-1}+\frac{103}{104}y_n-\frac{11}{104}y_{n+2}+\frac{17}{26}hf_{n+1}+\frac{17}{26}hf_n$ $y_{n+2}=\frac{4}{49}y_{n-1}-\frac{11}{49}y_n+\frac{8}{7}y_{n+1}+\frac{68}{147}hf_{n+2}+\frac{68}{147}hf_{n+1}$

Analysis of the Method

Zero Stability of the proposed formula (VSSMBBDF)

In this section, we analyze the stability property of the formula (10-11) for different step size ratio r.

Definition 1: A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple (Sulaiman *et al.*, 2013b)

Definition 2: A linear multistep Method is said to be an A-stable method if its stability region covers the entire negative half-plane (Sulaiman *et al.*, 2013b)

The stability of the method (10-11) can be obtains by applying the standard test equation of the form

$$y' = \lambda y \qquad \lambda \text{ is a complex number}, \qquad Re(\lambda) < 0 \tag{12}$$

$r = 1$

$$y_{n+1} = \frac{1}{9}y_{n-1} + y_n - \frac{1}{9}y_{n+2} + \frac{2}{3}hf_{n+1} + \frac{2}{3}hf_n \tag{13}$$

$$y_{n+2} = \frac{1}{13}y_{n-1} - \frac{3}{13}y_n + \frac{15}{13}y_{n+1} + \frac{6}{13}hf_{n+2} + \frac{6}{13}hf_{n+1} \tag{14}$$

Putting (12) into (13-14)

$$y_{n+1} = \frac{1}{9}y_{n-1} + y_n - \frac{1}{9}y_{n+2} + \frac{2}{3}h\lambda y_{n+1} + \frac{2}{3}h\lambda y_n \tag{15}$$

$$y_{n+2} = \frac{1}{13}y_{n-1} - \frac{3}{13}y_n + \frac{15}{13}y_{n+1} + \frac{6}{13}h\lambda y_{n+2} + \frac{6}{13}h\lambda y_{n+1} \tag{16}$$

(15) and (16) can be written in a matrix form as

$$\begin{bmatrix} 1-\frac{2}{3}h\lambda & \frac{1}{9} \\ -\frac{15}{13}-\frac{6}{13}h\lambda & 1-\frac{6}{13}h\lambda \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & 1+\frac{2}{3}h\lambda \\ \frac{1}{13} & -\frac{3}{13} \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} \tag{18}$$

where

$$A = \begin{bmatrix} 1-\frac{2}{3}h\lambda & \frac{1}{9} \\ -\frac{15}{13}-\frac{6}{13}h\lambda & 1-\frac{6}{13}h\lambda \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} \frac{1}{9} & 1+\frac{2}{3}h\lambda \\ \frac{1}{13} & -\frac{3}{13} \end{bmatrix} \tag{19}$$

To find the first characteristic polynomial for (15) & (16), we use

$\det[At - B] = 0$. To get the polynomial for $r = 1$ as follows

$$R_1(h,t) = \frac{44}{39}t^2 - \frac{14}{13}t^2h\lambda - \frac{40}{39}t + \frac{4}{13}t^2(h\lambda)^2 - \frac{4}{13}th\lambda - \frac{4}{39} - \frac{4}{13}t(h\lambda)^2 - \frac{2}{39}h\lambda \tag{20}$$

Using similar procedure as in above for the step sizes $r = 2, r = \frac{1}{2}$ *and* $r = \frac{5}{6}$ to obtain the following polynomial respectively

$$R_2(h,t) = \frac{64}{55}t^2 - \frac{432}{385}t^2h\lambda - \frac{416}{385}t + \frac{128}{385}t^2(h\lambda)^2 - \frac{576}{385}th\lambda - \frac{32}{385} - \frac{128}{385}t(h\lambda)^2 - \frac{16}{385}h\lambda \quad (21)$$

$$R_{\frac{1}{2}}(h,t) = \frac{95}{86}t^2 - \frac{44}{43}t^2h\lambda - \frac{85}{86}t + \frac{25}{86}t^2(h\lambda)^2 - \frac{105}{86}th\lambda - \frac{5}{43} - \frac{25}{86}t(h\lambda)^2 - \frac{5}{86}h\lambda \quad (22)$$

$$R_{\frac{5}{6}}(h,t) = \frac{102}{91}t^2 - \frac{680}{637}t^2h\lambda - \frac{646}{637}t + \frac{578}{1911}t^2(h\lambda)^2 - \frac{2482}{1911}th\lambda - \frac{68}{637} - \frac{578}{1911}t(h\lambda)^2 - \frac{34}{637}h\lambda \quad (23)$$

Put $h\lambda = H$ in (20-23), we have

$$R_1(H,t) = \frac{44}{39}t^2 - \frac{14}{13}t^2H - \frac{40}{39}t + \frac{4}{13}t^2H^2 - \frac{4}{13}tH - \frac{4}{39} - \frac{4}{13}tH^2 - \frac{2}{39}H \quad (24)$$

$$R_2(h,t) = \frac{64}{55}t^2 - \frac{432}{385}t^2H - \frac{416}{385}t + \frac{128}{385}t^2H^2 - \frac{576}{385}tH - \frac{32}{385} - \frac{128}{385}tH^2 - \frac{16}{385}H \quad (25)$$

$$R_{\frac{1}{2}}(h,t) = \frac{95}{86}t^2 - \frac{44}{43}t^2H - \frac{85}{86}t + \frac{25}{86}t^2H^2 - \frac{105}{86}tH - \frac{5}{43} - \frac{25}{86}tH^2 - \frac{5}{86}H \quad (26)$$

$$R_{\frac{5}{6}}(h,t) = \frac{102}{91}t^2 - \frac{680}{637}t^2H - \frac{646}{637}t + \frac{578}{1911}t^2H^2 - \frac{2482}{1911}tH - \frac{68}{637} - \frac{578}{1911}tH^2 - \frac{34}{637}H \quad (27)$$

Set $H = 0$ in (24-27) and solve for t in all the Polynomials. The following table is obtained with the respective roots (t) of the polynomials.

Table 2: Zero stability of the proposed formulae

Step Size Ratio (r)	Roots of the proposed methods
$r = 1$	$t = 1, -0.0909090909$
$r = 2$	$t = 1, -0.0714285714$
$r = \frac{1}{2}$	$t = 1, -0.1052631579$
$r = \frac{5}{6}$	$t = 1, -0.0952380952$

From table 2, it has been shown that the formula (10-11) with the step size ratios tested ($r = 1,\ r = 2,\ r = \frac{1}{2}$ *and* $r = \frac{5}{6}$) are Zero stable methods in accordance with definition 1.

A - Stability of the proposed formulae

The region for the stability of the proposed method is plotted, by considering the stability polynomials in (20), (21), (22) & (23). The set of point defined by $t = e^{i\theta}$, $0 \leq \theta \leq 2\pi$

describes the boundary of the stability region. The following stability region was the complex plot of the proposed method with the aid of Maple Software.

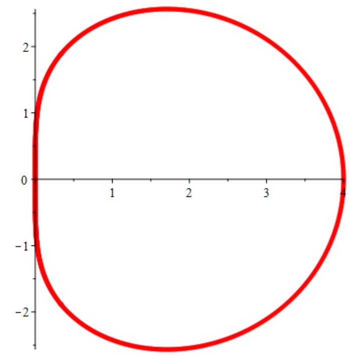

Figure 1: A-Stability region of the method with the step size ratio $r = 1$

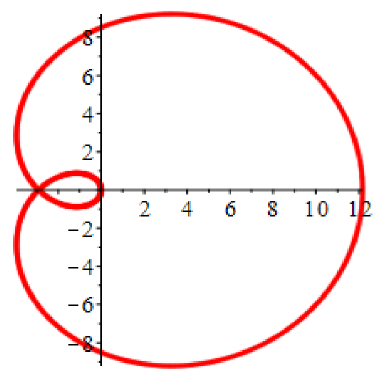

Figure 2: A-Stability region of the method with the step size ratio $r = 2$

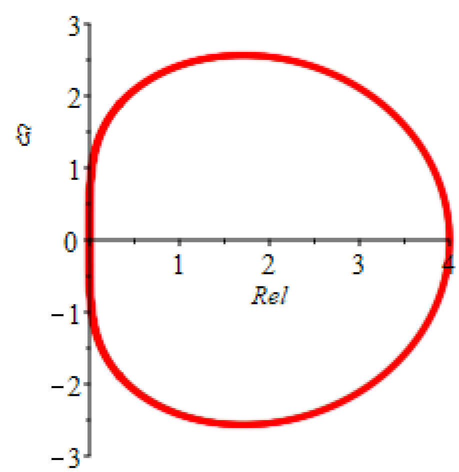

Figure 3: A-Stability region of the method with the step size ratio $r = \frac{1}{2}$

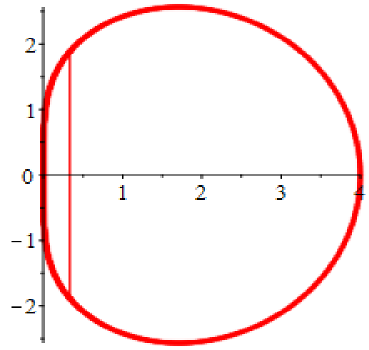

Figure 4: A-Stability region of the method with the step size ratio $r = \frac{5}{6}$

The region of the stability covered the entire negative left half plane in figure 1 - 4. Hence, the proposed formulae (VSSMBBDF) by definition 2 is an A-Stable method.

Implementation of the method

In this section Newton's iteration is considered for the implementation of the method (10)–(11) across different choice of the step size ratio. The method (10)–(11) can be written in the following form:

$$\left.\begin{aligned} y_{n+1} &= \theta_1 y_{n+2} + \gamma_1 h f_{n+1} + \gamma_2 h f_n + v_1 \\ y_{n+2} &= \theta_2 y_{n+1} + \gamma_3 h f_{n+1} + \gamma_4 h f_{n+2} + v_2 \end{aligned}\right\} \tag{28}$$

Where v_1 and v_2 are the back values. (28) will be transforming to the following form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} + h\begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} + h\begin{bmatrix} \gamma_1 & 0 \\ \gamma_3 & \gamma_4 \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{29}$$

(29) can also be written as

$$(I - B)Y = h(C_1 G_1 + C_2 G_2) + \in \tag{30}$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} \gamma_1 & 0 \\ \gamma_3 & \gamma_4 \end{bmatrix}, G_1 = \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}, G_2 = \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} \tag{31}$$

Let

$$G = (I - B)Y - h(C_1 G_1 + C_2 G_2) - \in = 0 \tag{32}$$

Therefore, the Newton's iteration for the proposed method (VSSMBBDF) is going to be

$$Y_{n+1,n+1}^{(i+1)} - Y_{n+1,n+1}^{(i)} = -\left[G_j'\left(Y_{n+1,n+1}^{(i)}\right)\right]^{-1}\left[G_j\left(Y_{n+1,n+1}^{(i)}\right)\right] \tag{33}$$

Equation (33) is represented as

$$Y_{n+1,n+1}^{(i+1)} - Y_{n+1,n+1}^{(i)} = [(I - B) - hC_1 \frac{\delta G_1}{\delta Y}\left(Y_{n+1,n+1}^{(i)}\right) - hC_2 \frac{\delta G_2}{\delta Y}\left(Y_{n+1,n+1}^{(i)}\right)]^{-1} \times [(I - B)\left(Y_{n+1,n+1}^{(i)}\right) - hC_1 G_1 - hC_2 G_2 - \in] \tag{34}$$

To compute the maximum error from the proposed algorithm, let $y_i \; and \; y(x_i)$ be the approximate and the exact solution of (3), respectively. The absolute error is given by

$(error_i)_t = |(y_i)_t - (y(x_i)_t)|$ (35)

The maximum error (MAXE) is given by

$$MAXE = \underbrace{max}_{1 \le i \le T} (\underbrace{max\,(error_i)_t}_{1 \le i \le N}). \quad (36)$$

where T is the total number of steps and N is the number of equations.

Let $Y_{n+1}^{(i+1)}$ denote the $(i+1)$th iteration and $E_{1,2}^{(i+1)} = Y_{n+1,n+1}^{(i+1)} - Y_{n+1,n+1}^{(i)}$ (37)

From (34) we have

$$\mathrm{E}_{1,2}^{(i+1)} = \overline{\mathrm{B}}^{-1} - \overline{\mathrm{C}} \quad (38)$$

which is equivalent to

$\overline{\mathrm{B}}\mathrm{E}_{1,2}^{(i+1)} = \overline{\mathrm{C}}$

(39)

where

$$\overline{\mathrm{B}} = [(I - B) - hC_1 \frac{\delta G_1}{\delta Y}\left(Y_{n+1,n+1}^{(i)}\right) - hC_2 \frac{\delta G_2}{\delta Y}\left(Y_{n+1,n+1}^{(i)}\right)]^{-1} \quad (40)$$

And

$$\overline{\mathrm{C}} = -[(I - B)\left(Y_{n+1,n+1}^{(i)}\right) - hC_1G_1 - hC_2G_2 - \in] \quad (41)$$

Newton's iteration would therefore be used to solve the system (39) for the different values of r,

$$\overline{\mathrm{B}} = \begin{bmatrix} 1 - \gamma_1 h \frac{\delta f_{n+1}}{\delta y_{n+1}} & -\theta_1 \\ -\theta_2 - \gamma_3 h \frac{\delta f_{n+1}}{\delta y_{n+1}} & 1 - \gamma_4 h \frac{\delta f_{n+2}}{\delta y_{n+2}} \end{bmatrix}, \quad \overline{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^i + \theta_1 y_{n+2}^i + \gamma_1 h f_{n+1}^i + \gamma_2 h f_n^i + v_1 \\ -y_{n+2}^i + \theta_2 y_{n+1}^i + \gamma_3 h f_{n+1}^i + \gamma_4 h f_{n+2}^i + v_2 \end{bmatrix}$$

when r = 1

$$\overline{\mathrm{B}} = \begin{bmatrix} 1 - \frac{2}{3} h \frac{\delta f_{n+1}}{\delta y_{n+1}} & \frac{1}{9} \\ -\frac{15}{13} - \frac{6}{13} h \frac{\delta f_{n+1}}{\delta y_{n+1}} & 1 - \frac{6}{13} h \frac{\delta f_{n+2}}{\delta y_{n+2}} \end{bmatrix}, \overline{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^i - \frac{1}{9} y_{n+2}^i + \frac{2}{3} h f_{n+1}^i + \frac{2}{3} h f_n^i + v_1 \\ -y_{n+2}^i + \frac{15}{13} y_{n+1}^i + \frac{6}{13} h f_{n+1}^i + \frac{6}{13} h f_{n+2}^i + v_2 \end{bmatrix}$$

When $r = 2$

$$\overline{\mathrm{B}} = \begin{bmatrix} 1 - \frac{8}{11} h \frac{\delta f_{n+1}}{\delta y_{n+1}} & \frac{3}{22} \\ -\frac{6}{5} - \frac{16}{35} h \frac{\delta f_{n+1}}{\delta y_{n+1}} & 1 - \frac{16}{35} h \frac{\delta f_{n+2}}{\delta y_{n+2}} \end{bmatrix}, \overline{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^i - \frac{3}{22} y_{n+2}^i + \frac{8}{11} h f_{n+1}^i + \frac{8}{11} h f_n^i + v_1 \\ -y_{n+2}^i + \frac{5}{6} y_{n+1}^i + \frac{16}{35} h f_{n+1}^i + \frac{16}{35} h f_{n+2}^i + v_2 \end{bmatrix}$$

When $r = \frac{1}{2}$

$$\overline{\mathrm{B}} = \begin{bmatrix} 1-\frac{5}{8}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{3}{32} \\ -\frac{48}{43}-\frac{20}{43}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & 1-\frac{20}{43}h\frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}, \ \overline{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^{i} - \frac{3}{32}y_{n+2}^{i} + \frac{5}{8}hf_{n+1}^{i} + \frac{5}{8}hf_{n}^{i} + v_1 \\ -y_{n+2}^{i} + \frac{48}{43}y_{n+1}^{i} + \frac{20}{43}hf_{n+1}^{i} + \frac{20}{43}hf_{n+2}^{i} + v_2 \end{bmatrix}$$

When $r = \frac{5}{6}$

$$\overline{\mathrm{B}} = \begin{bmatrix} 1-\frac{17}{26}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & \frac{11}{104} \\ -\frac{8}{7}-\frac{68}{147}h\frac{\partial f_{n+1}}{\partial y_{n+1}} & 1-\frac{68}{147}h\frac{\partial f_{n+2}}{\partial y_{n+2}} \end{bmatrix}, \ \overline{\mathrm{C}} = \begin{bmatrix} -y_{n+1}^{i} - \frac{11}{104}y_{n+2}^{i} + \frac{17}{26}hf_{n+1}^{i} + \frac{17}{26}hf_{n}^{i} + v_1 \\ -y_{n+2}^{i} + \frac{8}{7}y_{n+1}^{i} + \frac{68}{147}hf_{n+1}^{i} + \frac{68}{147}hf_{n+2}^{i} + v_2 \end{bmatrix}$$

Results and Discussion

Numerical Examples

In this section the proposed method a variable step size multi-block backward differentiation formula for solving stiff initial value problem of ordinary differential equations developed in (10-11) will be adopted to solve some IVPs and the results are compared with some existing scheme in the literature to evaluate the performance of the new method. Below are some of the numerical problems that were considered.

Problem 1: (Ibrahim *et al*, 2007)

$y' = -10xy \qquad y(0) = 1 \qquad 0 \leq x \leq 10$

Exact Solution

$y(x) = e^{-5x}$

Problem 2: (Ibrahim *et al*, 2007)

$y_1' = 198\mathrm{y}_1 + 199\mathrm{y}_1 y_1(0) = 1 0 \leq x \leq 10$

$y_2' = -398y_1 - 399y_2 y_2(0) = -1$

Exact Solution

$y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$

Below are symbols and notations used in the research with results of the problems solved as tabulated to depict the comparative differences among the methods considered.

h = step-size

MAXE = Maximum Error

NS = Number of Steps

TIME =Time in microseconds

VSSMBBDF = Variable Step Size Multi-Block Backward Differentiation Formula

3ESBBDF = Extended 3 Point Super Class of Block Backward Differentiation Formula

2BBDF = 2 Point Block Backward Differentiation Formula.

Table 3: Numerical results for problem 1 (with different step size ratio r)

h	Method	NS	MAXE	TIME
10^{-2}	VSSMBBDF ($r = 1$)	333	2.41547e-05	7.35331e-005
	VSSMBBDF ($r = 2$)	333	4.42218e-04	7.36289e-002
	VSSMBBDF ($r = 1/2$)	100	5.72116e-05	8.23891e-003
	VSSMBBDF ($r = 5/6$)	100	3.74336e-04	4.31415e-003
10^{-3}	VSSMBBDF ($r = 1$)	3,333	3.80282e-07	7.77357e-004
	VSSMBBDF ($r = 2$)	3,333	5.27560e-06	5.81512e-002
	VSSMBBDF ($r = 1/2$)	1,000	3.16238e-07	6.52651e-002
	VSSMBBDF ($r = 5/6$)	1,000	2.06882e-06	3.24902e-002
10^{-4}	VSSMBBDF ($r = 1$)	33,333	4.10508e-09	7.60323e-003
	VSSMBBDF ($r = 2$)	33,333	5.27691e-08	5.81491e-001
	VSSMBBDF ($r = 1/2$)	10,000	3.16292e-09	6.52624e-001
	VSSMBBDF ($r = 5/6$)	10,000	2.08874e-08	1.94953e-001
10^{-5}	VSSMBBDF ($r = 1$)	333,333	4.15121e-11	7.70577e-002
	VSSMBBDF ($r = 2$)	333,333	5.27942e-10	5.81122e+000
	VSSMBBDF ($r = 1/2$)	100,000	3.16631e-11	6.62791e+001
	VSSMBBDF ($r = 5/6$)	100,000	2.08146e-10	1.52607e+001
10^{-6}	VSSMBBDF($r = 1$)	3,333,333	4.15739e-13	7.65324e-001
	VSSMBBDF($r = 2$)	3,333,333	5.28009e-12	5.79987e+001
	VSSMBBDF($r = 1/2$)	1,000,000	3.21089e-13	6.52197e+000
	VSSMBBDF($r = 5/6$)	1,000,000	2.10218e-12	1.42317e+000

Table 4: Numerical results for problem 2 (with different step size ratio r)

h	Method	NS	MAXE	TIME
10^{-2}	VSSMBBDF ($r = 1$)	333	3.26548e-05	6.25231e-005
	VSSMBBDF ($r = 2$)	333	7.83217e-04	5.23172e-003
	VSSMBBDF ($r = 1/2$)	100	2.18419e-05	1.66324e-004
	VSSMBBDF ($r = 5/6$)	100	6.59871e-03	7.86022e-004
10^{-3}	VSSMBBDF ($r = 1$)	3,333	3.65283e-07	5.67257e-004
	VSSMBBDF ($r = 2$)	3,333	5.05338e-06	4.23012e-002
	VSSMBBDF ($r = 1/2$)	1,000	2.20071e-07	5.40174e-003
	VSSMBBDF ($r = 5/6$)	1,000	4.43161e-05	8.55109e-003
10^{-4}	VSSMBBDF ($r = 1$)	33,333	4.70027e-09	3.50223e-003
	VSSMBBDF ($r = 2$)	33,333	5.26692e-08	4.22943e-001
	VSSMBBDF ($r = 1/2$)	10,000	2.20445e-09	3.73910e-002
	VSSMBBDF ($r = 5/6$)	10,000	4.43208e-07	4.82591e-002
10^{-5}	VSSMBBDF ($r = 1$)	333,333	4.10002e-11	6.60477e-002
	VSSMBBDF ($r = 2$)	333,333	5.32740e-10	4.22582e+000
	VSSMBBDF ($r = 1/2$)	100,000	2.20472e-11	5.29043e-001
	VSSMBBDF ($r = 5/6$)	100,000	4.44001e-09	6.75021e-001
10^{-6}	VSSMBBDF ($r = 1$)	3,333,333	4.14240e-13	2.55224e-001
	VSSMBBDF ($r = 2$)	3,333,333	5.33362e-12	4.22172e+001
	VSSMBBDF ($r = 1/2$)	1,000,000	2.20493e-13	2.64921e+000
	VSSMBBDF ($r = 5/6$)	1,000,000	4.44108e-11	5.73214e+000

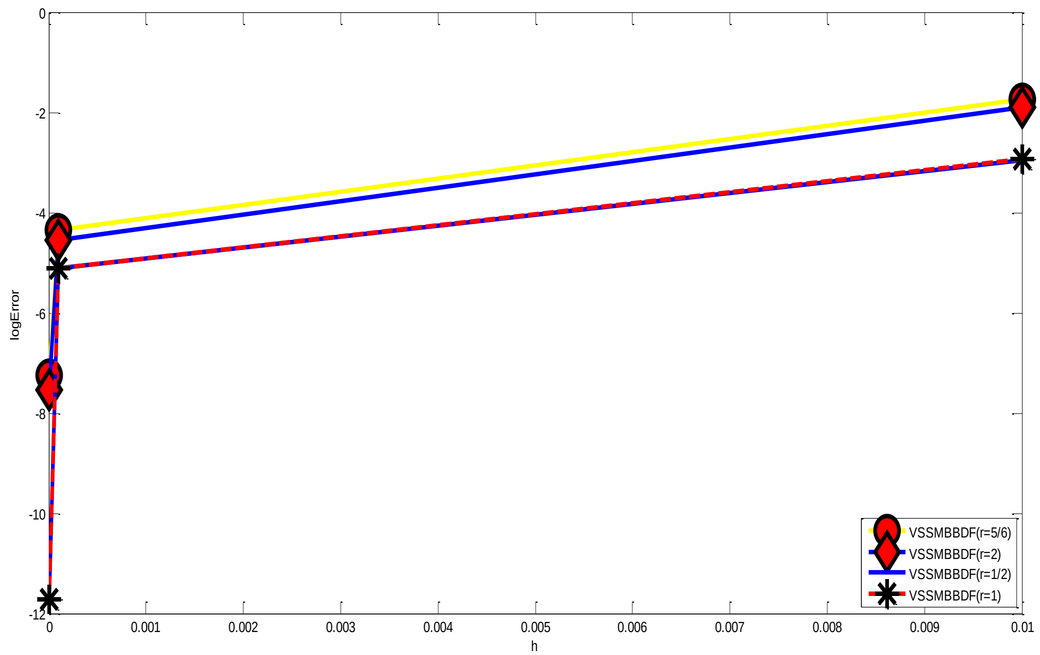

Figure 5: Comparison of $\mathbf{Log_{10}}(MAXE)$ against h for problem 1 (from table 3)

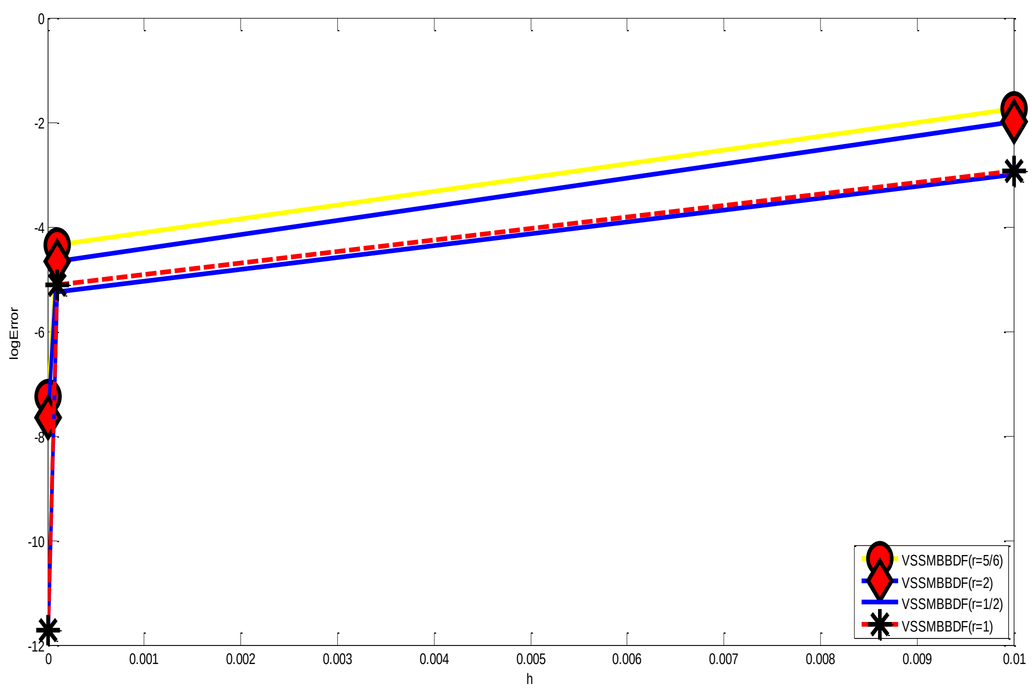

Figure 6: Comparison for $\mathbf{Log_{10}}(MAXE)$ against h for problem 2 (from table 4)

From figures 5 & 6, the proposed Formula (VSSMBBDF) possessed different degree of accuracy across different chosen step size ratio. However, the result indicated that the methods with $r = 1$ & $r = 1/2$ have least scale error compared to the methods $r = 2$ & $r = 5/6$ in all the problems considered in the research. The executional time also favors the schemes with $r = 1$ in problem 1 & 2. However, the methods with $r = 1/2$ & $r = 5/6$ are closely competing in the execution time. All the schemes have advantages over the scheme with $r = 2$ in all the problems solved.

Table 5: Comparison of results for problem 1 (with $r = 1$)

h	Method	NS	MAXE	TIME
10^{-2}	VSSMBBDF ($r = 1$)	333	2.41547e-05	7.35331e-005
	3ESBBDF	333	3.73308e-03	6.64306e-004
	2BBDF	500	2.47600e-02	15,328
10^{-3}	VSSMBBDF ($r = 1$)	3,333	3.80282e-07	7.77357e-004
	3ESBBDF	3,333	4.85429e-05	5.62257e-003
	2BBDF	5,000	2.86614e-03	127,105
10^{-4}	VSSMBBDF ($r = 1$)	33,333	4.10508e-09	7.60323e-003
	3ESBBDF	33,333	4.85783e-07	5.48808e-002
	2BBDF	55,555	2.90520e-04	125,5816
10^{-5}	VSSMBBDF ($r = 1$)	333,333	4.15121e-11	7.70577e-002
	3ESBBDF	333,333	4.85873e-09	5.46692e-001
	2BBDF	555,555	2.90911e-05	12,571,049
10^{-6}	VSSMBBDF ($r = 1$)	3,333,333	4.15739e-12	7.65324e-001
	3ESBBDF	3,333,333	5.08727e-11	5.47681e+000
	2BBDF	5,555,555	2.90951e-06	125,811,893

Table 6: Comparison of results for problem 2 (with $r = 1$)

h	Method	NS	MAXE	TIME
10^{-2}	VSSMBBDF ($r = 1$)	333	3.26548e-05	6.25231e-005
	3ESBBDF	333	034.58309e-04	5.54206e-004
	2BBDF	500	7.18323e-03	28,413
10^{-3}	VSSMBBDF ($r = 1$)	3,333	3.65283e-07	5.67257e-004
	3ESBBDF	3,333	044.70430e-05	8.52157e-003
	2BBDF	5,000	7.34012e-04	256,695
10^{-4}	VSSMBBDF ($r = 1$)	33,333	4.70027e-09	3.50223e-003
	3ESBBDF	33,333	054.90784e-07	4.38708e-002
	2BBDF	55,000	7.35584e-05	2,554,368
10^{-5}	VSSMBBDF ($r = 1$)	333,333	4.10002e-11	6.60477e-002
	3ESBBDF	333,333	065.70874e-09	5.36592e-001
	2BBDF	555,555	7.35741e-06	25,625,785
10^{-6}	VSSMBBDF ($r = 1$)	3,333,333	4.14240e-12	2.55224e-001
	3ESBBDF	3,333,333	075.77228e-11	4.37581e+000
	2BBDF	5,555,555	7.35747e-07	256,394,582

To visibly highlight the performance of the proposed method VSSMBBDF in relation to the other methods 3ESBBDF and 2BBDF. The graphs of *Log10(MAXE)* against *h* for the problems tested are plotted below.

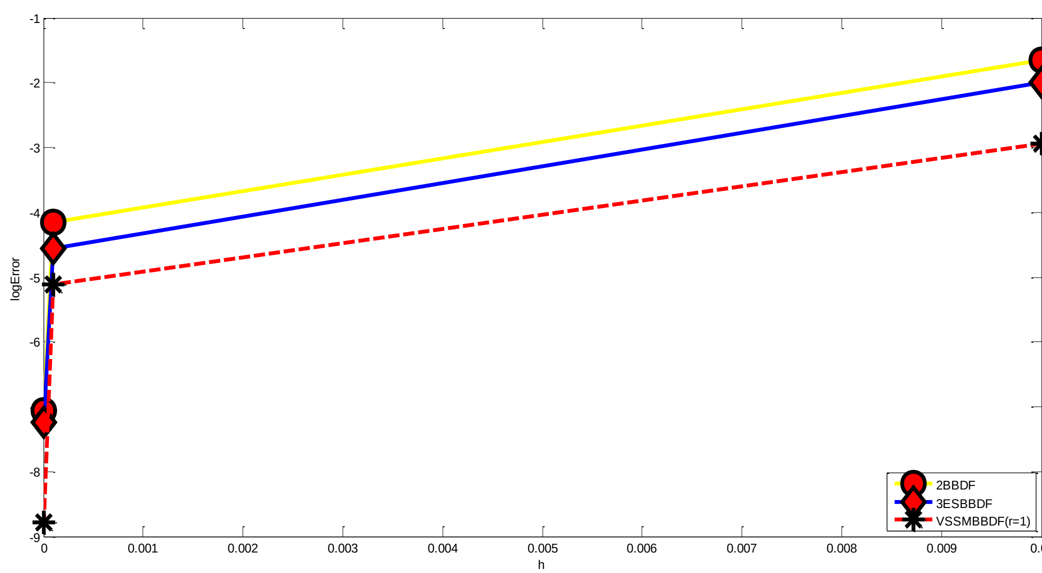

Figure 7: Graph of $\mathbf{Log_{10}}(MAXE)$ against h for problem 1

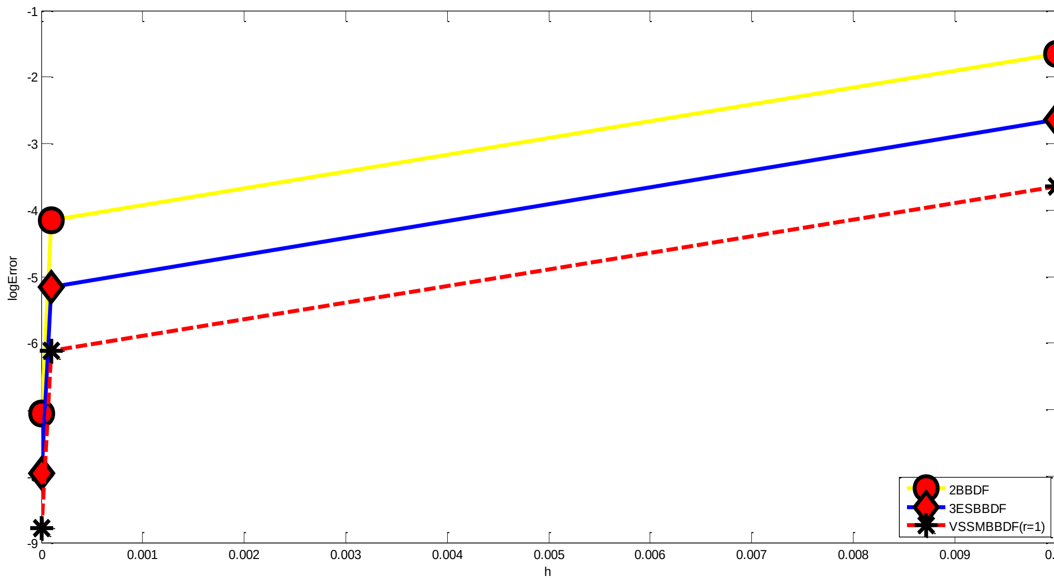

Figure 8: Graph of $\mathbf{Log_{10}}(\boldsymbol{MAXE})$ against h for problem 2

Considering the results in tables 5 and 6 of problems 1 and 2 have shown that the new method VSSMBBDF($r = 1$) outperformed the 3ESBBDF and 2BBDF in terms of accuracy, computational time in all problems tested, while the accuracy of the approximated solution increases as the step size decreases. However, 3ESBBDF outperformed the 2BBDF in terms of accuracy. But 3ESBBDF and 2BBDF competes closely in computational time with 2BBDF having advantage in problem 2. Similarly, the graphs in Figure 7 and 8 also shows clearly that the scaled errors for the VSSMBBDF($r = 1$) is smaller when compared with that in 3ESBBDF and 2BBDF Method based on the tested problems. However, the 3ESBBDF has advantage over 2BBDF in terms of accuracy.

Conclusion

A variable step size multi-block backward differentiation formula (VSSMBBDF) for solving stiff initial value problem of ordinary differential equations was derived. The proposed methods adopted a variable step size technique and possessed a very good stability property; the method is zero stable and A- stable across different chosen values of the step size ratio of $r = 1, r = 2,\ r = \frac{1}{2}\ and\ r = \frac{5}{6}$. The proposed methods solved samples of first order stiff IVPs of ODEs, the results are tabulated and the graphs are plotted, both have clearly highlighted the performance of the proposed methods in terms of accuracy of the scaled error and executional time compared to two other methods considered in the work. Hence, the proposed method can be used in solving a system of first order stiff initial value problem of ordinary differential equations.

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