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Approximate Maximum Likelihood Estimation in Fractional Stochastic Transport Equation

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Abstract. We estimate the drift of the fractional stochastic transport equation the by maximum likelihood and the minimum contrast methods. We show consistency and asymptotic normality of the estimators. We consider both continuous and discrete time observations.

1. Introduction

Stochastic transport equation (STE) has applications in biophysics, statistical physics, climate and weather sciences, interface growth, turbulence in fluid dynamics, polymer structure, finance and sports. The STE can be used to model air pollution, dye dispersion or traffic flow with the solution representing the density of the pollutant (or dye or traffic) at position x and time t. STE can be useful for modeling long-range correlations of DNA sequences. Molecular motors play a key role for generation of movements and forces in cells. STE can be useful for modeling in biophysis, e.g., what is the maximal excursion of a molecular motor against or in the average direction of the motor within a given time? How long does it take a motor to reach its maximum excursion against the chemical bias? What is the entropy production associated with an extreme fluctuation of a molecular motor? Other examples are microtubule catastrophes or a sperm winning a race against a billion competitors. Stochastic nonlinear transport equation has particular applications which involve a two-phase fluid flow, which has been used to study the flow of water through oil in a porous medium. For porous media flows, the spatial variations of porous formations occur on all length scales, but only variations at the largest length scales are reliably reconstructed from data. The heterogeneities occurring in the smaller lengths scales are incorporated stochastically.

A stochastic partial differential equation (SPDE) is a continuous version of simultaneous cross-section time series model. For a fixed spatial mode, it is an autoregressive time series (recall that an Ornstein-Uhlenbeck process is a continuous limit of Gaussian AR(1) process) and for a fixed time, it is a regression model. One can study asymptotic estimation for one fixed time point with

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large number of spatial observations or at a fixed spatial mode with large number of discrete time points, or simultaneous large spatial and temporal observations. We consider the estimation based on a fixed spatial mode with large number of randomly spaced time points, where the inter arrival times are exponentially distributed. Thus we have a random time sampling at a fixed space point. What mainly distinguishes SPDE from classical models is the type of sampling and the unusual rate of convergence of the estimators. Applications of SPDE model is numerous, e.g, in cell biology, neurophysiology, turbulence, oceanography and finance: see Itô [1], Walsh [2], Kallianpur and Xiong [3], Holden et al. [4], Adler et al. [5], Carmona and Rozovskii [6] and Bishwal [7]. Bishwal [7] studied asymptotic inference for fractional SPDE model for neurobiology. Recently SPDE has been used to model cell repolarization (stochastic Meinhardt model) and parameter estimation techniques developed for linear SPDE models have been applied to this model when the space resolution is finer, see Altmeyer et al. [8]. In this paper we study the model used in climate variability and predictability. Bishwal [9] studied estimation and hypothesis testing on nonlinear SPDEs from both continuous and discrete observations. Bishwal [10] studied estimation by the mixingale estimation function method for SPDEs with random sampling. Discrete observations in time of continuous models are important for practical applications, e.g. stochastic volatility models, see Bishwal [11].

Types of operators appearing in SPDE makes the estimation problem simple or difficult. Commuting operators have the same system of eigenvectors and make the corresponding finite dimensional projections as diffusion processes. The case is not so for noncommuting operators. Noncommuting operators appear in quantum mechanics. According to Heisenberg's uncertainty principle, if two operators representing a pair of variables do not commute, then the part of variables are mutually complementary, which means they can not be simultaneously measured or known precisely.

Parameter estimation is an inverse problem. Loges [12] initiated the study of parameter estimation in infinite dimensional stochastic differential equations. When the length of the observation time becomes large, he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges [13] extended the work of Loges [12] to minimum contrast estimators. Koski and Loges [14] applied the work to a stochastic heat flow problem. See Bishwal [15] for estimation results on likelihood asymptotics and Bayesian asymptotics for drift estimation of finite and infinite dimensional stochastic differential equations. See Bishwal [16] for asymptotic statistical results for discretely sampled diffusions.

Huebner et al. [17] started statistical investigation in SPDEs. They gave two contrast examples of parabolic SPDEs in one of which they obtained consistency, asymptotic normality and asymptotic efficiency of the MLE as noise intensity decreases to zero under the condition of absolute continuity of measures generated by the process for different parameters (the situation is similar to the classical finite dimensional case) and in the other they obtained these properties as the finite dimensional projection becomes large under the condition of singularity of the measures generated by the process for different parameters. The second example was extended by Huebner and Rozovskii [18] and the first example was extended by Huebner [19] to MLE for general parabolic SPDEs where the partial differential operators commute and satisfy different order conditions in the two cases.

Huebner [20] extended the problem to the ML estimation of multidimensional parameter. Lototsky and Rozovskii [21] studied the same problem without the commutativity condition. Small noise asymptotics of the nonparmetric estimation of the drift coefficient was studies by Ibragimov and Khasminskii [22].

The Bernstein-von Mises theorem (BVT, in short), concerning the convergence of suitably normalized and centered posterior distribution to normal distribution, plays a fundamental role in asymptotic Bayesian inference, see Le Cam and Yang [23]. Borwanker et al. [24] obtained the BVT for discrete time Markov processes. Bose [25] extended the BVT to the homogeneous nonlinear diffusions. As a further refinement in BVT, Bishwal [26] obtained sharp rates of convergence to normality of the posterior distribution and the Bayes estimators for the Ornstein-Uhlenbeck process.

All these above work on BVT are concerned with finite dimensional SDEs. Bishwal [27] proved the BVT and obtained asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding ergodic diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE.

Bishwal [28] obtained BVT and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In that case, the measures generated by the process for different parameters are singular. Here we treat the case when the measures generated by the process for different parameters are absolutely continuous under some conditions on the order of the partial differential operators. Bishwal [29] studied the asymptotic properties of the posterior distributions and Bayes estimators when we have either fully observed process or finite-dimensional projections. The asymptotic parameter is only the intensity of noise. In this paper we consider estimation for the fractional stochastic transport equation by the maximum likelihood and the minimum contrast method. Note that for the finite dimensional fractional Ornstein-Uhlenbeck process, Berry-Esseen inequalities of minimum contrast estimators based on continuous and discrete observations was studied in Bishwal [30].

We need the following preliminary results on LLN and CLT to prove our main theorems.

Lemma 1.1 *(LLN)*
Let $\xi_n, n \geq 1$ be a sequence of random variables and $b_n, n \geq 1$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$ and

$$\sum_{n=1}^{\infty} \frac{Var(\xi_n)}{b_n^2} < \infty.$$

i) If the random variables ξ_n are independent then,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}(\xi_k - E\xi_k)}{b_n} = 0 \ a.s.$$

ii) If the random variables ξ_n are uncorrelated then,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}(\xi_k - E\xi_k)}{b_n} = 0 \ in\ probability.$$

Part (i) is from Shiryayev [31]. Part (ii) can be proved by using Markov inequality.

Lemma 1.2 *(CLT for Stochastic Integrals)*
Let $\mathcal{S} = (\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{t\geq 0}, \{W_k\}_{k\geq 1})$ be a stochastic basis. Suppose that $\sigma_k \in L^2(\Omega; L^2([0,T]))$ be a sequence of real valued predictable processes such that

$$\frac{\sum_{k=1}^{n}\int_0^T \sigma_k dW_k(t)}{\left(\sum_{k=1}^{n} E\int_0^T \sigma_k^2 dt\right)^{1/2}} \to 1 \ in\ probability\ as\ n \to \infty.$$

Then

$$\frac{\sum_{k=1}^{n}\int_0^T \sigma_k dW_t}{\left(\sum_{k=1}^{n} E\int_0^T \sigma_k^2 dt\right)^{1/2}} \to^{\mathcal{D}} \mathcal{N}(0,1) \ as\ n \to \infty.$$

2. Approximate Maximum Likelihood Estimation

Consider the fractional stochastic transport equation (fSTE) which governs the transport of a substance which is dispersing in a moving medium in the d-dimensional space:

$$dU_t(x) = (-\nu u - (\mathbf{v}\cdot\nabla)U_t(x) + D\nabla^2 U_t(x))dt + dW_t^H(x)$$

where the average velocity $\mathbf{v} = (v_1, v_2, \ldots, v_d)$, diffusivity $D > 0$, the leakage rate (Newton's cooling coefficient or feedback factor) $\nu > 0$ are constants, ∇ is the gradient, ∇^2 is the horizontal Laplacian and $W_t^H(x)$ is a cylindrical fractional Brownian motion with Hurst parameter $H > 0.5$.

A fractional Brownian motion (fBM) has the covariance

$$\widetilde{C}_H(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right], \quad s, t > 0.$$

For $H > 0.5$ the process has long range dependence or long memory and the process is self-similar. For $H \neq 0.5$, the process is neither a Markov process nor a semimartingale. For $H = 0.5$, the process reduces to standard Brownian motion. Fractional Brownian motion can be represented as a Riemann-Liouville (fractional) derivative of Gaussian white noise, see Decreusefond and Ustunel [32] and Jumarie [33]. For deterministic fractional calculus, see Samko et al. [34].

The process $W_t^H(x)$ is the total heat flux through the upper and lower boundaries of the ocean mixed layer. The process $U_t(x) = U(t,x)$ denotes the concentration of the substance at time t at the point x in a bounded domain G. Δ is the Laplacian and ∇ is the gradient with respect to the spatial variable x.

The problem is to estimate a component of the velocity $\mathbf{v}$, diffusivity D and the feedback parameter ν under observations of the field $U(t,x)$ in some region for few time moments. In other words, the problem is how to evaluate the current and lateral and vertical heat interchange from a small number of satellite images of sea surface temperature (SST).

Let us introduce the partial differential operators. Let

$$A_k\phi_m := \nu_{km}\phi_m, \quad k = 0, 1$$

where

$$\nu_{0m} = \alpha_{0m} + i\beta_{0m}, \quad \nu_{1m} = \alpha_{1m} + i\beta_{1m}, \quad i = \sqrt{-1}$$

and set

$$\alpha_m(\theta) := \alpha_{0m} + \alpha_{1m}\theta, \quad m \geq 1.$$

where $\{\phi_m, m \geq 1\}$ is a fixed orthogonal basis The most important observation we will focus on is that roughly speaking the condition

$$\sum_{m=1}^{\infty}\left[\frac{\alpha_{1m}^2}{\alpha_m(\theta)^2} + h^2\beta_{1m}^2 e^{-2h\alpha_m(\theta)}\right] = \infty$$

is necessary and sufficient for the consistency of the minimum contrast estimator (MCE) where the time interval $h = \min_{1\leq j\leq N-1}(t_{j+1} - t_j)$. In particular, for self-adjoint operators this becomes

$$\sum_{m=1}^{\infty}\frac{\alpha_{1m}^2}{\alpha_m(\theta)^2} = \infty.$$

If A_0 and A_1 are elliptic differential operators then this is equivalent to

$$order(A_0 + \theta_1) - order(A_1) \leq d/2.$$

MCE of D is consistent because $order(A_\theta) = order(A_1) = d = 2$ while the MCE of ν is not consistent because in this case $order(A_1) = 0$.

Here $A^\nu := \nu + \mathbf{v}\cdot\nabla - D\nabla^2$. Our aim is to estimate ν by the maximum likelihood and minimum contrast method based on continuous and discrete observations of the amplitudes when v and D are known. The amplitudes

$$U_m(t) := \int_G U(t,x)\phi_m(x)dx$$

are independent complex-valued processes obeying the Ornstein-Uhlenbeck equations

$$\dot{U}_m + \nu_m(\theta)U_m = \sigma_m\dot{w}_m, \quad m \geq 1.$$

where $\dot{w}_m$ is color noise. The random field $U(t,x)$ is observed at discrete time points t and discrete positions x. Equivalently, the Fourier coefficients $U_m(t)$ are observed at discrete time points. Thus the spatial resolution is a stochastic interacting particle system, see Ligget [35]. We obtain the asymptotics with increasing space resolution provided the time interval is finite.

Let

$$U_m^1(t_j) := \mathrm{Re}(U_m(t)), \quad U_m^2(t_j) := \mathrm{Im}(U_m(t)).$$

The invariant distribution is given by

$$U_m|_{t=0} \sim \mathcal{N}\left(0, \frac{\sigma_m^2}{2\alpha_m(\theta)}\right).$$

Now we focus on the fundamental semimartingale behind the fSTE model. Define

$$\begin{aligned}
\kappa_H &:= 2H\Gamma(3/2-H)\Gamma(H+1/2), \quad k_H(t,s) := \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H},\\
\lambda_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, \quad v_t \equiv v_t^H := \lambda_H^{-1}t^{2-2H}, \quad \mathcal{M}_t^H := \int_0^t k_H(t,s)dW_s^H.
\end{aligned}$$

From Norros et al. [36] it is well known that $\mathcal{M}_t^H$ is a Gaussian martingale, called the fundamental martingale whose variance function $\langle \mathcal{M}^H\rangle_t$ is v_t^H. Moreover, the natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the fBm W^H since

$$W_t^H := \int_0^t K(t,s)d\mathcal{M}_s^H$$

holds for $H \in (1/2, 1)$ where

$$K_H(t,s) := H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, \quad 0 \le s \le t$$

and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define

$$Q_t := \frac{d}{dv_t}\int_0^t k_H(t,s)U_s ds.$$

It is easy to see that

$$Q_t = \frac{\lambda_H}{2(2-2H)}\left\{t^{2H-1}Z_t + \int_0^t r^{2H-1}dZ_s\right\}.$$

Define the process $Z = (Z_t, t \in [0,T])$ by

$$Z_t := \int_0^t k_H(t,s)dU_s.$$

The following facts are known from Kleptsyna and Le Breton [37]:

(i) Z is the fundamental semimartingale associated with the process X.

(ii) Z is a $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_t = \theta\int_0^t Q_s dv_s + \mathcal{M}_t^H.$$

(iii) U admits the representation

$$U_t = \int_0^t K_H(t,s)dZ_s.$$

(iv) The natural filtration $(\mathcal{Z}_t)$ of Z and $(\mathcal{U}_t)$ of U coincide.

We have

$$\begin{aligned} Q_t &= \frac{d}{dv_t}\int_0^t k_H(t,s)U_s ds \\ &= \kappa_H^{-1}\frac{d}{dv_t}\int_0^t s^{1/2-H}(t-s)^{1/2-H}U_s ds \\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\frac{d}{dt}\int_0^t s^{1/2-H}(t-s)^{1/2-H}U_s ds \\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t \frac{d}{dt}s^{1/2-H}(t-s)^{1/2-H}U_s ds \\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t s^{1/2-H}(t-s)^{-1/2-H}U_s ds. \end{aligned}$$

The process Q depends continuously on U and therefore, the discrete observations of U does not allow one to obtain the discrete observations of Q. The process Q can be approximated by

$$\widetilde{Q}_n = \kappa_H^{-1}\lambda_H n^{2H-1}\sum_{j=0}^{n-1} j^{1/2-H}(n-j)^{-1/2-H}U_j.$$

It is easy to show that $\widetilde{Q}_n \to Q_t$ almost surely as $n \to \infty$, see Tudor and Viens [38].

Define a new partition $0 \le r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k, \quad k = 1, 2, \cdots, n.$

Define

$$\widetilde{Q}_i(t_k) = \kappa_H^{-1}\eta_H t_k^{2H-1}\sum_{j=1}^{m_k} r_j^{1/2-H}(r_{m_k}-r_j)^{-1/2-H}u_i(r_j)(r_j - r_{j-1}),$$

$k = 1, 2, \cdots, n.$

It is easy to show that $\widetilde{Q}_i(t_k) \to Q_i(t)$ almost surely as $m_k \to \infty$ for each $k = 1, 2, \cdots, n$.

We use this approximate observation in the calculation of our estimators. Thus our observations are

$$U_i(t) \approx \int_0^t K_H(t,s)d\widetilde{Z}_i(s) \ \text{ where } \ \widetilde{Z}_i(t) = \theta\int_0^t \widetilde{Q}_i(s)dv_s + \mathcal{M}_t^H$$

observed at $t_1, t_2, \ldots, t_n$.

Note that for equally spaced data

$$\Delta v_{t_i} := v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[i^{2-2H} - (i-1)^{2-2H}].$$

For $H = 0.5$,

$$v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[i^{2-2H} - (i-1)^{2-2H}] = \frac{T}{n}, \quad i = 1, 2, \ldots, n$$

the standard equispaced partition. In this paper we do not need to assume $T/n \to 0$ unlike the finite dimensional diffusion models as we take advantage of the increasing spatial dimension $M \to \infty$ in this paper.

The following is the fractional Girsanov theorem (see Decreusefond and Ustunel [32, 39]) which will be useful for the calculation of the likelihood ratio.

Theorem 2.1 *Let $T > 0$ and let $u : [0, T] \to \mathbb{R}$ be continuous. Suppose K satisfies the equation*

$$\int_0^T K(s)\phi(s,t)ds = u(t); \quad 0 \le t \le T$$

and extend K to $\mathbb{R}$ by putting $K(s) = 0$ outside $[0, T]$. Define the probability measure $\hat{\mu}_H$ on $\mathcal{F}_T^{(H)}$ by

$$d\widehat{\mu}_H(\omega) = \exp\left\{-\int_0^T K(s)dW_s^H - \frac{1}{2}|K|_\phi^2\right\} d\mu_H(\omega).$$

Then

$$\widehat{W}_H(t) := \int_0^t u(s)ds + W_H(t)$$

is a fractional Brownian motion with respect to $\hat{\mu}_H$.

Let the observations $Q_M^N := \{(Q_m^1(t_j),\ Q_m^2(t_j)),\ m = 1, \ldots, M,\ j = 1, \ldots, N\}$ be the discrete data set. The measures P_M^θ and $P_M^{\theta_0}$ generated by the observations Q_M^N corresponding to θ and θ_0 respectively are *singular* if and only if

$$\sum_{m=1}^{\infty}\left[\frac{\alpha_{1m}^2}{\alpha_m(\theta)^2} + \sin^2[\beta_{1m}h(\theta-\theta_0)]e^{-h(\alpha_m(\theta)+\alpha_m(\theta_0))}\right] = \infty.$$

So consistency follows from the singularity of the measures and is equivalent to it for a wide class of elliptic operators. Main contribution of the paper is that we have a time series representation of the model. It is well known that the discretized version of the O-U process is an first order autoregressive process (AR(1)). Hence we have

$$Q_m(t_{n+1}) = e^{-\nu_m(\theta)\Delta t_n}Q_m(t_n) + \epsilon_n$$

where

$$\epsilon_n \sim \mathcal{N}\left(0, \frac{1-e^{-2\nu_m(\theta)\Delta t_n}}{\nu(\theta)}\sigma_m^2\right), \ m \ge 1.$$

Based on the discrete observations $Q_m(t_n), m = 1, 2, \ldots, M, n = 1, 2, \ldots, N,$ the likelihood ratio is given by

$$\begin{aligned} L_{M,N}(\theta,\theta_0) \quad = \quad & \sum_{m=1}^{M}\left\{N\ln\frac{\nu_m(\theta)}{\nu_m(\theta_0)} - \sum_{n=1}^{N-1}\ln\frac{1-e^{-2\nu_m(\theta)\Delta v_{t_n}}}{1-e^{\nu_m(\theta_0)\Delta v_{t_n}}} - (\theta-\theta_0)\frac{\nu_{1m}}{\sigma_m^2}Q_m(t_1)\right. \\ & -\frac{1}{\sigma_m^2}\sum_{n=1}^{N-1}\left[\frac{\nu(\theta)}{1-e^{-2\nu_m(\theta)\Delta v_{t_n}}}\left(Q_m(t_{n+1}) - e^{-\nu_m(\theta)\Delta v_{t_n}}Q_m(t_n)\right)^2\right. \\ & \left.\left. - \frac{\nu(\theta_0)}{1-e^{-2\nu_m(\theta_0)\Delta v_{t_n}}}\left(Q_m(t_{n+1}) - e^{-\nu_m(\theta_0)\Delta t_n}Q_m(t_n)\right)^2\right]\right\}. \end{aligned}$$

where $\Delta t_n = t_{n+1} - t_n, n = 1, 2, \ldots, N-1$ and $t_1 = 0$. The MLE is defined as

$$\hat{\theta}_{M,N} = \text{armax}_{\theta} L_{M,N}(\theta, \theta_0).$$

The MLE $\hat{\theta}_{M,N}$ can not be expressed as an explicit form. It can be computed by numerically solving the likelihood equation

$$\frac{\partial L_{M,N}(\theta, \theta_0)}{\partial \theta} = 0.$$

If

$$\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m^2(\theta)} = \infty,$$

then the MLE $\hat{\theta}_{M,N}$ is consistent.

Another advantage of minimum contrast estimator over MLE is it is easier to simulate as it does not involve the stochastic integral like the MLE. Also MCE is efficient and robust.

We consider observations at one single time point $\{Q_m(t_1), m = 1, \ldots, M\}$ for some $t_1 > t_0$. The observation $Q_m(t_1)$ is Gaussian with zero mean and variance

$$Var(Q_m(t_1)) = \frac{\sigma_m^2}{2\nu_m(\theta)}.$$

Let P_M^{θ} be the measure generated by the sample $U_M^N = (Q_m^1(t_j), Q_m^2(t_j)), m = 1, \ldots, M, j = 1, \ldots, N$ with the parameter θ and let θ_0 be the true value. Using sample independence of $Q_1, Q_2, \ldots, Q_M$ and their Gaussianity, one can obtain that the likelihood is given by the Radon-Nikodym derivative

$$\frac{dP_M^{\theta}}{dP_M^{\theta_0}} = \exp\left\{-\frac{1}{2}\left[2(\theta - \theta_0)\sum_{m=1}^{M} \frac{\nu_{1m} Q_m^2}{\sigma_m^2} - \sum_{m=1}^{M} \ln \frac{\nu_m(\theta)}{\nu_m(\theta_0)}\right]\right\}$$

where $Q_m = Q_m(t_1)$.

Let the corresponding MLE be denoted by $\hat{\theta}_M$, i.e.,

$$\hat{\theta}_M = \arg\max \frac{dP_M^{\theta}}{dP_M^{\theta_0}}(Q_M^{\theta_0}).$$

Note that P_M^{θ} goes to the measure P^{θ} generated by $Q(t_1, x)$ as $M \to \infty$. From the general results concerning the absolute continuity of Gaussian measures, it follows that P^{θ} and P^{θ_0} are absolutely continuous $(\theta \neq \theta_0)$ if and only if

$$\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta)\nu_m(\theta_0)} < \infty.$$

In this case

$$\frac{dP^{\theta}}{dP^{\theta_0}} = \exp\left\{-\frac{1}{2}\left[2(\theta - \theta_0)\sum_{m=1}^{\infty} \frac{\nu_{1m} Q_m^2}{\sigma_m^2} - \sum_{m=1}^{\infty} \ln \frac{\nu_m(\theta)}{\nu_m(\theta_0)}\right]\right\}.$$

If

$$\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m^2(\theta)} = \infty,$$

then extending Piterbarg and Rozovskii [40, 41] it can be shown that

$$\hat{\theta}_M \to \theta_0 \quad \text{a.s.} \quad \text{as } M \to \infty$$

and if additionally

$$\lim_{M\to\infty} \frac{\max_{1\le m\le M} \frac{\nu_{1m}^2}{\nu_m^2(\theta)}}{\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m^2(\theta)}} = 0,$$

then

$$\sqrt{2\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\nu_m^2}}(\hat{\theta}_M - \theta_0) \to^{\mathcal{D}} \mathcal{N}(0,1) \;\; \text{as } M \to \infty$$

since

$$q_M(\hat{\theta}_M, \theta_0)(\hat{\theta}_M - \theta_0)\sqrt{2\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\nu_m(\theta_0)^2}} = \frac{\sum_{m=1}^{M} \zeta_m}{\sqrt{\sum_{m=1}^{M} E\zeta_m^2}}.$$

But $q_M(\hat{\theta}_M, \theta_0) \to 1$ almost surely as $M \to \infty$. By the central limit theorem for i.i.d. random variables, we obtain the result.

Let θ^* be the MLE with respect to $\frac{dP^\theta}{dP^{\theta_0}}$, i.e.,

$$\theta^* = \arg\max_{\theta\in\Theta} \frac{dP^\theta}{dP^{\theta_0}}$$

which is given by

$$\theta^* = G_{\theta_0}^{-1}\left(\sum_{m=1}^{\infty} \nu_{1m}(Q_m^2 - EQ_m^2)\right)$$

where

$$G_{\theta_0}(\theta) = (\theta - \theta_0)\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta)\nu_m(\theta_0)}$$

which is a continuously differentiable function of $\theta \in \Theta$ for any fixed θ_0.

$$\frac{dG_{\theta_0}(\theta)}{d\theta} \ge \delta > 0$$

where δ is independent of θ and θ_0.

Indeed,

$$0 < C_1 < \nu_m(\theta)/\nu_m(\theta_0) < C_2 \;\forall\; \theta, \theta_0, m,$$

it follows that the series converges uniformly with respect to $\theta \in \Theta$ and can be differentiated because the series

$$\frac{dG_\theta(\theta)}{d\theta} = \sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta)^2}$$

converges uniformly as well. We can set

$$\delta = \inf_\theta \sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta)^2}.$$

Then $\hat{\theta}_M \to \theta^*$ as $M \to \infty$.

In the case of continuous observation $Q_M^T := \{Q_m(t),\ m = 1, \dots, M,\ 0 \le t \le T\}$, the likelihood function is given by

$$L_{M,T}(\theta, \theta_0) = \frac{dP_M^{\theta}}{dP_M^{\theta_0}} = \exp\left\{\left[(\theta - \theta_0)\sum_{m=1}^{M}\int_0^T\left(\frac{\nu_{1m}\nu_{0m}}{\sigma_m^2}Q_m(t)dZ_m(t) - \frac{\nu_{1m}\nu_{0m}}{\sigma_m^2}Q_m^2(t)dv_t\right)\right.\right.$$
$$\left.\left. -\frac{1}{2}(\theta^2 - \theta_0^2)\sum_{m=1}^{M}\int_0^T \frac{\nu_{1m}^2}{\sigma_m^2}Q_m^2(t)dv_t\right]\right\}.$$

The MLE is given by

$$\hat{\theta}_{M,T} = \frac{\displaystyle\sum_{m=1}^{M}\frac{\nu_{1m}}{\sigma_m^2}\int_0^T Q_m(t)dZ_m(t) - \sum_{m=1}^{M}\frac{\nu_{1m}\nu_{0m}}{\sigma_m^2}\int_0^T Q_m^2(t)dv_t}{\displaystyle\sum_{m=1}^{M}\int_0^T \frac{\nu_{1m}^2}{\sigma_m^2}Q_m^2(t)dv_t}.$$

In the case of continuous observation, the contrast function is given by

$$K_{M,T}(\theta, \theta_0) = \left\{\left[(\theta - \theta_0)\sum_{m=1}^{M}\left(\frac{\nu_{1m}\nu_{0m}}{\sigma_m^2}\frac{T}{2} + \frac{\nu_{1m}\lambda_{0m}}{\sigma_m^2}\int_0^T Q_m^2(t)dv_t\right)\right.\right.$$
$$\left.\left. +\frac{1}{2}(\theta^2 - \theta_0^2)\sum_{m=1}^{M}\int_0^T \frac{\nu_{1m}^2}{\sigma_m^2}Q_m^2(t)dv_t\right]\right\}.$$

The minimum contrast estimator (MCE) is defined as

$$\tilde{\theta}_{M,T} := \arg\min_{\theta\in\Theta} K_{M,T}(\theta, \theta_0)$$

which is given by

$$\widetilde{\theta}_{M,T} = \frac{\displaystyle -\frac{T}{2}\sum_{m=1}^{M}\frac{\nu_{1m}\nu_{0m}}{\sigma_m^2} - \sum_{m=1}^{M}\frac{\nu_{1m}\nu_{0m}}{\sigma_m^2}\int_0^T Q_m^2(t)dv_t}{\displaystyle\sum_{m=1}^{M}\int_0^T \frac{\nu_{1m}^2}{\sigma_m^2}Q_m^2(t)dv_t}.$$

Using the approximation $e^x \approx 1 + x$ (Euler scheme) for small x, we construct the approximate contrast function.

In the case of discrete observations $\{Q_m(t_n),\ m = 1, 2, \dots, M,\ n = 1, 2, \dots, N\}$, the contrast function is given by

$$K_{M,N}(\theta, \theta_0) \ = \ \sum_{m=1}^{M}\left\{-(\theta - \theta_0)\frac{\nu_{1m}}{\sigma_m^2}Q_m(t_1) - \frac{1}{\sigma_m^2}\sum_{n=1}^{N-1}\left[\frac{\nu(\theta)}{\nu_m(\theta)\Delta v_{t_n}}\left(Q_m(t_{n+1}) + \nu_m(\theta)\Delta v_{t_n}Q_m(t_n)\right)^2\right.\right.$$
$$\left.\left. - \frac{\nu(\theta_0)}{\nu_m(\theta_0)\Delta v_{t_n}}\left(Q_m(t_{n+1}) + \nu_m(\theta_0)\Delta v_{t_n}Q_m(t_n)\right)^2\right]\right\}.$$

The approximate minimum contrast estimator (AMCE) is defined as

$$\widetilde{\theta}_{M,N} := \arg\min_{\theta\in\Theta} K_{M,N}(\theta, \theta_0)$$

which is given by

$$\widetilde{\theta}_{M,N} = \frac{\sum_{m=1}^{M}[\frac{\nu_{1m}}{\sigma_m^2}Q_m(t_1) + \sum_{n=1}^{N-1} Q_m(t_n)(Q_m(t_{n+1}) - Q_m(t_n))]}{\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\sigma_m^2} \sum_{n=1}^{N-1} Q_m^2(t_n)\Delta v_{t_n}}.$$

We study the asymptotic behavior of $\widetilde{\theta}_{M,T}$ and $\widetilde{\theta}_{M,N}$ as $M \to \infty$.

Remark: Suppose that the diffusivity D is given. Then the MCE $\widetilde{\nu}_{M,T}$ is consistent for ν if and only if $d \geq 4$ for continuous observation and the AMCE $\widetilde{\nu}_{M,N}$ is consistent for ν if and only if $d \geq 2$ for discrete observations. By passing to discrete observations, we lose two units of dimension. If $D = 0$, then the MCE is consistent for any d. The condition of zero diffusivity $D = 0$ is necessary and sufficient for the consistency of the MCE of the velocity component when other components with ν and D are given.

Suppose the velocity v is given. In this case $p = 2$ and $p_1 = 0$. The MCE $\widetilde{\nu}_{M,T}$ is consistent for ν if and only if $d \geq 2$ for continuous observation and the AMCE $\widetilde{\nu}_{M,N}$ is consistent for ν if and only if $d \geq 4$ for discrete observations. Thus passing to the discrete observations, we lose two units of dimension.

In the purely dissipative case $\mathbf{v} = 0$, the MLE of the diffusivity is consistent for all dimensions, and for all kinds of observations, both continuous and discrete.

For discrete observations, we have the strong consistency and the asymptotic normality of the AMLE:

Theorem 2.2 a) For $d \geq 2(p - p_1)$ and fixed $N \geq 1$, $\hat{\theta}_M \to \theta_0$ almost surely as $M \to \infty$.

b) For $d = 2(p - p_1)$ and fixed $N \geq 1$, we have

$$(\log M)^{1/2}(\hat{\theta}_M - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

c) For $d > 2(p - p_1)$ and fixed $N \geq 1$, we have

$$M^{\frac{2p_1-p+d}{2d}}(\hat{\theta}_M - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

Proof. Recall that we have the likelihood ratio which is the Radon-Nikodym derivative of P_M^{θ} with respect to $P_M^{\theta_0}$ given by

$$\ln \frac{dP_M^{\theta}}{dP_M^{\theta_0}} = \left\{-\frac{1}{2}\left[2(\theta - \theta_0)\sum_{m=1}^{M}\frac{\nu_{1m}U_m^2}{\sigma_m^2} - \sum_{m=1}^{M}\ln\frac{\nu_m(\theta)}{\nu_m(\theta_0)}\right]\right\}.$$

Differentiating with respect to θ, we obtain that $\hat{\theta}_M$ is the solution of the equation

$$2\sum_{m=1}^{M}\frac{\nu_{1m}Q_m^2}{\sigma_m^2} = \sum_{m=1}^{M}\frac{\nu_{1m}}{\nu_m(\theta)}.$$

Put

$$\xi_m := \frac{Q_m^2 - EQ_m^2}{\sqrt{Var(Q_m^2 - EQ_m^2)}} = \frac{\sqrt{2}Q_m^2\nu_m(\theta_0)}{\sigma_m^2} - \frac{1}{\sqrt{2}}.$$

Note that

$$E(\xi_m) = 0,\ \ E(\xi_m^2) = 1,\ \ E\xi_m^4 < \infty$$

and

$$\frac{2Q_m^2}{\sigma_m^2} = \frac{1 + \xi_m\sqrt{2}}{\lambda_m(\theta_0)}.$$

On substitution

$$\sum_{m=1}^{M} \frac{\nu_{1m}\sqrt{2}}{\nu_m(\theta_0)}\xi_m = (\theta - \theta_0)\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\nu_m(\theta)\nu_m(\theta_0)}.$$

where ξ_m are i.i.d. random variables satisfying the previous conditions. Dividing both sides by $\sum_{m=1}^{M} \frac{2\nu_{1m}}{\nu_m(\theta_0)^2}$, we have

$$\frac{\sum_{m=1}^{M} \zeta_m}{\sum_{m=1}^{M} E\zeta_m^2} = (\theta - \theta_0)q_M(\theta, \theta_0)$$

where

$$\zeta_m := \frac{\nu_{1m}\sqrt{2}}{\nu_m(\theta_0)}\xi_m, \quad q_M(\theta, \theta_0) := \frac{\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\nu_m(\theta)\nu_m(\theta_0)}}{\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\nu_m(\theta_0)^2}}.$$

By the law of large numbers for i.i.d. random variables $(\zeta_m, m \geq 1)$, we obtain the result.

Further,

$$(\hat{\theta}_M - \theta_0)q_M(\hat{\theta}_M, \theta_0)\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta_0)^2} = \frac{\sum_{m=1}^{M} \zeta_m}{\sum_{m=1}^{M} E\zeta_m^2}.$$

By the central limit theorem for i.i.d. random variables $(\zeta_m, m \geq 1)$, we obtain the result since $q_M(\theta, \theta_0) = 1$ and $\partial q_M/\partial\theta$ is uniformly bounded as shown below:

$$\frac{\partial q_M}{\partial \theta} = -\sum_{m=1}^{M} \frac{\nu_{1m}^3}{\nu_m(\theta)^2\nu_m(\theta_0)} \Big/ \sum_{m=1}^{M} \frac{2\nu_m^2}{\nu_m(\theta_0)^2}.$$

Hence

$$\left|\frac{\partial q_M}{\partial \theta}\right| \leq (2C_1)^{-1} \max_m \frac{|\nu_{1m}|}{\nu_m(\theta)}.$$

Let $\theta_1 \in \Theta$ be arbitrary. Then

$$\frac{|\nu_{1m}|}{\nu_m(\theta)} = (\theta_1 - \theta)^{-1}\left|\frac{\nu_{1m}}{\nu_m(\theta)} - 1\right|.$$

It follows that

$$\frac{|\nu_{1m}|}{\nu_m(\theta)} < |\theta_1 - \theta|^{-1}(C_2 + 1).$$

The arbitrariness of θ_1 and the boundedness of $|\partial q_M/\partial\theta|$ imply the uniform boundedness of $|\partial q_M/\partial\theta|$.

Let

$$G_M(\theta) := (\theta - \theta_0)\sum_{m=1}^{M} \frac{\nu_{1m}^2}{\nu_m(\theta)\nu_m(\theta_0)}.$$

As earlier for $G_{\theta_0}(\theta)$, one can show that

$$\left|\frac{dG_M(\theta)}{d\theta}\right| \geq \delta_1 > 0$$

Assuming that the series $\sum_{m=1}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta)}$ converges and $M \to \infty$, we get

$$2\sum_{m=1}^{\infty} \frac{\sqrt{2}\nu_{1m}}{\nu_m(\theta_0)}\xi_m = (\theta^* - \theta_0)\sum_{m=M}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta^*)\nu_m(\theta_0)} + G_M(\theta^*).$$

$$2\sum_{m=1}^{\infty} \frac{\sqrt{2}\nu_{1m}}{\nu_m(\theta_0)}\xi_m - (\theta^* - \theta_0)\sum_{m=M}^{\infty} \frac{\nu_{1m}^2}{\nu_m(\theta)\nu_m(\theta_0)} = G_M(\theta^*) - G_M(\hat{\theta}_M).$$

Since the left side of the above equation goes to zero and the derivative of $G_M(\theta)$ is positive, by Theorem 2 in Kasonga [42] (which is similar to Theorem 1 in Frydman [43]), we conclude that $\hat{\theta}_M \to \theta^*$ almost surely as $M \to \infty$. □

The proofs of the following theorems are similar to the previous theorem. We omit the details.

For discrete observations, we have the asymptotic normality for the MLE:

Theorem 2.3 a) For $d \geq 2(p - p_1)$ and fixed $N \geq 1$, we have $\widetilde{\theta}_{M,N} \to \theta_0$ almost surely as $M \to \infty$.
b) For $d = 2(p - p_1)$ and fixed $N \geq 1$, we have

$$(\log M)^{1/2}(\hat{\theta}_{M,N} - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

c) For $d > 2(p - p_1)$ and fixed $N \geq 1$, we have

$$M^{\frac{2p_1 - p + d}{2d}}(\hat{\theta}_{M,N} - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

In the case of continuous observation, for fixed $T > 0$, we have the strong consistency and the asymptotic normality of the MCE:

Theorem 2.4 a) For $d \geq p - 2p_1$ and fixed $T > 0$, we have $\widetilde{\theta}_{M,T} \to \theta_0$ almost surely as $M \to \infty$.
b) For $d = p - 2p_1$ and fixed $T > 0$, we have

$$(\log M)^{1/2}(\widetilde{\theta}_{M,T} - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

c)) For $d > p - 2p_1$, we have

$$M^{\frac{2p_1 - p + d}{2d}}(\widetilde{\theta}_{M,T} - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

Note that for $d = 3, p = 2, p_1 = 0$, the rate is $M^{1/6}$.

For discrete observations, we have the strong consistency and the asymptotic normality of the MCE:

Theorem 2.5 a) For $d \geq 2(p - p_1)$ and fixed $N \geq 1$, we have $\widetilde{\theta}_{M,N} \to \theta_0$ almost surely as $M \to \infty$.
b) For $d = 2(p - p_1)$ and fixed $N \geq 1$, we have

$$(\log M)^{1/2}(\widetilde{\theta}_{M,N} - \theta) \to^{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as} \quad M \to \infty.$$

c) For $d > 2(p - p_1)$, we have

$$M^{\frac{2p_1-2p+d}{2d}}(\widetilde{\theta}_{M,N} - \theta) \rightarrow^{\mathcal{D}} \mathcal{N}(0,1) \quad \text{as} \quad M \rightarrow \infty.$$

Note that for $d = 5, p = 2, p_1 = 0$, the rate is $M^{1/8}$.

3. Example: Complex-valued Fractional Ornstein-Uhlenbeck Process

Arato et al. [44] studied parameter estimation in the complex valued Ornstein-Uhlenbeck process. They used the model for geophysical problem. Remember that Kolmogorov [45] was the founder of fractional Brownian motion. Hence their model can be extended to fractional Brownian motion. The complex valued fractional Ornstein-Uhlenbeck process is given by

$$dU(t) = -[(\alpha_1 + i\beta_1)\theta + (\alpha_0 + i\beta_0)\theta]U(t)dt + \sigma[dW_1^H(t) + idW_2^H(t)]$$

where W_1^H, W_2^H are independent fractional Brownian motions, $U(t) = (U_1(t), U_2(t))$ where U_1 and U_2 are the real and imaginary parts of $U(t)$. So this paper is infinite dimensional generalization of Kolmogorov's model. In the paper they used a time transformation to reduce the general problem to a fixed time case and the asymptotics were studied in large parameter case, see also Bishwal [46] in this context.

Let $\Delta t_i = h, i = 1, 2, \dots, n, \alpha(\theta) = \alpha_0 + \alpha_1\theta$. For $H = 0.5$, the Fisher information based on the data $U(t_1), U(t_2), \dots, U(t_n)$ is given by

$$I(\theta) = \frac{\alpha_1^2}{\alpha(\theta)^2}\left\{1 + (n-1)\left[\left(\frac{1 - e^{-2h\alpha(\theta)} - 2h\alpha(\theta)e^{-2h\alpha(\theta)}}{1 - e^{-2h\alpha(\theta)}}\right)^2 + \frac{(2h\alpha(\theta))^2 e^{-2h\alpha(\theta)}}{1 - e^{-2h\alpha(\theta)}}\right]\right\}$$
$$+\beta_1^2 h^2 (n-1)\frac{e^{-2\alpha(\theta)}}{1 - e^{-2\alpha(\theta)}}.$$

It is easy to verify results of Theorems 2.1 – 2.4 showing the consistency and asymptotic normality of the estimators.

Remarks We considered fractional Brownian motion driving term in this paper whose increments are stationary. Using fractional Levy process as the driving term which include jumps, maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [47]. Recently, sub-fractional Brownian (sub-FBM) motion which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s, t > 0$$

for $0 < H < 1$ introduced by Bojdecki et al. [48] has received some attention recently in finite dimensional models. The interesting feature of this process is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. It would be interesting to see extension of this paper to sub-FBM case. We generalize sub-fBM to Sub-fractional Levy process (sub-FLP).

Sub-fractional Levy process (SFLP) is defined as

$$S_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dM_s, \quad t \in \mathbb{R}$$

where $M_t, t \in \mathbb{R}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0$, $E(M_1^2) < \infty$ and without Brownian component. SFLP has the following properties:

1) The covariance of the process is given by

$$\text{Cov}(S_{H,t}, S_{H,s}) = s^{2H} + t^{2H} + \frac{E[L(1)^2]}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) S_H is not a martingale. For a large class of Levy processes, S_H is neither a semimartingale nor a Markov process. 3) S_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) S_H has nonstationary increments. 5) S_H is symmetric. 6) S_H is self similar. 7) S_H has infinite total variation on compacts.

It would be interesting to investigate QML estimation in SPDE driven by subfractional Levy processes which incorporate both jumps and long memory apart from nonstationarity.

Funaki [49] studied random transport equation in bounded domains with regular coefficients. Flandoli et al. [50] studied linear stochastic transport equation. Fang and Luo [51] studied Wong-Zakai type approximations of the stochastic transport equation. Guillet et al. [52] studied extreme-value statistics of STE. Another application is in nuclear physics. Recently Chapron et al. [53] studied stochastic transport in upper ocean dynamics. Tian and Tang [54] studied stochastic entropy solutions for stochastic nonlinear transport equations.

Another possible generalization is the following: Hawkes processes (see Hawkes [55]) are an efficient generalization of the Poisson processes to model a sequence of arrivals over time of some types of events, that present self-exciting feature, in the sense that each arrival increases the rate of future arrivals for some period of time. This class of counting processes allows one to capture self-exciting phenomena in a more accurate way compared to inhomogeneous Poisson processes or Cox processes. This is the case with aftershocks of earthquakes; an earthquake increases the geophysical tension in the region and can cause a second earthquake. In finance, they are accurate to model for example credit risk contagion, order book or microstructure noises's feature of financial markets.

A Hawkes process is a counting process A_t with stochastic intensity λ_t given by $\lambda_t = \mu + \int_0^t \Phi(t-s)dA_s$ where $\mu > 0$ and $\Phi : \mathbb{R} \to \mathbb{R}_+$ are two parameters. The parameter $\mu > 0$ is called the *background intensity* and the function Φ is called the *excitation function*. When $\Phi = 0$, this a homogeneous Poisson process.

A fractional Hawkes process $\{A_H(t), t > 0\}$ with Hurst parameter $H \in (1/2, 1)$ is defined as

$$A_H(t) = \frac{1}{\Gamma(H-\frac{1}{2})}\int_0^t u^{\frac{1}{2}-H}\left(\int_u^t \tau^{H-\frac{1}{2}}(\tau-u)^{H-\frac{3}{2}}d\tau\right)dR(u)$$

where $R(u) = \frac{A(u)}{\sqrt{\lambda_t}} - \sqrt{\lambda_t}u$ and $A(u)$ is a Hawkes process with stochastic intensity λ_t.

It would be interesting to investigate QML estimation in SPDE driven by fractional Hawkes process which would incorporate self-excitation, jumps and long memory of sea surface temperature.

Concluding Remarks: In this paper we studied the estimation of the component of velocity, diffusivity and the feedback parameter of the fractional STE. Also results on long range dependence in our model would be useful for measuring sea surface temperature which in general are non-Markovian. Strong consistency and asymptotic normality of the estimators were studied. The rates of convergence to normality depends on the dimension of the space, and for some special cases, the rates are one sixth in the case of continuous observation and one eighth on the case of discrete observations which are unusual in classical statistics. Results in this paper very useful as we have studied the discrete approximations of the STE which are spatial autoregressive processes. Also we have generalised asymptotic results on real-valued Ornstein-Uhlenbeck processes, which has been a benchmark for short-term interest rate modeling in finance, to complex-valued Ornstein-Uhlenbeck processes, which in turn are continuous analogues of Gaussian autoregressive processes. Thus our model is infinite dimensional generalization of Kolmogorov's model [44] in geophysics.

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