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Quasi-likelihood and Quasi-Bayes Estimation in Noncommutative Fractional SPDEs

Jaya P. N. Bishwal

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 376 Fretwell Bldg, 9201 University City Blvd. Charlotte, NC 28223-0001, USA
J.Bishwal@uncc.edu

Abstract. We study the quasi-likelihood and quasi Bayes estimator of the drift parameter in the stochastic partial differential equations when the process is observed at the arrival times of a Poisson process. Unlike the previous work, no commutativity condition is assumed between the operators in the equation. We use a two stage estimation procedure. We first estimate the intensity of the Poisson process. Then we plug-in this estimate in the quasi-likelihood to estimate the drift parameter. Under certain non-degeneracy assumptions on the operators, we obtain the consistency and the asymptotic normality of the estimators.

1. Introduction

A stochastic partial differential equation (SPDE) is a continuous version of simultaneous cross-section time series model. For a fixed spatial mode, it is an autoregressive time series (recall that an Ornstein-Uhlenbeck process is a continuous limit of Gaussian AR(1) process) and for a fixed time, it is a regression model. One can study asymptotic estimation for one fixed time point with large number of spatial observations or at a fixed spatial mode with large number of discrete time points, or simultaneous large spatial and temporal observations. We consider the estimation based on a fixed spatial mode with large number of randomly spaced time points, where the inter arrival times are exponentially distributed. Thus we have a random time sampling at a fixed space point. What mainly distinguishes SPDE from classical models is the type of sampling and the unusual rate of convergence of the estimators. Applications of SPDE model is numerous, e.g, in cell biology, neurophysiology, turbulence, oceanography and finance: see Itô [42], Walsh [61], Kallianpur and Xiong [44], Holden *et al.* [32], Adler *et al.* [1], Carmona and Rozovskii [23] and Bishwal [15]. Bishwal [15] studied asymptotic inference for fractional SPDE model for neurobiology. Recently SPDE has been used to model cell repolarization (stochastic Meinhardt model) and parameter estimation techniques developed for linear SPDE models have been applied to this model when the space resolution is finer, see Altmeyer *et al.* [2].

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Types of operators appearing in SPDE makes the estimation problem simple or difficult. Commuting operators have the same system of eigenvectors and make the corresponding finite dimensional projections as diffusion processes. The case is not so for noncommuting operators. Noncommuting operators appear in quantum mechanics. According to Heisenberg's uncertainty principle, if two operators representing a pair of variables do not commute, then the par of varibles are mutually complementary, which means they can not be simultaneously measured or known precisely.

Parameter estimation is an inverse problem. Loges (1984) initiated the study of parameter estimation in infinite dimensional stochastic differential equations. When the length of the observation time becomes large, he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges [51] extended the work of Loges (1984) to minimum contrast estimators. Koski and Loges [50] applied the work to a stochastic heat flow problem. See Bishwal [11] for recent results on likelihood asymptotics and Bayesian asymptotics for drift estimation of finite and infinite dimensional stochastic differential equations. See Bishwal [10] for asymptotic statistical results for discretely sampled diffusions. For the finite dimensional fractional Ornstein-Uhlenbeck process, Berry-Esseen inequalities of minimum contrast estimators based on continuous and discrete observations was studied in Bishwal [13]. Bishwal [17] studied estimation and hypothesis testing on nonlinear SPDEs from both continuous and discrete observations. Bishwal [19] studied estimation by the mixingale estimation function method for SPDEs with random sampling. Discrete observations in time of continuous models are important for practical applications,e.g. stochastic volatility models, see Bishwal [18].

Huebner, Khasminskii and Rozovskii [36] started statistical investigation in SPDEs. They gave two contrast examples of parabolic SPDEs in one of which they obtained consistency, asymptotic normality and asymptotic efficiency of the MLE as noise intensity decreases to zero under the condition of absolute continuity of measures generated by the process for different parameters (the situation is similar to the classical finite dimensional case) and in the other they obtained these properties as the finite dimensional projection becomes large under the condition of singularity of the measures generated by the process for different parameters. The second example was extended by Huebner and Rozovskii [38] and the first example was extended by Huebner [35] to MLE for general parabolic SPDEs where the partial differential operators commute and satisfy different order conditions in the two cases.

Huebner [37] extended the problem to the ML estimation of multidimensional parameter. Lototsky and Rozovskii [55] studied the same problem without the commutativity condition. Small noise asymptotics of the nonparmetric estimation of the drift coefficient was studies by Ibragimov and Khasminskii (1998).

The Bernstein-von Mises theorem (BVT, in short), concerning the convergence of suitably normalized and centered posterior distribution to normal distribution, plays a fundamental role in asymptotic Bayesian inference, see Le Cam and Yang (1990). Borwanker *et al.* [21] obtained the

BVT for discrete time Markov processes. Bose [22] extended the BVT to the homogeneous nonlinear diffusions. As a further refinement in BVT, Bishwal [8] obtained sharp rates of convergence to normality of the posterior distribution and the Bayes estimators for the Ornstein-Uhlenbeck process.

All these above work on BVT are concerned with finite dimensional SDEs. Bishwal [7] proved the BVT and obtained asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding ergodic diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE.

Bishwal (2002) obtained BVT and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In that case, the measures generated by the process for different parameters are singular. Here we treat the case when the measures generated by the process for different parameters are absolutely continuous under some conditions on the order of the partial differential operators. Bishwal [16] studied the asymptotic properties of the posterior distributions and Bayes estimators when we have either fully observed process or finite-dimensional projections. The asymptotic parameter is only the intensity of noise. Recently Cheng *et al.* [24] studied BVT and Bayesian estimation for a large class of prior distributions and loss functions (of at most polynomial growth) for diagonalizable bilinear SPDEs driven by a multiplicative noise. In this paper we treat the more general model and study estimation by martingale estimation function method.

The rest of the paper is organized as follows : Section 2 contains model, assumptions and preliminaries. In Section 3 we prove the asymptotic properties of the discretely sampled quasi-likelihood estimator. In Section 4, we prove the Bernstein-von Mises theorem. In section 5, we study the asymptotics of quasi-Bayes estimators for smooth priors are loss functions. In section 6, we study fixed accuracy sequential estimation. Section 7 provides several examples of noncommutative SPDEs.

2. Model and Preliminaries

Let $\mathbb{G}$ be a smooth bounded domain in $\mathbb{R}^d$. We assume that the boundary $\partial\mathbb{G}$ of this domain is a C^∞-manifold of dimension $(d-1)$ and locally $\mathbb{G}$ is totally on one side of $\partial\mathbb{G}$. For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_d)$ we write

$$D^\gamma f(x) := \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}} f(x)$$

where $|\gamma| = \gamma_1 + \gamma_2 + \ldots + \gamma_d$.

Let A_0 and A_1 be partial differential operators of order m_0 and m_1 (the order of the highest derivative in it) respectively, written in the form

$$A_i(x)u := -\sum_{|\alpha|,|\beta| \le m_i} (-1)^{|\alpha|} D^\alpha (a_i^{\alpha\beta}(x) D^\beta(u))$$

where $a_i^{\alpha\beta}(x) \in C^\infty(\overline{\mathbb{G}})$. For $\theta \in \mathbb{R}$, write $A^\theta = \theta A_1 + A_0$ and $a^{\alpha\beta}(\theta, x) = \theta a_1^{\alpha\beta}(x) + a_0^{\alpha\beta}(x)$. Let us fix θ_0, the unknown true value of the parameter θ. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $W(t, x)$ be a cylindrical Brownian motion on this space with values in the Schwarz space of distributions $\mathcal{D}'(\mathbb{G})$.

A cylindrical fractional Brownian motion (C.F.B.M) is $W = W^H(t, x)$ is a distribution valued process such that for every such that for every $\phi \in C_0^\infty(\mathbb{G})$ with $\|\phi\|_{L^2(\mathbb{G})} = 1$ the inner product $\langle W(t, \cdot), \phi(\cdot)\rangle$ is a one dimensional fractional Brownian motion and for every $\phi_1, \phi_2 \in C_0^\infty(\mathbb{G})$,

$$E(\langle W^H(s, \cdot), \phi_1(\cdot)\rangle\langle W^H(t, \cdot), \phi_2(\cdot)\rangle) = (s \wedge t)(\phi_1, \phi_2)_{L^2(\mathbb{G})}.$$

The C.F.B.M. W^H can be expanded in the series $W^H(t, x) = \sum_{i=1}^\infty W_i^H(t) h_i(x)$ where $\{W_i^H(t)\}_{i=1}^\infty$ are independent one dimensional fractional Brownian motions and $\{h_i\}_{i=1}^\infty$ is complete orthonormal system in $L_2(\mathbb{G})$. The latter series converges P-a.s.

Recall that a fractional Brownian motion (fBM) has the covariance

$$\widetilde{C}_H(s, t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |s - t|^{2H}\right], \quad s, t > 0.$$

For $H > 0.5$ the process has long range dependence or long memory and the process is self-similar. For $H \neq 0.5$, the process is neither a Markov process nor a semimartingale. For $H = 0.5$, the process reduces to standard Brownian motion. Fractional Brownian motion can be represented as a Riemann-Liouville (fractional) derivative of Gaussian white noise, see Decreusefond and Ustunel [26] and Jumarie [43]. For deterministic fractional calculus, see Samko *et al.* [59].

We will consider the Dirichlet problem for a parabolic SPDE associated with the operator A^θ, and driven by the C.F.B.M. W^H :

$$\frac{\partial u(t, x)}{\partial t} = (A_0 + \theta A_1)u(t, x) + \frac{\partial}{\partial t} W^H(t, x) \tag{2.1}$$

$$u(0, x) = u_0(x) \tag{2.2}$$

$$D^\gamma u(t, x)|_{\partial G} = 0 \tag{2.3}$$

$$\text{for all multi-indices } \gamma \text{ with } |\gamma| \leq m - 1.$$

The problem (2.1) - (2.3) is understood in the sense of distributions.

We assume that

$$order(A_1) \geq \frac{1}{2} order(A_0 + \theta A_1) - d \tag{2.4}$$

For non-diagonalizable case, one can use Galerkin approximation of the solution of the SPDE. We do not assume anything about the eigenfunctions of the operators in the equation. The equation is considered in a compact d-dimensional manifold so that there are no boundary conditions involved. The main assumption is that the operators A_0 and A_1 are of different orders and the operator $A_0 + \theta A_1$ is *elliptic* for admissible values of θ.

Let M be a d-dimensional compact orientable C^∞ manifold with a smooth positive measure dx. Let L be a elliptic positive definite self-adjoint differential operator of order $2m$ on M. Let A, B, N be differential operators on M with $\max(order(A), order(B), order(N)) < 2m$.

Consider the random field u defined by the evolution equation

$$du(t,x)+[\theta_1(L+A)+\theta_2 B+N]u(t,x)dt=dW^H(t,x),\quad 0\le t\le T,\quad u(0,x)=0$$

where $\theta_1>0$ and $\theta_2>0$. We estimate θ_1 when θ_2 is known and we estimate θ_2 when θ_1 is known.

If θ_2 is known, then $A_0=\theta_2 B+N, \theta=\theta_1, \Theta=(0,\infty), A_1=L+A$. If θ_1 is known, $A_0=\theta_1(L+A)+N, \theta=\theta_2, \Theta=\mathbb{R}, A_1=B$. For every $f\in L_2(M,dx)$ the representation $f=\sum_{i\ge 1}\psi_i(f)e_i$ holds, where $\psi_i(f)=\int_M f(x)e_i(x)dx$ and $\psi_k(W^H(t))=W_k^H(t)$. Let the finite dimensional projection operator be Π^K. For every $f=\{\psi_i(f)\}_{i\ge 1}$,

$$\Pi^K(f)=\sum_{i=1}^{K}\psi_i(f)e_i. \tag{2.5}$$

The finite dimensional processes $\Pi^K u$, $\Pi^K A_0 u$, $\Pi^K A_1 u$ will be used to estimate the unknown parameter. By (2.1)

$$d\Pi^K u(t)+\Pi^K A^\theta(x)u(t)dt=dW^{K,H}(t) \tag{2.6}$$

where $W^{K,H}(t)=\Pi^K W^H(t)$. Let $\Pi^K u(t):=u_{k,t}$.

Now we focus on the fundamental semimartingale behind the SPDE model. Define

$$\begin{aligned}\kappa_H &:= 2H\Gamma(3/2-H)\Gamma(H+1/2),\quad k_H(t,s):=\kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H},\\ \lambda_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)},\quad v_t\equiv v_t^H:=\lambda_H^{-1}t^{2-2H},\quad \mathcal{M}_t^H:=\int_0^t k_H(t,s)dW_s^H.\end{aligned}$$

From Norros *et al.* [57] it is well known that $\mathcal{M}_{k,t}^H$ is a Gaussian martingale, called the fundamental martingale whose variance function $\langle\mathcal{M}_k^H\rangle_t$ is v_t^H. Moreover, the natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the fBm W^H since $W_{k,t}^H:=\int_0^t K(t,s)d\mathcal{M}_{k,s}^H$ holds for $H\in(1/2,1)$ where $K_H(t,s):=H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr,\quad 0\le s\le t$ and for $H=1/2$, the convention $K_{1/2}\equiv 1$ is used.

Define $Q_{k,t}:=\frac{d}{dv_t}\int_0^t k_H(t,s)u_{k,s}ds$. Define the process $Z_k=(Z_{k,t},t\in[0,T],k\ge 1)$ by $Z_{k,t}:=\int_0^t k_H(t,s)du_{k,s}$. It is easy to see that $Q_{k,t}=\frac{\lambda_H}{2(2-2H)}\left\{t^{2H-1}Z_{k,t}+\int_0^t r^{2H-1}dZ_{k,s}\right\}$.

The following facts are known from Kleptsyna and Le Breton [46]:

(i) Z_k is the fundamental semimartingale associated with the process u_k. (ii) Z_k is a $(\mathcal{F}_t)$ - semimartingale with the decomposition $Z_{k,t}=\theta\int_0^t Q_{k,s}dv_s+\mathcal{M}_{k,t}^H$. (iii) u_k admits the representation $u_{k,t}=\int_0^t K_H(t,s)dZ_{k,s}$. (iv) The natural filtration $(\mathcal{Z}_t)$ of Z_k and $(\mathcal{U}_t)$ of u_k coincide.

We have

$$\begin{aligned}Q_{k,t} &= \frac{d}{dv_t}\int_0^t k_H(t,s)u_{k,s}ds=\kappa_H^{-1}\frac{d}{dv_t}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds\\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\frac{d}{dt}\int_0^t s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds\\ &= \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t \frac{d}{dt}s^{1/2-H}(t-s)^{1/2-H}u_{k,s}ds\end{aligned}$$

$$= \quad \kappa_H^{-1}\lambda_H t^{2H-1}\int_0^t s^{1/2-H}(t-s)^{-1/2-H}u_{k,s}ds.$$

The process Q_k depends continuously on u_k and therefore, the discrete observations of u_k does not allow one to obtain the discrete observations of Q. The process Q_k can be approximated by

$$\widetilde{Q}_{k,n} = \kappa_H^{-1}\lambda_H n^{2H-1}\sum_{j=0}^{n-1} j^{1/2-H}(n-j)^{-1/2-H}u_{k,j}$$

where $0 \le t_1 < t_2 < \ldots < t_n = t$. It is easy to show that $\widetilde{Q}_{k,n} \to Q_{k,t}$ almost surely as $n \to \infty$, see Tudor and Viens (2007).

Define a new partition $0 \le r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k, \quad k = 1, 2, \cdots, n$. Define for each $i \ge 1$,

$$\widetilde{Q}_i(t_k) = \kappa_H^{-1}\eta_H t_k^{2H-1}\sum_{j=1}^{m_k} r_j^{1/2-H}(r_{m_k}-r_j)^{-1/2-H}u_i(r_j)(r_j - r_{j-1}),$$

$k = 1, 2, \cdots, n$. It is easy to show that $\widetilde{Q}_i(t_k) \to Q_i(t)$ almost surely as $m_k \to \infty$ for each $k = 1, 2, \cdots, n$ and $i \ge 1$, see Tudor and Viens (2007).

We use this approximate observation in the calculation of our estimators. Thus our observations are

$$u_i(t) \approx \int_0^t K_H(t,s)d\widetilde{Z}_i(s) \quad \text{where} \quad \widetilde{Z}_i(t) = \theta\int_0^t \widetilde{Q}_i(s)dv_s + \mathcal{M}_{i,t}^H$$

observed at $t_1, t_2, \ldots, t_n$.

Note that for equally spaced data $\Delta v_{t_i} := v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[i^{2-2H} - (i-1)^{2-2H}]$. For $H = 0.5$, $v_{t_i} - v_{t_{i-1}} = \lambda_H^{-1}\left(\frac{T}{n}\right)^{2-2H}[i^{2-2H} - (i-1)^{2-2H}] = \frac{T}{n}, \quad i = 1, 2, \ldots, n$ the standard equispaced partition. In this paper we do not need to assume $T/n \to 0$ unlike the finite dimensional diffusion models as we take advantage of the increasing spatial dimension $K \to \infty$ in this paper.

The process $\Pi^K u$ is finite dimensional, continuous in the mean, and Gaussian process but *not in general a diffusion process* because the operators A_0 and A_1 need not commute with Π^K. Let

$$q_t^{\theta,K}(\Pi^K Q) = E(\Pi^K A^\theta Q|\mathcal{F}_t^K)$$

where $\mathcal{F}_t^K$ is the sigma-algebra generated by $\Pi^K Q(s), 0 \le s \le t$. But $q_t^{\theta,K}(\Pi^K u)$ is not known explicitly. One can apply Kitagawa algorithm for the computation of the estimator.

By Theorem 7.12 in Liptser and Shiryayev [54], $\Pi^K u$ satisfies

$$d\Pi^K Q(t) = q_t^K(\Pi^K Q)dt + d\tilde{W}^{K,H}(t),\ \Pi^K Q(0) = 0 \tag{2.7}$$

where $\tilde{W}^{K,H}(t) = \sum_{i=1}^K \tilde{W}_i^H(t)e_i$ and $\tilde{W}_i(t), i = 1, 2, \ldots, K$ are independent one-dimensional fractional Wiener processes in general different for different θ. Since $\left\{\Pi^K A^\theta Q,\ W^{K,H}\right\}$ is a Gaussian system for every $\theta \in \Theta$, from Theorem 7.16 and Lemma 4.10 in Liptser and Shiryayev [54], the likelihood is given by

$$\begin{aligned}\Lambda_K^\theta(Q) := \tfrac{dP_\theta}{dP_{\theta_0}}(\Pi^K Q) = \exp\Big\{&\textstyle\int_0^T \left(q_t^{\theta,K}(\Pi^K Q) - q_t^{\theta_0,K}(\Pi^K Q), d\Pi^K Q\right)_0 \\ &-\tfrac{1}{2}\textstyle\int_0^T (\|q_t^{\theta,K}(\Pi^K Q)\|_0^2 - \|q_t^{\theta_0,K}(\Pi^K Q)\|_0^2)dv_t\Big\}.\end{aligned} \tag{2.8}$$

The maximum likelihood estimate (MLE) is defined as

$$\hat{\theta}^K = \arg\max_{\theta} \frac{dP_\theta}{dP_{\theta_0}}(\Pi^K Q).$$

But since the function $q^{\theta,K}(\Pi^K Q)$ is not known explicitly, this estimate can not be computed. The situation is much simpler if the operators A_0 and A_1 commute with Π^K so that $\Pi^K A_i = \Pi^K A_i \Pi^K$, $i = 0, 1$, and $U_t^K(X) = \Pi^K A^\theta X(t)$, that is, they have a *common set* of eigenfunctions. In this case the MLE is computable and is given by

$$\hat{\theta}^K = \frac{\int_0^T (\Pi^K A_1 Q(t), d\Pi^K Q(t) - \Pi^K A_0(x)Q(t)dv_t)_0}{\int_0^T \|\Pi^K A_1 Q(t)\|_0^2 dv_t}. \tag{2.9}$$

Of course this expression (2.9) is well defined when the operators A_0 and A_1 do not commute with Π^K, and if the whole trajectory of u is observed, then the values of $\Pi^K A_1 Q(t)$ and $\Pi^K A_1 Q(t)$ can be evaluated, making (2.9) computable. Even though (2.9) is not in general maximum likelihood estimate of θ, it looks like a natural estimate to consider.

For sufficiently large K, note that $P\left(\int_0^T \|\Pi^K A_1 Q(t)\|_0^2 dv_t > 0\right) = 1$. Since the operator A_1 is not identically zero, $\left(\Pi^K A_1 W^H\right)_{t\geq 0}$ is a continuous nonzero square integrable martingale, while $\left(\int_0^t \Pi^K A_1[\theta_1(L+A) + \theta_2 B]Q(s)\ dv_s\right)_{t\geq 0}$ is a continuous process with bounded variation. Then it follows that

$$\hat{\theta}^K = \theta_0 + \frac{\int_0^T (\Pi^K A_1 Q(t), dW^{K,H}(t))_0}{\int_0^T \|\Pi^K A_1 Q\|_0^2 dv_t} \quad \text{P-a.s.} \tag{2.10}$$

In order that $\int_0^T \|\Pi^K A_1 Q(t)\|_0^2 dv_t \to \infty$ as $K \to \infty$, the operator A_1 should be essentially non-degenerate.

Recall that the random field u defined on M satisfies the evolution equation

$$du(t,x) + [\theta_1(L+A) + \theta_2 B + N]u(t,x)dt = dW^H(t,x), 0 \leq t \leq T, u(0,x) = 0$$

where $\theta_1 > 0$ and $\theta_2 \in \mathbb{R}$. We estimate θ_1 when θ_2 is known and we estimate θ_2 when θ_1 is known. We supress the dependence of Q and W on x. According to (2.10)

$$\hat{\theta}_1^K = \frac{\int_0^T (\Pi^K(L+A)Q(t), d\Pi^K Z(t) - d\Pi^K(\theta_2 B + N)Q(t)dv_t)_0}{\int_0^T \|\Pi^K(L+A)Q(t)\|_0^2 dv_t}, \tag{2.11}$$

$$\hat{\theta}_2^K = \frac{\int_0^T (\Pi^K BQ(t), d\Pi^K Z(t) - d\Pi^K \theta_1(L+A) + N)Q(t)dv_t)_0}{\int_0^T \|\Pi^K BQ(t)\|_0^2 dv_t}. \tag{2.12}$$

Theorem 2.1 *The operator L is a positive definite self-adjoint elliptic operator of order $2m$ and $c := \max(order(A), order(B), order(N)) < 2m$. When θ_2 is known, $\hat{\theta}_1^K$ is consistent and asymptotically normal :*

$$a)\quad \hat{\theta}_1^K \to^P \theta_1 \quad as \quad K \to \infty$$

$$b)\ \ \Psi_{K,1}^{1/2}(\hat{\theta}_1^K - \theta_1) \to^{\mathcal{D}} \mathcal{N}(0,1) \ \ as \ \ K \to \infty$$

where $\Psi_{K,1}^{1/2} =: \sqrt{(T/2\theta_1)\sum_{i=1}^K l_i}$.

Let B *be essentially non-degenerate and* $order(B) = b \geq m - d/2$. *When* θ_1 *is known,* $\hat{\theta}_2^K$ *is consistent and asymptotically normal:*

$$c)\ \ \hat{\theta}_2^K \to^P \theta_2 \ \ as \ \ K \to \infty$$

$$d)\ \ \Psi_{K,2}^{1/2}(\hat{\theta}_2^K - \theta_2) \to^{\mathcal{D}} \mathcal{N}(0,1) \ \ as \ \ K \to \infty$$

where $\Psi_{K,2}^{1/2} \asymp \sqrt{\sum_{i=1}^K l_i^{(b-m)/m}}$.

Remarks Since $l_i \asymp i^{2m/d}$, the rate of convergence for $\hat{\theta}_1^K$ is $\Psi_{K,1}^{1/2} \asymp K^{m/d+1/2}$, and for $\hat{\theta}_2^K$ is

$$\Psi_{K,2}^{1/2} \asymp \begin{cases} K^{(b-m)/d+1/2} & : \ \ b > m - d/2 \\ \sqrt{\ln K} & : \ \ b = m - d/2 \end{cases}$$

Theorem 2.2 *If* θ_1 *is known and* $order(B) = b < m - d/2$, *the estimator* $\hat{\theta}_2^K$ *is inconsistent:*

$$\hat{\theta}_2^K \to^P \theta_2 + \frac{\int_0^T (BQ(t), dW^H(t))_0}{\int_0^T \|BQ(t)\|_0^2 dv_t} \ \ as \ \ K \to \infty.$$

Remark $\hat{\theta}_1^K$ is consistent and asymptotically normal for $b \geq m - d/2$ and $b < m - d/2$. But $\hat{\theta}_2^K$ is consistent and asymptotically normal for $b \geq m - d/2$ and inconsistent $b < m - d/2$.

Proof. If $\psi_k(t) := \psi_k(u(t))$, then

$$d\psi_k(t) = -\theta_1 l_k \psi_k(t)dt - \psi_k((\theta_1 A + \theta_2 B + N)u(t))dt + dW_k^H(t), \ \ \psi_k(0) = 0.$$

The solution of this equation is given by $\psi_k(t) = \zeta_k(t) + \eta_k(t)$ where

$$\zeta_k(t) := \int_0^t e^{-\theta_1 l_k(t-s)} dW_k^H(s), \quad \eta_k(t) := -\int_0^t e^{-\theta_1 l_k(t-s)} \psi_k((\theta_1 A + \theta_2 B + N))u(t)dt$$

Thus the solution process can be written as $u(t) = \zeta(t) + \eta(t)$ where $\zeta(t)$ is defined by the sequence $\{\zeta_k(t)\}_{k\geq 1}$ and $\eta(t)$ defined by the sequence $\{\eta_k(t)\}_{k\geq 1}$.

It can be shown by direct computation that if $\mathcal{P}$ is an essentially nondegenerate operator of order $p \geq m - d/2$, then the asymptotics of $E\int_0^T \|\Pi^K A_1 \mathcal{P} u(t)\|_0^2 dv_t$ is determined by the asymptotics of $E\int_0^T \|\Pi^K A_1 \mathcal{P}\zeta(t)\|_0^2 dv_t$.

Specifically

$$E\int_0^T \|\Pi^K A_1 \mathcal{P} Q(t)\|_0^2 dv_t \asymp E\int_0^T \|\Pi^K A_1 \mathcal{P}\zeta(t)\|_0^2 dv_t \asymp \sum_{i=1}^K l_i^{(p-m)/m}, \ \ K \to \infty$$

which implies

$$P - \lim_{K\to\infty} \int_0^T \|\Pi^K A_1 \mathcal{P} Q(t)\|_0^2 dv_t = \infty$$

and also

$$\lim_{K\to\infty}\frac{E\int_0^T\|\Pi^K A_1\mathcal{P}\eta(t)\|_0^2 dv_t}{E\int_0^T\|\Pi^K A_1\mathcal{P}\zeta(t)\|_0^2 dv_t}=0,\quad P-\lim_{K\to\infty}\frac{\int_0^T\|\Pi^K A_1\mathcal{P}\zeta(t)\|_0^2 dv_t}{E\int_0^T\|\Pi^K A_1\mathcal{P}\zeta(t)\|_0^2 dv_t}=1$$

which imply

$$P-\lim_{K\to\infty}\frac{\int_0^T\|\Pi^K A_1\mathcal{P}Q(t)\|_0^2 dv_t}{E\int_0^T\|\Pi^K A_1\mathcal{P}Q(t)\|_0^2 dv_t}=1.$$

Hence by the martingale LLN the following result holds:

$$P-\lim_{K\to\infty}\frac{\int_0^T(\Pi^K A_1\mathcal{P}Q(t),d\mathcal{M}_k^H)_0}{\int_0^T\|\Pi^K A_1\mathcal{P}Q(t)\|_0^2 dv_t}=0$$

and by the martingale CLT the following result holds:

$$\lim_{K\to\infty}\frac{\int_0^T(\Pi^K A_1\mathcal{P}Q(t),d\mathcal{M}_k^H)_0}{\sqrt{E\int_0^T\|\Pi^K A_1\mathcal{P}Q(t)\|_0^2 dv_t}}=\mathcal{N}(0,1)$$

in distribution. Thus the statement of the theorem follows by setting $\mathcal{P}=L+A$ and $\mathcal{P}=B$. □

Remark The coefficients $\psi_k(u(t))$ for different k are *dependent* processes because the eigenfunctions of the operators are *different.* This is the *noncommutative* case.

If the operators A, B, N have the same eigenfunctions as L, then the coefficients $\psi(u_k)$ for different k are *independent* OU processes. This is the well studied *commutative* case.

3. Quasi-Likelihood Estimation

The computation of the estimators $\hat\theta_1^K$ and $\hat\theta_2^K$ requires the knowledge of the whole field u rather than its projection. One option is to replace u by $\Pi^K u$. This can simplify the computation but the estimators are far from the maximum likelihood estimators because some information is lost and asymptotic properties of the resulting estimators are more difficult to study. In general, the construction of the estimate depending only on the projection $\Pi^K u$ is equivalent to the parameter estimation for a partially observed system with observations being given by (2.6). This is the reason we take the alternative route of estimating function approach so that the estimators will be computable. We take random sampling as it produces optimal discretization, see Gobet and Stazhynski [30] who consider observations at random stopping times for multidimensional diffusion processes which include Poisson-like random times. Another motivation of using random times is from mathematical finance where one can do almost sure optimal hedging when the discrete rebalancing dates (or trading dates) are stopping times, see Gobet and Landon [29]. Their scheme also includes Karandikar scheme of discretization of stochastic integrals.

If an operator A_i in the equation does not commute with the corresponding projection operator Π^K, then to evaluate $\Pi^k A_i u$, it is not enough to know only $\Pi^k u$. Another approach is to assume Galerkin approximation of the solution is observed, see Huebner (1997) and Huebner *et al.* (1997).

The ideas behind of martingale estimation function, (see Bibby and Srensen [4]), quasi-likelihood and M-estimation are similar. We consider mixingale estimation function.

We need some preliminary results in this section. The following is the strong law of large numbers for triangular array of mixingales.

Lemma 3.1 *(Mixingale SLLN) (De Jong [27]) : Suppose the triangular array* $\{X_{n,i}, \mathcal{F}_{n,i}\}$ *is a* L_2*-mixingale and for a positive integer-valued sequence* m_n*, we have* $\sum_{n=1}^{\infty}(n^{-1}\sum_{i=1}^{n} c_{n,i}\psi_{m_n})^2 < \infty$*. Then* $\bar{X}_n \to 0$ *almost surely as* $n \to \infty$*.*

Lemma 3.2 *Mixing CLT (Peligrad and Utev [58]): Let* $\{X_{n,i}, 1 \le i \le k_n\}$ *be a tringular array of random variables satisfying:*

a) $var(\sum_{j=a}^{b} X_{n,j}) \le C \sum_{j=a}^{b} var(X_{n,j})$ *for every* $0 \le a \le b \le k_n$ *where* C *is a universal constant;*

b) $\liminf_{n\to\infty} \dfrac{var(\sum_{j=1}^{k_n} X_{n,j})}{\sum_{j=1}^{k_n} var(X_{n,j})} > 0$*;*

c) $\left|cov\left(\exp\left(it\sum_{j=a}^{b} X_{n,j}\right), \ \exp\left(it\sum_{j=b+u}^{c} X_{n,j}\right)\right)\right| \le h_t(u)\sum_{j=a}^{c} var(X_{n,j})$

for every $0 \le a \le b \le c \le k_n$ *where* $h_t(u) \ge 0, \sum h_t(2^j) < \infty$ *and* u *is of the form* $u = [(c-a)^{1-\epsilon}]$ *for certain* $0 < \epsilon < 1$*;*

d) $\sigma_n^{-2}\sum_{j=1}^{k_n} EX_{n,j}^2 I(|X_{n,j}| > \epsilon\sigma_n)$ *as* $n \to \infty$ *for every* $\epsilon > 0$ *where* σ_n^2 *denotes* $var(\sum_{j=1}^{k_n} X_{n,j})$*. Then* $S_n/\sigma_n \to^{\mathcal{D}} \mathcal{N}(0,1)$ *as* $n \to \infty$ *where* $S_n = \sum_{j=1}^{k_n} X_{n,j}$*.*

The following is the central limit theorem for triangular array of mixingales.

Lemma 3.3 *(Mixingale CLT) (Ikeda [41]) : Suppose the triangular array* $\{X_{n,i}, \mathcal{F}_{n,i}\}$ *is a uniformly integrable* L_1*-mixingale. Then*

$$\sum_{i=1}^{nt} X_{n,i} \to^{\mathcal{D}} \int_0^t \delta_v^{1/2} dW_v$$

as $n \to \infty$ *where* $h^{-1}E\left[\left(\sum_{i=\lfloor nt\rfloor}^{\lfloor n(t+h)\rfloor} X_{n,i}\right)^2 |\mathcal{F}_{n,\lfloor ns\rfloor}\right] - \delta_t \to^{P} 0$ *as* $n^{-1} + h + (nh)^{-1} \to 0$ *for some* s, t *such that* $0 \le s < t < t+h < 1$ *with* $(\delta_t)_{t\in[0,1]}$ *is non-negative,* t*-continuous, uniformly integrable and* $\mathcal{F}$*-measurable and* $\int_0^1 \delta_s ds$ *is uniformly bounded away from zero, and* W *is standard Brownian motion*

For simplicity of presentation, we assume $\theta_2 = 0$ and $A = 0, B = I, N = 0$ and we estimate the parameter θ_1. We denote θ_1 by θ. We keep $k = 1$ fixed. Thus

$$d\psi_k(t) = -\theta l_k \psi_k(t)dt + dW_k^H(t), \ \psi_k(0) = 0, \ k \ge 1.$$

Consider the Fourier expansion of the process

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t)\phi_k(x) \tag{3.1}$$

corresponding to some orthogonal basis $\{\phi_k(x)\}_{k=1}^{\infty}$.

Note that $\{u_k^\theta(t), k \geq 1\}$ are *dependent* one dimensional Ornstein-Uhlenbeck processes

$$du_k^\theta(t) = \mu_k^\theta u_k^\theta(t)dt + \lambda_k^{-\alpha}(\theta)dW_k^H(t), \quad u_k^\theta(0) = u_{0k}^\theta \tag{3.2}$$

Recall that $\mu_k = -\lambda_k^{2m} + \kappa(\theta)$.

Thus

$$du_k^\theta(t) = (\kappa(\theta) - \lambda_k^{2m})u_k^\theta(t)dt + \lambda_k^{-\alpha}(\theta)dW_k^H(t), k \geq 1. \tag{3.3}$$

The Fourier coefficients $u_k^\theta(t)$ are observed at discrete time points. As an alternative approach, random field $u^\theta(t,x)$ could be observed at discrete time points t and discrete positions x. Thus the spatial resolution could be an stochastic interacting particle system, see Ligget [53]. Recently Hu *et al.* [33] studied existence and uniqueness of interacting system of SPDEs. However, we do not pursue this approach here.

We have random temporal discretization. We study the parameter estimation in two steps: The rate λ of the Poisson process can be estimated given the jump times t_i, therefore it is done at a first step. Since we observe total number of jumps n of the Poisson process over the T intervals of length one, the MLE of λ is given by $\hat{\lambda}_n := \frac{n}{T}$.

Theorem 3.1 a) $\hat{\lambda}_n \to \lambda$ a.s. as $n \to \infty$.
b) $\sqrt{n}(\hat{\lambda}_n - \lambda) \to^{\mathcal{D}} \mathcal{N}(0,\ e^{\lambda}(1-e^{-\lambda}))$ as $n \to \infty$.

Proof. Let V_i be the number of jumps in the interval $(i-1, i]$. Then $V_i,\ i = 1, 2, \ldots, n$ are i.i.d. Poisson distributed with parameter λ. Since Φ is continuous, we have $I_{\{0\}}(V_i) = I_{\{0\}}(Q_{t_i}),$ a.s. $i = 1, 2, \ldots, n$. Note that

$$\frac{1}{n}\sum_{i=1}^{n} I_{\{0\}}(Q_{k,t_i}) \to^{a.s.} E(I_{\{0\}}V_1) = P(V_1 = 0) = e^{-\lambda} \text{ as } n \to \infty.$$

LLN, CLT and delta method applied to the sequence $I_{\{0\}}(Q_{t_i}),\ i = 1, 2, \ldots, n$ produce the results. □

The CLT result above allows us to construct confidence interval for the jump rate λ. A $100(1-\alpha)\%$ confidence interval for λ is given by $\left[\frac{n}{T} - \varepsilon_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}},\ \ \frac{n}{T} + \varepsilon_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}}\right]$ where $\varepsilon_{1-\frac{\alpha}{2}}$ is the $(1-\frac{\alpha}{2})$-quantile of the standard normal distribution.

We have a time series representation of the model. It is well known that the discretized version of the O-U process is an first order autoregressive process (AR(1)). Hence we have

$$Q_{k,t_{i+1}} = e^{-\mu_k(\theta)\Delta t_i} Q_{k,t_i} + \epsilon_{k,i}$$

where

$$\epsilon_{k,i} \sim \mathcal{N}\left(0, \frac{1-e^{-2\mu(\theta)\Delta t_i}}{\nu(\theta)}\sigma_i^2\right), \quad i \geq 1, k \geq 1.$$

Define $\rho := \frac{\lambda}{\lambda+\theta}$. Mixingale estimation function (MEF) estimator, which is also the quasi maximum likelihood estimator (QMLE) is the solution of the estimating equation: $G^*_{n,K}(\theta) = 0$ where

$$G^*_{n,K}(\theta) = \sum_{k=1}^{K}\sum_{i=1}^{n} \frac{\beta_k^{2\alpha}\lambda(\rho(\lambda,\theta))^2}{\rho(\lambda,2\theta)} Q_{k,t_{i-1}}\left[(Q_{k,t_{i-1}}\theta\rho(\lambda,\theta))^2+\lambda\right]^{-1}[Q_{k,t_i}-\rho(\lambda,\theta)Q_{k,t_{i-1}}].$$

We call the solution of the estimating equation the quasi maximum likelihood estimator (QMLE). There is no explicit solution for this equation.

The optimal estimating function for estimation of the unknown parameter θ is given by

$$G_{n,K}(\theta) = \sum_{k=1}^{K}\sum_{i=1}^{n}\beta_k^{2\alpha}Q_{k,t_{i-1}}[Q_{k,t_i}-\rho(\lambda,\theta)Q_{k,t_{i-1}}].$$

The mixingale estimation function (MEF) estimator of ρ is the solution of $G_{n,N}(\theta) = 0$ and is given by

$$\hat{\rho}_{K,n} := \frac{\sum_{k=1}^{K}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{k,t_i}}{\sum_{k=1}^{K}\sum_{i=1}^{n} Q^2_{k,t_{i-1}}}. \tag{3.4}$$

We obtain the strong consistency and asymptotic normality of the estimator.

Theorem 3.2 a) $\hat{\rho}_{K,n} \to^P \rho$ as $n \to \infty$ and $K \to \infty$, such that $\frac{K}{n} \to 0$.
b) $\sqrt{n\Psi_K}(\hat{\rho}_{K,n}-\rho) \to^{\mathcal{D}} \mathcal{N}(0, \ \lambda^{-i}(1-e^{-\rho}))$ as $n \to \infty$ and $K \to \infty$ such that $\frac{K}{\sqrt{n}} \to 0$.

Proof: By using the fact that every stationary mixing process is ergodic, it is easy to show that if $Q_k(t)$ is a stationary ergodic O-U Markov process and t_i is a process with nonnegative i.i.d. increments which is independent of $Q_k(t)$, then $\{Q_{k,t_i}, i \geq 1, k \geq 1\}$ is a stationary ergodic Markov process. Hence $\{Q_{j,t_i}, i \geq 1\}$ is a stationary ergodic Markov process. Thus the extra randomness of the sampling instants preserves the stationarity and ergodicity of the Markov process in order for the law of large numbers to be applicable.

Observe that $Q^\theta_j(t) := v_j$ is a stationary ergodic Markov chain and $v_j \sim \mathcal{N}(0,\sigma^2)$ where σ^2 is the variance of Q_{1,t_0}. Thus by SLLN for zero mean square integrable mixingales (Lemma 3.1), Peligrad and Utev ([58], Theorem B) and arguments in Bibinger and Trabs ([5], Proposition 7.6), we have

$$\sum_{k=1}^{K}\sum_{i=1}^{n} Q_{k,t_{i-1}}Q_{k,t_i} \to^{a.s.} E(Q_{1,t_1}Q_{k,t_0}) = \rho E(Q^2_{1,t_0}) \tag{3.5}$$

and

$$\frac{1}{n\Psi_K}\sum_{k=1}^{K}\sum_{i=1}^{n} Q^2_{k,t_{i-1}} \to^{a.s.} E(Q^2_{1,t_0}). \tag{3.6}$$

Further $Q_{k,i}(t) := S_i$ is a stationary ergodic Markov chain and $S_i \sim \mathcal{N}(0,\sigma^2)$ where σ^2 is the variance of $Q_{k,0}$. SLLN for martingales proves the result.

Thus

$$\frac{\sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}}{\sum_{i=1}^{n} Q_{k,t_{i-1}}^2} \to^P \rho. \tag{3.7}$$

Further,

$$\sqrt{n\Psi_k}(\hat{\rho}_n - \rho) = \frac{(n\Psi_K)^{-1/2} \sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}(Q_{k,t_i} - \theta Q_{k,t_{i-1}})}{(n\Psi_K)^{-1} \sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}^2}. \tag{3.8}$$

Since

$$E(Q_{k,t_2} Q_{k,t_1} | Q_{k,t_1}) = \theta Q_{k,t_1}^2 \tag{3.9}$$

it follows by Lemma 3.3 and Lemma 3.2 which an generalization of Peligrad and Utev ([58], Theorem B), along with the arguments in Bibinger and Trabs [6], that

$$(n\Psi_K)^{-1/2} \sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}(Q_{k,t_i} - \theta Q_{k,t_{i-1}})$$

converges in distribution to normal distribution with mean zero and variance equal to

$$E[(Q_{k,t_1} Q_{k,t_2}) - E(Q_{k,t_1} Q_{k,t_2} | Q_{k,t_1})]^2 = (1 - e^{2(\theta - \lambda_i \delta)})\{2(\lambda_i - \theta)(\lambda_i + 1)\}^{-1}. \tag{3.10}$$

Applying delta method, the result follows. □

In the next step, we use the estimator of λ to estimate θ. Note that

$$\frac{1}{\hat{\rho}_{n,K}} = \frac{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{j,t_{i-1}}^2}{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{j,t_{i-1}} Q_{j,t_i}}. \tag{3.11}$$

Thus

$$1 + \frac{\beta_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}^2}{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}}. \tag{3.12}$$

Hence

$$\frac{\beta_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}^2}{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}} - 1 = -\frac{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}[Q_{k,t_i} - Q_{k,t_{i-1}}]}{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{j,t_i}} \tag{3.13}$$

Now replace λ by its estimator MLE $\hat{\lambda}_n = \frac{n}{T}$.

$$\beta_1^{2m} - \kappa(\theta) = -\frac{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{j,t_{i-1}}[Q_{k,t_i} - Q_{j,t_{i-1}}]}{\frac{T}{n} \sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}} \tag{3.14}$$

Thus

$$\hat{\theta}_{K,n} = \kappa^{-1}\left(\beta_1^{2m} + \frac{\sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}}[Q_{k,t_i} - Q_{k,t_{i-1}}]}{\frac{T}{n} \sum_{k=1}^{K} \sum_{i=1}^{n} Q_{k,t_{i-1}} Q_{k,t_i}}\right). \tag{3.15}$$

Since the function $\kappa^{-1}(\cdot)$ is a continuous differentiable function, applying delta method the following result follows is a consequence of Theorem 3.2.

Theorem 3.3 a) $\hat{\theta}_{K,n} \to^P \theta$ as $n \to \infty$ and $K \to \infty$ such that $\frac{K}{n} \to 0$.
b) $\sqrt{n\Psi_K}(\hat{\theta}_{K,n} - \theta) \to^{\mathcal{D}} \mathcal{N}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta) - \beta_1^{2m})}))$ as $n \to \infty$ and $K \to \infty$ such

that $\frac{K}{\sqrt{n}} \to 0$. In the second stage, we substitute λ by its estimator $\hat{\lambda}_n$.

4. Bernstein-von Mises Theorem

The Bernstein-von Mises theorem states that the posterior distribution of the normalized distance between the randomized parameter θ and the corresponding MLE is asymptotically normal. This implies that the posterior distribution measure approaches the Dirac measure as the data size increases. It also serves as an essential tool in derivation of some asymptotic properties of Bayes estimators. It also gives the equivalence of the MLE and the Bayes estimator. Recently Cheng *et al.* [24] studied BVT and Bayesian estimation for diagonalizable bilinear SPDEs driven by a multiplicative noise.

Here $\Psi_K = \Psi_{K,1}$ for $\theta = \theta_1$ and $\Psi_K = \Psi_{K,2}$ for $\theta = \theta_2$. Also $\hat{\theta}^K = \hat{\theta}^{K,1}$ for $\theta = \theta_1$ and $\hat{\theta}^K = \hat{\theta}^{K,2}$ for $\theta = \theta_2$.

Suppose that Π is a prior probability measure on $(\Theta, \mathcal{D})$, where $\mathcal{D}$ is the σ-algebra of Borel subsets of Θ. Assume that Π has a density $\pi(\cdot)$ w.r.t. the Lebesgue measure and the density is continuous and positive in an open neighbourhood of θ_0.

The posterior density of θ given in Q^n is given by

$$p(\theta|Q^K) := \frac{\Lambda_K^\theta(u)\pi(\theta)}{\int_\Theta \Lambda_K^\theta(u)\pi(\theta)d\theta}. \tag{4.1}$$

Let $\tau = \Psi_K^{1/2}(\theta - \hat{\theta}^K)$. Then the posterior density of $\Psi_K^{1/2}(\theta - \hat{\theta}^K)$ is given by

$$p^*(\tau|Q^K) := \Psi_K^{-1/2} p(\hat{\theta}^K + \Psi_K^{-1/2}\tau|Q^K).$$

Let

$$\nu_K(\tau) := \frac{dP^K_{\hat{\theta}^K + \Psi_K^{-1/2}\tau}/dP^K_{\theta_0}}{dP^K_{\hat{\theta}^K}/dP^K_{\theta_0}} = \frac{dP^K_{\hat{\theta}^K + \psi_K^{-1/2}\tau}}{dP^K_{\hat{\theta}^K}}, \quad C_K := \int_{-\infty}^{\infty} \nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau)d\tau.$$

Clearly

$$p^*(\tau|Q^K) = C_K^{-1}\nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau).$$

Let $\kappa(\cdot)$ be a non-negative measurable function satisfying the following two conditions :

(K1) There exists a number η, $0 < \eta < 1$, for which

$$\int_{-\infty}^{\infty} \kappa(\tau)\exp\{-\frac{1}{2}\tau^2(1-\eta)\}d\tau < \infty.$$

(K2) For every $\epsilon > 0$ and $\delta > 0$

$$e^{-\epsilon\Psi_K}\int_{|\tau|>\delta} \kappa(\tau\Psi_K^{1/2})\pi(\hat{\theta}^K + \tau)d\tau \to 0 \ \text{ a.s. } [P_{\theta_0}] \text{ as } K \to \infty.$$

We need the following Lemma to prove the Bernstein-von Mises theorem.

Lemma 4.1 Under the assumptions (K1) – (K2),
(i) There exists a $\delta_0 > 0$ such that

$$\lim_{K\to\infty} \int_{|\tau|\le \delta_0 \Psi_K^{1/2}} \kappa(\tau) \left| \nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) - \pi(\theta_0)\exp(-\frac{1}{2}\tau^2) \right| d\tau = 0 \quad \text{a.s. } [P_{\theta_0}].$$

(ii) For every $\delta > 0$,

$$\lim_{K\to\infty} \int_{|\tau|\ge \delta \Psi_K^{1/2}} \kappa(\tau) \left| \nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) - \pi(\theta_0)\exp(-\frac{1}{2}\tau^2) \right| d\tau = 0 \quad \text{a.s. } [P_{\theta_0}].$$

Proof. From (2.7) and (2.8), it is easy to check that

$$\log \nu_K(\tau) = -\frac{1}{2}\tau^2 \Psi_K^{-1} \int_0^T \|A_1 Q^K(s)\|_0^2 d\nu_s.$$

Now (i) follows by an application of dominated convergence theorem.

For every $\delta > 0$, there exists $\epsilon > 0$ depending on δ and β such that

$$\begin{aligned}
& \int_{|\tau|\ge \delta \Psi_K^{1/2}} \kappa(\tau) \left| \nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) - \pi(\theta_0)\exp(-\frac{1}{2}\tau^2) \right| d\tau \\
\le\ & \int_{|\tau|\ge \delta \Psi_K^{1/2}} \kappa(\tau)\nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) d\tau + \int_{|\tau|\ge \delta \Psi_K^{1/2}} \pi(\theta_0)\exp(-\frac{1}{2}\tau^2) d\tau \\
\le\ & e^{-\epsilon\Psi_K} \int_{|\tau|\ge \delta \psi_K^{1/2}} \kappa(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) d\tau + \pi(\theta_0) \int_{|\tau|\ge \delta \Psi_K^{1/2}} \exp(-\frac{1}{2}\tau^2) d\tau \\
=:\ & F_K + G_K
\end{aligned}$$

By condition (K2), it follows that $F_K \to 0$ a.s. $[P_{\theta_0}]$ as $K \to \infty$ for every $\delta > 0$. Condition K(1) implies that $G_K \to 0$ as $K \to \infty$. This completes the proof of the Lemma. □

Now we are ready to prove the generalized version of the Bernstein-von Mises theorem for parabolic SPDEs.

Theorem 4.1 Under the assumptions (K1) - (K2), we have

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} \kappa(\tau) \left| p^*(\tau|Q^K) - (\frac{1}{2\pi})^{1/2}\exp(-\frac{1}{2}\tau^2) \right| d\tau = 0 \quad \text{a.s. } [P_{\theta_0}].$$

Proof From Lemma 4.1, we have

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} \kappa(\tau) \left| \nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) - \pi(\theta_0)\exp(-\frac{1}{2}\tau^2) \right| d\tau = 0 \quad \text{a.s. } [P_{\theta_0}]. \tag{4.2}$$

Putting $\kappa(\tau) = 1$ which trivially satisfies (K1) and (K2), we have

$$C_K = \int_{-\infty}^{\infty} \nu_K(\tau)\pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) d\tau \to \pi(\theta_0) \int_{-\infty}^{\infty} \exp(-\frac{1}{2}\tau^2) d\tau \quad \text{a.s. } [P_{\theta_0}]. \tag{4.3}$$

Therefore, by (4.2) and (4.3), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \kappa(\tau) \left| p^*(\tau|Q^K) - (\frac{1}{2\pi})^{1/2} \exp(-\frac{1}{2}\tau^2) \right| d\tau \\
\leq\ & \int_{-\infty}^{\infty} \kappa(\tau) \left| C_K^{-1} \nu_K(\tau) \pi(\hat{\theta}^K + \Psi_K^{-1/2}\tau) - C_K^{-1}\pi(\theta_0) \exp(-\frac{1}{2}\tau^2) \right| d\tau \\
&+ \int_{-\infty}^{\infty} \kappa(\tau) \left| C_K^{-1}\pi(\theta_0) \exp(-\frac{1}{2}\tau^2) - (\frac{1}{2\pi})^{1/2} \exp(-\frac{1}{2}\tau^2) \right| d\tau \to 0 \ \text{ a.s. } [P_{\theta_0}] \text{ as } K \to \infty. \quad \square
\end{aligned}$$

Theorem 4.2 Suppose for some non-negative integer r $\int_{-\infty}^{\infty} |\theta|^r \pi(\theta) d\theta < \infty$ holds. Then

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} |\tau|^r \left| p^*(\tau|Q^K) - (\frac{1}{2\pi})^{1/2} \exp(-\frac{1}{2}\tau^2) \right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

Proof. For $r = 0$, the verification of (K1) and (K2) is easy and the theorem follows from Theorem 3.1. Suppose $r \geq 1$. Let $\kappa(\tau) = |\tau|^r, \delta > 0$ and $\epsilon > 0$. Using $|a+b|^r \leq 2^{r-1}(|a|^r + |b|^r)$, we have

$$\begin{aligned}
& e^{-\epsilon\Psi_K} \int_{|\tau|>\delta} \kappa(\tau\Psi_K^{1/2}) \pi(\hat{\theta}^K + \tau) d\tau \leq \Psi_K^{r/2} e^{-\epsilon\Psi_K} \int_{|\tau-\hat{\theta}^K|>\delta} \pi(\tau)|\tau - \hat{\theta}^K|^r d\tau \\
\leq\ & 2^{r-1}\Psi_K^{r/2} e^{-\epsilon\Psi_K} [\int_{|\tau-\hat{\theta}^K|>\delta} \pi(\tau)|\tau|^r d\tau + \int_{|\tau-\hat{\theta}^K|>\delta} \pi(\tau)|\hat{\theta}^K|^r d\tau] \\
\leq\ & 2^{r-1}\Psi_K^{r/2} e^{-\epsilon\Psi_K} [\int_{-\infty}^{\infty} \pi(\tau)|\tau|^r d\tau + |\hat{\theta}^K|^r] \to 0 \text{ a.s. } [P_{\theta_0}] \text{ as } K \to \infty
\end{aligned}$$

from the strong consistency of $\hat{\theta}^K$ and hypothesis of the theorem. Thus the theorem follows from Theorem 3.1. $\square$

Remark 4.1 For $r = 0$ in Theorem 4.2, we have

$$\lim_{K\to\infty} \int_{-\infty}^{\infty} \left| p^*(\tau|Q^K) - (\frac{1}{2\pi})^{1/2} \exp(-\frac{1}{2}\tau^2) \right| d\tau = 0 \ \text{ a.s. } [P_{\theta_0}].$$

This is the classical form of Bernstein-von Mises theorem for parabolic SPDEs in its simplest form.

As a special case of Theorem 4.2, we obtain $E_{\theta_0}[\Psi_K^{1/2}(\hat{\theta}^K - \theta_0)]^r \to E[\xi^r]$ as $K \to \infty$ where $\xi \sim \mathcal{N}(0,1)$.

5. Quasi-Bayes Estimation

As an application of Theorem 4.1, we obtain the asymptotic properties of a regular Bayes estimator of θ. Suppose $l(\theta, \phi)$ is a loss function defined on $\Theta \times \Theta$. Assume that $l(\theta, \phi) = l(|\theta - \phi|) \geq 0$ and $l(\cdot)$ is non decreasing. Suppose that J is a non-negative function on $\mathbb{N}$ and $\kappa(\cdot)$ and $\widetilde{G}(\cdot)$ are functions on $\mathbb{R}$ such that

(B1) $J(K)l(\tau\Psi_K^{-1/2}) \leq \widetilde{G}(\tau)$ for all K,
(B2) $J(K)l(\tau\Psi_K^{-1/2}) \to \kappa(\tau)$ as $K \to \infty$ uniformly on bounded subsets of $\mathbb{R}$.
(B3) $\int_{-\infty}^{\infty} \kappa(\tau + s) \exp\{-\frac{1}{2}\tau^2\} d\tau$ has a strict minimum at $s = 0$.

(B4) $\widetilde{G}(\cdot)$ satisfies (K1) and (K2).

Let

$$B_K(\phi) = \int_\theta l(\theta, \phi) p(\theta | Q^K) d\theta.$$

A regular Bayes estimator $\tilde{\theta}^n$ based on Q^K is defined as

$$\tilde{\theta}^K := \arg \inf_{\phi \in \Theta} B_K(\phi).$$

Assume that such an estimator exists. The following Theorem shows that MLE and Bayes estimators are asymptotically equivalent as $n \to \infty$.

Theorem 5.1 Assume that (K1) - (K2) and (B1) - (B4) hold. Then we have

(i) $\Psi_K^{1/2}(\tilde{\theta}^K - \hat{\theta}^K) \to 0$ a.s.-$[P_{\theta_0}]$ as $K \to \infty$,

(ii) $\lim_{n\to\infty} J(K) B_K(\tilde{\theta}^K) = \lim_{n\to\infty} J(K) B_K(\hat{\theta}^K) = (\frac{1}{2\pi})^{1/2} \int_{-\infty}^{\infty} \kappa(\tau) \exp(-\frac{1}{2}\tau^2) d\tau$ a.s. $[P_{\theta_0}]$.

Proof. The proof is analogous to Theorem 4.1 in Borwanker *et al.* (1972). We omit the details. □

Corollary 5.2 Under the assumptions of Theorem 5.1, we have

(i) $\tilde{\theta}^K \to \theta_0$ a.s. $[P_{\theta_0}]$ as $K \to \infty$.

(ii) $\Psi_K^{1/2}(\tilde{\theta}^K - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$ as $K \to \infty$.

Proof. (i) and (ii) follow easily by combining Theorem 5.1 and the strong consistency and asymptotic normality results of the QMLE in Theorem 2.1 and 2.2. □

The following theorem shows that Bayes estimators are locally asymptotically minimax in the Hajek-Le Cam sense, i.e., equality is achieved in Hajek-Le Cam inequality.

Theorem 5.3 Under the assumptions of Theorem 5.1, we have

$$\lim_{\delta\to\infty} \lim_{K\to\infty} \sup_{|\theta-\theta_0|<\delta} E\omega\left(\psi_K^{1/2}(\tilde{\theta}^K - \theta_0)\right) = E\omega(\xi), \quad \mathcal{L}(\xi) = \mathcal{N}(0, 1),$$

where $\omega(\cdot)$ is a bowl shaped loss function.

Proof. The Theorem follows from Theorem III.2.1 in Ibragimov-Has'minskii (1981) since here conditions (N1) - (N4) of the said theorem are satisfied using Lemma 3.1 and local asymptotic normality (LAN) property obtained in Huebner and Rozovskii [38]. □

6. Sequential Estimation

Sequential estimation in Hilbert space valued SDE was first studied in Bishwal [7] based on continuous stopping time.

Consider the stopping time

$$\tau = \tau_R = \inf\{r : \int_0^T \|\Pi^r A_1 Q(t)\|_0^2 dv_t \geq R\} \tag{6.1}$$

where $0 < R < \infty$ is a preassigned level of precision.

Thus we observe the process until the observed Fisher information of the process reaches a preassigned level of precision.

The random vector $(\tau_R, \hat{\theta}^R)$ is called the sequential plan in which τ_R is the special stopping rule and $\hat{\theta}^R$ is the estimate of the parameter θ at the moment τ_R.

We show that the sequential plan is closed, i.e, $P(\tau_R < \infty) = 1$.

Then

$$\hat{\theta}^R = \frac{\int_0^T (\Pi^\tau A_1 u(t), dQ^K(t))_0 + \delta_R \|\Pi^\tau A_1 Q(\tau)\|_0^2 dv_\tau}{R} \quad \text{P-a.s.} \tag{6.2}$$

Then

$$\hat{\theta}^R = \theta_0 + \frac{\int_0^T (\Pi^\tau A_1 Q(t), dW^K(t))_0}{R} \quad \text{P-a.s.} \tag{6.3}$$

because of the discrete nature of the stopping time, it may not reach the precision at τ. We adjust the stopping time using the bias adjustment procedure of Konev and Pergamenshchikov (1990) used for the least squares method in autoregressive model to reach the precision R.

Let δ_R be a fraction of the observed Fisher information such that

$$\int_0^T \|\Pi^\tau A_1 Q(t)\|_0^2 dv_t + \delta_R \|\Pi^\tau A_1 Q(\tau_R)\|_0^2 v_{\tau_R} = R \tag{6.4}$$

giving

$$\delta_R = \frac{R - \int_0^T \|\Pi^\tau A_1 Q(t)\|_0^2 dv_t}{\|\Pi^\tau A_1 Q(\tau_R)\|_0^2 v_{\tau_R}}. \tag{6.5}$$

The proof of the following theorem is standard.

Theorem 6.1 *The sequential estimator $\hat{\theta}^R$ is an unbiased estimator of θ and $\sqrt{R}(\hat{\theta}^R - \theta_0)$ is uniformly (in the parameter) $\mathcal{N}(0,1)$ distributed for fixed R.*

7. Examples

Example 1 Complex-valued OU Process

Arato, Kolmogorov and Sinai [3] studied parameter estimation in the complex valued Ornstein-Uhlenbeck process. They used the model for geophysical problem. Remember that Kolmogorov [47] was the founder of fractional Brownian motion. Hence their model can be extended to fractional Brownian motion. The complex valued fractional Ornstein-Uhlenbeck process is given by

$$du(t) = -[(\alpha_1 + i\beta_1)\theta + (\alpha_0 + i\beta_0)\theta]u(t)dt + \sigma[dW_1^H(t) + idW_2^H(t)] \tag{7.1}$$

where W_1^H, W_2^H are independent fractional Brownian motions, $u(t) = (u_1(t), u_2(t))$ where u_1 and u_2 are the real and imaginary parts of $u(t)$. So this paper is an infinite dimensional generalization of Kolmogorov's model. In the paper they used a time transformation to reduce the general problem to a fixed time case and the asymptotics were studied in large parameter case, see also Bishwal [12] in this context.

Let $\Delta t_i = h, i = 1, 2, \ldots, n,\ \alpha(\theta) = \alpha_0 + \alpha_1\theta$. The Fisher information based on the data $u(t_1), u(t_2), \ldots, u(t_n)$ is given by

$$I(\theta) = \frac{\alpha_1^2}{\alpha(\theta)^2}\left\{1 + (n-1)\left[\left(\frac{1 - e^{-2h\alpha(\theta)} - 2h\alpha(\theta)e^{-2h\alpha(\theta)}}{1 - e^{-2h\alpha(\theta)}}\right)^2 + \frac{(2h\alpha(\theta))^2 e^{-2h\alpha(\theta)}}{1 - e^{-2h\alpha(\theta)}}\right]\right\}$$

$$+\beta_1^2 h^2 (n-1)\frac{e^{-2\alpha(\theta)}}{1 - e^{-2\alpha(\theta)}}$$

It is easy to verify the results of the previous sections.

Example 2 Heat Balance Equation

$$du^\theta(t,x) = \theta_1 \frac{\partial^2}{\partial x^2}u^\theta(t,x)dt + \theta_2 u^\theta(t,x)dt + dW^H(t,x) \tag{7.2}$$

Here $d = 2$, $A = 0$, $B = I$, $order(L) = 2$ so that m=1, $order(A) = 0$, $order(B) = 0$ so that b=0, and $order(N) = 1$. The rates are $\Psi_{K,1} = K$ and $\Psi_{K,2} = \sqrt{\ln K}$, since $b = 0 = m - d/2$.

$$K(\hat\theta_1^K - \theta_1) \to^{\mathcal{D}} \mathcal{N}(0,1) \text{ as } K \to \infty,$$

$$\sqrt{\ln K}(\hat\theta_2^K - \theta_2) \to^{\mathcal{D}} \mathcal{N}(0,1) \text{ as } K \to \infty.$$

Example 3 Stochastic Heat Equation

Consider the stochastic heat equation

$$du^\theta(t,x) = \theta \frac{\partial^2}{\partial x^2}u^\theta(t,x)dt + dW^H(t,x) \tag{7.3}$$

for $0 \le t \le T$ and $x \in (0,1)$ and $\theta > 0$ with periodic boundary conditions.

Here $2m = m_1 = 2$ and $\mu_j = -\theta\pi^2 j^2, \gamma > 1/2$. $\psi_K = K^3$.

$$\sqrt{nK^3}(\hat\theta_{n,K} - \theta) \to^{\mathcal{D}} \mathcal{N}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta) - \beta_1^{2m})}))$$

as $n \to \infty$ and $K \to \infty$.

Es-Sebaiy *et al.* [28] obtained Berry-Esseen bound of the order $O(K^{-3/2})$ for the MLE there by improving the bound $O(K^{-1})$ of Kim and Park [45].

Example 4 Linear Parabolic Equation

As another example of the evolution equation consider the linear parabolic equation

$$du^\theta(t,x) = \theta u^\theta(t,x) + \frac{\partial^2}{\partial x^2}u^\theta(t,x)dt + dW^H(t,x),\ t \ge 0,\ x \in [0,1] \tag{7.4}$$

$$u(0,x) = u_0(x) \in L_2([0,1]) \tag{7.5}$$

$$u^\theta(t,0) = u^\theta(t,1),\ t \in\ [0,T], \tag{7.6}$$

If $d = 2$, then we have

$$\sqrt{n \log K}(\hat{\theta}_{n,K} - \theta) \to^{\mathcal{D}} \mathcal{N}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\beta_i^{2m})}))$$

as $n \to \infty$ and $K \to \infty$.

If $d > 2$, then we have

$$\sqrt{nK^{(d-2)/d}}(\hat{\theta}_{n,K} - \theta) \to^{\mathcal{D}} \mathcal{N}(0,\ (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\beta_1^{2m})}))$$

as $n \to \infty$ and $K \to \infty$.

Concluding Remarks We considered fractional Brownian motion driving term in this paper whose increments are stationary. Using fractional Levy process as the driving term which include jumps, maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [14]. Recently, sub-fractional Brownian (sub-FBM) motion which is a centered Gaussian process with covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0$$

for $0 < H < 1$ introduced by Bojdecki, Gorostiza and Talarczyk [20] has received some attention recently in finite dimensional models. The interesting feature of this process is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. It would be interesting to see extension of this paper to sub-FBM case. We generalize sub-fBM to Sub-fractional Levy process (sub-FLP).

Sub-fractional Levy process (SFLP) is defined as

$$S_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dM_s, \quad t \in \mathbb{R}$$

where $M_t, t \in \mathbb{R}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0,\ E(M_1^2) < \infty$ and without Brownian component. SFLP has the following properties:

1) The covariance of the process is given by

$$\text{Cov}(S_{H,t}, S_{H,s}) = s^{2H} + t^{2H} + \frac{E[L(1)^2]}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) S_H is not a martingale. For a large class of Levy processes, S_H is neither a semimartingale nor a Markov process. 3) S_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) S_H has nonstationary increments. 5) S_H is symmetric. 6) S_H is self similar. 7) S_H has infinite total variation on compacts.

It would be interesting to investigate QML estimation in SPDE driven by subfractional Levy processes which incorporate both jumps and long memory apart from nonstationarity.

Recently Ichiba *et al.* [39, 40] studied generalized fractional Brownian motion (GFBM). A generalized fractional Brownian motion is a Gaussian self-similar process whose increments are not necessarily stationary. It appears in the scaling limit of a shot-noise process with a power law shape function and non-stationary noises with a power law variance function. They studied semi-martingale properties of the mixed process made up of an independent Brownian motion and a GFBM for the persistent Hurst parameter. It would be interesting to extend the current paper to GFBM noise.

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