Eur. J. Stat. 4 (2024) 7
doi: 10.28924/ada/stat.4.7

Measuring Asymmetric Tails Under Copula Distributions

Paramahnasa Pramanik

Department of Mathematics and Statistics, University of South Alabama, Mobile, AL, 36608, United States
ppramanik@southalabama.edu

Abstract. Extensive evidence has been gathered showcasing the prevalence of heavy-tailed distributions and asymmetric tail interdependence within equity and foreign exchange markets, particularly during times of crisis. Tail interdependence in financial markets often manifests as financial contagion, characterized by periods where declining prices and heightened volatility propagate across economic and financial sectors. This phenomenon causes markets that typically exhibit minimal or no correlation to behave similarly, often in opposition to fundamental principles. Instances of unwarranted contagion present a perplexing challenge, suggesting irrationality among market participants and defying conventional risk management strategies and optimal portfolio selections. Our objective is to construct a comprehensive framework for dissecting such occurrences, utilizing a metric of tail-nonexchangeable dependencies employing various copulas with diverse marginal distributions. We offer analytical insights into our measurement of tail order dependence to aid in understanding these events.

1. Introduction

The aim of the research is to suggest effective methods for measuring the extent of non-exchangeability in the extreme ends of a two-variable combined distribution. Investigating non-exchangeability in the tails can offer valuable insights into tail-specific statistical models, which are crucial for understanding non-exchangeability patterns observed in various fields like risk management, quantitative finance, psychometrics, econometrics, and environmetrics [23]. When we describe data as heavy-tailed or fat-tailed, we are indicating that it exhibits a significant portion of notably substantial fluctuations. Here, large and relatively big denote proportions and fluctuations comparable to those seen in a normally distributed random variable. These sizable fluctuations tend to occur concurrently across multiple markets, despite each market typically displaying distinct behavior, meaning they are heterogeneous [13].

Take, for instance, the stock market returns during October 2008. Within just a few days, from October 6 to 10, the S&P 500 experienced a loss of approximately 15%. Were the S&P 500 to follow a normal distribution, such an event would occur no more frequently than once in a million years. Now, observe the performance of other global markets during the same timeframe. The

Received: 4 Apr 2024.

Key words and phrases. non-exchangeable copulas; heavy tail; marginal distribution; tail behavior.

FTSE 100, a prominent European stock index, saw a decline of around 14%, while the Nikkei 225, a major Asian stock index, plummeted by about 21% [13]. Similar or even more significant drops occurred previously, such as on October 19, 1987, famously known as Black Monday, albeit within a single day. By utilizing estimations of the mean and standard deviation of these indices, it becomes evident that if returns adhered to a normal distribution, the likelihood of such drops would be on the order of 10^{-107}. To contextualize, this probability is immensely smaller than the chance of selecting a specific atom from the entirety of atoms in the observable universe, estimated to be $10^{8}0$! [33].

There have recently been quite a few papers focusing on the study of quantifying the degree of overall non-exchangeability. For example, [18] and [19] study the extremal cases of bivariate non-exchangeability using copulas, [3] provides some axioms for measures of bivariate non-exchangeability, [4] and [4] study the cases for some specific bivariate copula families, [6] proposes a test for bivariate non-exchangeability, [7] extends the study of extremal non-exchangeability to multivariate cases. However, all of these studies are for overall non-exchangeability.

This paper focuses on evaluating the extent of tail non-exchangeability in a bivariate random vector with identical marginal distributions. The concept of tail non-exchangeability herein relies on the asymptotic properties of bivariate copulas [12]. Initially, we introduce a suitable metric for measuring the intensity of tail non-exchangeability. Subsequently, we outline various non-exchangeable bivariate copula families derived from commonly employed methodologies. While existing literature explores several straightforward non-exchangeable copulas like the Marshall-Olkin copula and the generalized Clayton Copula as referenced in [5], along with copulas constructed through comonotonic latent variables [11], our paper restricts its focus to two methods for generating non-exchangeable copulas: Khoudraji's device [17], and extreme value copulas utilizing dependence functions featuring non-exchangeable structures.

A bivariate non-exchangeable-transformed Clayton copula is studied in [9], which leads to non-exchangeable structures. In what follows, we refer to this copula as KB4 copula. The cumulative Distribution Function (CDF) of a bivariate KB4 copula is

$$\zeta(u,v) = u^{1-\alpha_1} v^{1-\alpha_2}\left(u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1\right)^{-1/\delta}, \quad \delta \geq 0, \quad (\alpha_1, \alpha_2) \in \ [0,1]^2.$$

Thus, this specific example offers an avenue for investigating tail non-exchangeability. Furthermore, further exploration can be conducted on copulas formed via comonotonic latent variables [11], where non-exchangeable copulas can be readily established, necessitating the modeling of tail non-exchangeability levels. One viable approach involves examining the disparity between certain conditional quantities upon interchanging Z_1 and Z_2. Without loss of generality, for identical non-negative random variables Z_1 and Z_2, we leverage the asymptotic property of $\Gamma(t) = \mathbb{E}[\mathbb{Z}_{\not\vDash}|\mathbb{Z}_{\neq} > \approx]/\mathbb{E}[\mathbb{Z}_{\neq}|\mathbb{Z}_{\not\vDash} > \approx]$ as t tends to infinity to evaluate the intensity of tail non-exchangeability. Conditional quantiles, as utilized in [1], serve as a tool for investigating the robustness of tail dependence.

2. Preliminaries

In the realm of dependence modeling, copula functions are frequently utilized to capture dependency patterns observed in the extreme regions of joint distributions, especially when these patterns cannot be adequately represented by commonly employed multivariate models such as multivariate Normal or Student t distributions. The majority of commonly used bivariate copulas adhere to an exchangeable structure, implying that $\zeta(u, v) \equiv \zeta(v, u)$ for any pairs of $(u, v) \in [0, 1]^2$. Given the pivotal role copula modeling plays in addressing tail dependencies, there is a particular interest in understanding non-exchangeable structures within the joint tails. Inspired by [10], where the tail behavior of $\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx]$ is examined to capture the strength of tail dependence between the bivariate random variable $\{Z_1, Z_2\}$,

Definition 1. *Let $\{Z_1, Z_2\}$ be a bivariate random vector with identically distributed marginals, supported on $[0, \infty)^2$. Then the vector $\{Z_1, Z_2\}$ is said to be tail exchangeable if the following condition holds:*

$$\lim_{t\to\infty} \Gamma(t) := \lim_{t\to\infty} \frac{\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx]}{\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx]} = 1. \tag{1}$$

Remark 1. *We define "tail exchangeability" as a limiting property between two random variables when both of them take large values. When the condition (1) does not hold, the vector is said to be "tail non-exchangeable". The departure of functions $\Gamma(t)$ to 1 as $t \to \infty$ captures the degree of tail non-exchangeability.*

Without loss of generality, assume that the $\{Z_1, Z_2\}$ has a unique copula $\zeta(\cdot, \cdot)$, of which the survival copula denoted as $\hat{\zeta}(\cdot, \cdot)$. Therefore the above condition can be written as:

$$\lim_{t\to\infty} \Gamma(t) = \lim_{t\to\infty} \frac{\int_0^\infty \hat{\zeta}(\overline{F}(z), \overline{F}(t))dz}{\int_0^\infty \hat{\zeta}(\overline{F}(t), \overline{F}(z))dz} = 1, \tag{2}$$

where F is the cdf of the identical univariate marginals. The tail behavior of $\Gamma(t)$ is evidently influenced by both the copula ζ and the marginal distribution F. Our objective is to investigate how various marginal distributions impact the extent of tail non-exchangeability.

2.1. Notations.

In this section, we introduce various notations and symbols, presenting them individually. To begin, we denote distribution functions as $F(z)$, where the argument is enclosed in parentheses. For defining survival functions, we employ $\overline{F}(z)$, again with the argument enclosed in parentheses. It is well-established in traditional literature that the first-order derivative of the distribution function is equivalent to the density function [22]. Hence, in subsequent sections, when we differentiate $F(z)$ with respect to its argument, we denote the derivative as $f(z)$ rather than $F'(z)$. In simpler terms, $f(z) = F'(z) = \partial F(z)/\partial z$.

Secondly, throughout our paper we define the survival copula of an ordinary Copula ζ^* as $\hat{\zeta}^*(. , .)$. Important thing in this case is that $\hat{\zeta}^*(. , .)$ is the Copula before non-exchangeable transformation. After non-exchangeable transformation, we have the survival Copula as $\hat{\zeta}(. , .)$. Here, throughout our paper by Copula we actually mean Survival Copula where it itself is a function of survival functions [i.e. $\overline{F}(z)$]. In order to calculate the first order derivative we further use $\hat{\zeta}^*_{1|2}(u|v)$ instead of $\partial\hat{\zeta}^*(u,v)/\partial v$. From the literature we know that, $\hat{\zeta}^*_{1|2}(u|v)$ is a Cumulative distribution function(cdf) if $u, v \in [0,1]^2$. In this paper we put $u = \overline{F}(z_1)$ and $v = \overline{F}(z_2)$ to make them vary in $[0,1]$. At the tail, when we derive conditional expectation by Laplace approximation, we need to calculate second order derivative of our survival Copula. We use the notation $\hat{\zeta}^*_{1|2,2}(u|v)$t o define $\hat{\zeta}^*_{1|2,2}(u|v) = \partial^2\hat{\zeta}^*(u,v)/\partial v^2 = \partial\hat{\zeta}^*_{1|2}(u|v)/\partial v$.

In the chapter of Extreme value Copula we use the traditional notation of the [21] dependence function as $A(\nu)$. Furthermore, after using non-exchangeable transformation in general extreme value dependence function becomes A^*. In the first part of this section we define our own non-exchangeable extreme value Copula. Inside of the parenthesis we define $\nu = \log(\overline{F}(t))/\log(\overline{F}(z)\overline{F}(t))$. An extreme value survival Copula is non-exchangeable if $A^*(\nu) \neq A^*(1-\nu)$ [32]. In our case, we define $A^*_\alpha(\nu) = A^*(1-\nu)$. Our notation implies that, one survival extreme value Copula is non-exchangeable if $A^*(\nu) \neq A^*_\alpha(\nu)$.

Assumption 1. *Assume that following conditions are fulfilled:*

(1) *Let $f(z)$ be a real valued function on the finite or semi-infinite interval $[\alpha,\beta)$ and in an interval $(\alpha,\alpha+\epsilon]$ with $\epsilon > 0$ it is continuously differentiable, and*

$$\sup_{\alpha+\epsilon\leq z\leq\beta} f(z) \leq f(\alpha) - \delta, \quad \delta > 0.$$

For the derivative $f'(z)$ we have that

$$f'(z) < 0 \text{ for all } z \in (\alpha,\alpha+\epsilon],$$
$$f'(z) = -a(z-\alpha)^{r-1} + o((z-\alpha)^{r-1}) \text{ with } r > 0.$$

(2) *$h(x)$ is a continuous real valued function on the interval $[\alpha,\beta)$ with*

$$h(z) = b(z-\alpha)^{s-1} + o((z-\alpha)^{s-1}) \text{ with } s > 0.$$

(3) *Let the integral be*

$$\int_\alpha^\beta |h(z)| \exp(f(z))dz < \infty.$$

Then

$$I(\lambda) = \int_\alpha^\beta h(z)\exp(\lambda f(z))dz$$

with $\lambda \geq 1$ are all finite and have the asymptotic approximation

$$I(\lambda) \sim \frac{b}{r}\Gamma\left(\frac{s}{r}\right)\left(\frac{r}{a}\right)^{s/r}\lambda^{-s/r}\exp(\lambda f(\alpha)), \quad \lambda \to \infty.$$

3. Khoudraji Device

Assume that Z_1 and Z_2 have identical marginal distribution functions with the cdf F being continuous on $[0, \infty)$, and density functions and moments exist whenever they are used. Following Khoudraji's device, write

$$\hat{\zeta}(\overline{F}(z), \overline{F}(t)) = \overline{F}(z)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\hat{\zeta}^*(\overline{F}(z)^{\alpha_1}, \overline{F}(t)^{\alpha_2}), \ (\alpha_1, \alpha_2) \in [0, 1]^2, \tag{3}$$

where $\hat{\zeta}^*$ is the survival copula of ζ^* that is exchangeable.

Now, by using methods in [10] we are trying to see the tail non-exchangeability of survival copula which we are explaining in the next steps. As the conditional expectations do not have any closed form solutions, [10] suggests to use either Laplace approximation or Watson's lemma for asymptotic approximation when $t \to \infty$ [29]. They used these approximations in exchangeable copulas. In our paper, we are using the same method after transforming a copula into a non-exchangeable structure [33].

Proposition 2. *For all $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\beta > 1$, and if Z_1 and Z_2 are two random variables which exhibits a dependent structure with $g'(0, T) > 0$ then,*

$$\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\succeq} > \approx] \sim -\log(\overline{\mathbb{F}}(\approx))^{\ -\log(\overline{\mathbb{F}}(\approx))\ \eth(\gtrsim, -\log[\overline{\mathbb{F}}(\approx)])}\sqrt{\frac{\not\succeq\succsim\!\!\!\approx}{\log[\overline{\mathbb{F}}(\approx)]\ \eth''(\gtrsim, -\log[\overline{\mathbb{F}}(\approx)])}},$$

for all $t \to \infty$, $\gamma = \lim_{T\to\infty} \max_s g(s, T)$, $T = -\log(\overline{F}(t))$, $\alpha_1 \neq \alpha_2$, and $s \in (0, 0 + \epsilon]$

Proof. Following [10], together with (3),

$$\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\succeq} > \approx] = \int_0^\infty \frac{\hat{\zeta}(\overline{F}(z), \overline{F}(t))}{\overline{F}(t)}\ dz, \ \forall\ t$$
$$= \int_0^\infty \frac{\overline{F}(z)^{1-\alpha_1}\overline{F}(t)^{1-\alpha_2}\hat{\zeta}^*(\overline{F}(z)^{\alpha_1}, \overline{F}(t)^{\alpha_2})}{\overline{F}(t)}\ dz\ \forall\ t \tag{4}$$

Since $y = -\log \overline{F}(z) \implies dy = -\frac{\partial \overline{F}(z)/\partial z}{\overline{F}(z)}dz \implies \overline{F}(z)dy = -\frac{\partial \overline{F}(z)}{\partial z}dz \implies \overline{F}(z)dy = f(F^{-1}(1 - \overline{F}(z)))dz$, after changing of variables we get, $e^{-y}dy = f(F^{-1}(1 - e^{-y}))dz \implies e^{-y}[f(F^{-1}(1 - e^{-y}))]^{-1}dy = dz$. Using this condition into (4) yields,

$$\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\succeq} > \approx] = \int_{\not\vdash}^\infty \mathbb{T}^{\not\succeq_{\not\succeq} - \curvearrowright(\not\vdash - \not\succeq_{\not\Vdash})}\hat{\approx}^*(^{-\not\succeq_{\not\Vdash}\curvearrowright}, ^{-\not\succeq_{\not\succeq}\mathbb{T}})[\mho(\mathbb{F}^{-\not\Vdash}(\not\Vdash - ^{-\curvearrowright}))]^{-\not\Vdash} \curvearrowright \tag{5}$$

Let $y = sT$, and thus $dy = T ds$. Using this condition in (5) yields,

$$\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\succeq} > \approx] = \mathbb{T}^{-\precsim}\int_{\not\vdash}^\infty \mathbb{T}^{\eth(\sim,\mathbb{T})}\eqsim(\sim)\sim, \ \text{where}\eqsim(\sim) = \not\Vdash, \ \forall \sim \in [\not\vdash, \infty), \tag{6}$$

where $w := \lim_{z\to 0^+} \log(f(F^{-1}(z)) < \infty$, a real constant depending on marginal distribution F, and

$$g(s, T) = \alpha_2 - s(2 - \alpha_1) + \frac{1}{T}\{\log[\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})[f(F^{-1}(1 - e^{-sT}))]^{-1}] + w\}. \tag{7}$$

It can be easily verified that $g(0, T) = 0$.

From the previous paragraphs we know, $g(0, T) = 0$ and $g(\infty, T) = -\infty$ as $T \to \infty$. To implement Laplace approximation, we have to check if $g^{'}(0, T) > 0$ for all $T \to \infty$. In this case the asymptotic rate might be $T^{1/2} e^{T\gamma} h(s_0(T); T)[-g''(s_0(T); T)]^{1/2}$. Again, here also $\gamma = \lim_{T\to\infty} \max_s g(s, T)$ and $-g''(s_0(T); T) > 0$ exist and continuous as $T \to \infty$. From (6) we know, $g(s, T) = \alpha_2 - s(2 - \alpha_1) + \frac{1}{T}[\log \frac{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})}{f(F^{-1}(1-e^{-sT}))} + w]$ [in case $\mathbb{E}[\mathbb{Z}_{\not\Bbbk}|\mathbb{Z}_{\not=} > \approx]$]. If we differentiate this function with respect to s, we get,

$$
\begin{aligned}
g'(s, T) &= -(2 - \alpha_1) - \frac{\alpha_1 e^{-\alpha_1 sT} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad - \frac{e^{-sT} f'(F^{-1}(1 - e^{-sT}))[f(F^{-1}(1 - e^{-sT}))]^{-1}}{f(F^{-1}(1 - e^{-sT}))} \\
&= -(2 - \alpha_1) - \frac{\alpha_1 e^{-\alpha_1 sT} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} \\
&\quad - \frac{e^{-sT} f'(F^{-1}(1 - e^{-sT}))}{f^2(F^{-1}(1 - e^{-sT}))} \qquad (8)
\end{aligned}
$$

To satisfy the Laplace approximation we must have

$$
\begin{aligned}
\lim_{T\to\infty} g'(0, T) &= (\alpha_1 - 2) - \lim_{T\to\infty} \frac{\alpha_1 \hat{\zeta}^*_{2|1}(e^{-\alpha_1 sT}|1)}{e^{-\alpha_2 T}} \qquad (9) \\
&\quad - \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0 \qquad (10)
\end{aligned}
$$

In (8), the first order condition determines α_1. If all the above conditions are satisfied with $h(s) = 1$ then, Laplace approximation implies,

$$
\begin{aligned}
\mathbb{E}[\mathbb{Z}_{\not\Bbbk}|\mathbb{Z}_{\not=} > \approx] &= T \int_0^\infty e^{Tg(s,T)} h(s) ds \\
&\sim T \int_0^\infty exp\{Tg(\gamma, T) + \frac{1}{2}(s - \gamma)^2 g''(\gamma, T)\} ds \\
&\sim T e^{Tg(\gamma,T)} \sqrt{\frac{2\pi}{-Tg''(\gamma, T)}}. \qquad (11)
\end{aligned}
$$

In order to satisfy the final expression $g''(\gamma, T)$ must exist and $-g''(\gamma, T) > 0$. This completes the proof. □

Remark 2. *When $\alpha_1 = \alpha_2 = \alpha$, we obtain an exchangeable survival copula. Additionally, setting $\alpha = 1$ yields identical outcomes to those presented in Hua (2014). These specified conditions on the g function are prerequisites before employing Laplace approximation, Watson's lemma, or both. Notably, copulas with Pareto, Weibull, or Exponential margins fulfill both of these conditions.*

Remark 3. *Calculations for the case of $\hat{\zeta}^*$ (where $\hat{\zeta}^*$ is a Clayton Copula with Pareto margins) yield $w = \log \beta$. Apart from that after calculation we get $\frac{1}{T} \log \hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T}) = \frac{1}{T} \log e^{-\alpha_2 T} = -\alpha_2$. Thus, $g(0, T) = 0$ and $g(\infty, T) = -\infty$.*

We know, marginal cumulative distribution function (CDF) of Pareto distribution is $F(x) = 1-(1+x)^{-\beta}$. Thus, the density function is $f(x) = F'(x) = \beta(1+x)^{-(1+\beta)}$ and $f'(x) = -\beta(1+\beta)(1+x)^{-(2+\beta)}$, $\forall\ \beta > 1$. Thus, the extreme right hand side of (9) becomes, $-\lim_{T\to\infty} f'(F^{-1}(0))f^{-2}(F^{-1}(0)) = -\lim_{T\to\infty} -\beta(1+\beta)\beta^{-2}$. Thus, $-\lim_{T\to\infty} f'(F^{-1}(0))f^{-2}(F^{-1}(0)) = 1+\beta^{-1} > 1$. Now, let us solve the second term of the right hand side of equation (9). At first let us write the original survival Clayton copula before [17] transformation. The numerator part is just the first order derivative of Clayton Copula with respect to its first argument and the denominator term is just the copula itself.

We know, Clayton Copula is $\hat{\zeta}^*(u,v) = (u^{-\delta}+v^{-\delta}-1)^{-1/\delta}$. Hence,

$$\begin{aligned}\hat{\zeta}^*_{2|1}(v|u) &= -\frac{1}{\delta}(u^{-\delta}+v^{-\delta}-1)^{-(1+1/\delta)}(-\delta)u^{-(1+\delta)} \\ &= u^{-(1+\delta)}(u^{-\delta}+v^{-\delta}-1)^{-(1+1/\delta)},\ \forall\ \delta>0 \end{aligned} \tag{12}$$

From (9) we know that the middle term is $-\lim_{T\to\infty}\frac{\alpha_1\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}(1,e^{-\alpha_2 T})}$, that is,

$$\begin{aligned} -\lim_{T\to\infty}\frac{\alpha_1\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}^*(1,e^{-\alpha_2 T})} &= -\lim_{T\to\infty}\frac{\alpha_1 e^{-\alpha_2\delta T(1+1/\delta)}}{e^{\alpha_2\delta T}} \\ &= -\lim_{T\to\infty}\alpha_1\ e^{-\alpha_2\delta T} = 0, \end{aligned} \tag{13}$$

From our above discussion it is clear that we can use Laplace Approximation under Clayton copula with Pareto margins when $\lim_{T\to\infty} g'(0,T) > 0$. After using conditions (12),(13) we get $-(2-\alpha_1)+1+1/\beta > 0$ or, $1+\alpha_1+\beta^{-1} > 2 \implies \alpha_1+\beta^{-1} > 1$ otherwise we have to use Watson's lemma [25]. In the next section we will see that we need exactly the same condition before applying Laplace approximation and we will see in the cases of Weibull and exponential margins, $\lim_{T\to\infty} g'(0,T) < 0$.

Remark 4. *From the above example it is clear that, when $\alpha_1+\beta^{-1} > 1$ we can use Laplace Approximation. Furthermore, under this case, $\gamma \to\ \alpha_2\ (\alpha_1)^{-1}$ as $T\to\infty$. On the other hand, in general $\gamma \to\ \alpha_2\delta\left\{\alpha_1\delta+(1-\alpha_1)\right\}^{-1}$, as $T\to\infty$, $\forall\ (\alpha_1,\alpha_2)\in[0,1]^2$, and $\delta>0$.*

Proposition 3. *Let Z_1 and Z_2 be two random variables which exhibit dependent structure and $\int_0^\infty e^{Tg(s,T)}\ ds < \infty$. Then for all $(\alpha_1,\alpha_2)\in[0,1]^2$ and if $g'(0,T) \ngtr 0$ Watson's lemma implies,*

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{\nVdash}{\left[(\nvDash - \ngeq_{\nVdash}) + \mathbb{B}_{\nVdash} + \mathbb{B}_{\nvDash}\right]},\ \ as \approx\to\infty,$$

where $B_1 = \frac{\alpha_1\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}^*(1,e^{-\alpha_2 T})}$, $B_2 = \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}$, and $T = -\log\ \overline{F}(t)$.

Proof. Let us assume $\int_0^\infty e^{Tg(s,T)} < \infty$. From our definition we know that, $T = -\log\ \overline{F}(t)$. This implies, $T\to\infty \iff t\to\infty$. Thus, if we are able to prove,

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{\nVdash}{\left[(\nvDash - \ngeq_{\nVdash}) + \mathbb{B}_{\nVdash} + \mathbb{B}_{\nvDash}\right]},$$

as $T \to \infty$; where

$$B_1 = \frac{\alpha_1 e^{-\alpha_1 sT} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})},$$

and

$$B_2 = \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}.$$

Furthermore, since $g'(0,T) \leq 0$, then Watson's lemma works as $g'(s;T) < 0$. From our previous results , that is enough to prove the proposition. We know, $g(0,T) = 0$ and $g(\infty, T) = -\infty$.

Since $g(s,T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in an interval $(0, 0+\epsilon]$ with $\epsilon > 0$, this function is continuously differentiable, and

$$\sup_{0+\epsilon \leq s \leq \infty} g(s,T) \leq g(0,T) - \psi, \tag{14}$$

with $\psi > 0$.

For $g'(s,T)$ we have $g'(s,T) < 0$ as $\alpha_1 + \beta^{-1} \leq 1$, and $s \to 0$. We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall r > 0.$$

If $r = 1$, then

$$g'(s,T) = -a = -\left[(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}\right].$$

Watson's lemma requires,

$$\lim_{s^+ \to 0} g'(s,T) = -[(2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}],$$

which is a constant if

$$\frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}$$

goes to a constant as $s^+ \to 0$. Thus,

$$-a = -[(2-\alpha_1) + \frac{\alpha_1 \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}^*(1, e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}]$$

or, $a = (2-\alpha_1) + \frac{\alpha_1 \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}^*(1, e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = 2 + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0.$

Assume there exists another real and continuous function $h(s) \in [0,\infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1 \tag{15}$$

when $m = 1$. Assume $\int_0^\infty e^{g(s,T)} ds < \infty$ then, by Watson's lemma the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$. The asymptotic approximation is

$$I(T) \sim \left(\frac{1}{(2-\alpha_1) + D_1 + D_2}\right) \left(T^{-1}\right) e^{Tg(0,T)} \ as, \quad T \to \infty$$

Therefore,

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{\nVdash}{[(\nvDash - \nsucceq_{\nVdash}) + \mathbb{B}_{\nVdash} + \mathbb{B}_{\nvDash}]} \tag{16}$$

as $t \to \infty$, $\forall\ (\alpha_1, \alpha_2) \in [0,1]^2$, $\beta > 1$ and where $B_1 = \frac{\alpha_1 \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}^*(1, e^{-\alpha_2 T})} = \alpha_1$ and $B_2 = \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))}$. This completes the proof. □

Example 1. Above proposition implies,

$$a = (2 - \alpha_1) + \frac{\alpha_1 \widehat{C}^*_{2|1}(e^{-\alpha_2 T}|1)}{\hat{\zeta}^*(1, e^{-\alpha_2 T})} + \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} > 0.$$

In our case, the marginal CDF of the Pareto distribution is $F(x) = 1 - (1+x)^{-\beta}$. Thus, the density function is $f(x) = F'(x) = \beta(1+x)^{-(1+\beta)}$ and $f'(x) = -\beta(1+\beta)(1+x)^{-(2+\beta)}$, $\forall\, \beta > 1$. Thus, $\lim_{T\to\infty} \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = \lim_{T\to\infty} \frac{-\beta(1+\beta)}{\beta^2}$ for $s \in (0, 0+\epsilon]$. Hence, for Pareto margins we have $\lim_{T\to\infty} \frac{f'(F^{-1}(0))}{f^2(F^{-1}(0))} = -\left[1 + \frac{1}{\beta}\right]$ around $s = 0$ [i.e. $s \in (0, 0+\epsilon]$].

In the case of Clayton copula we know that,

$$\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT}) = \alpha_1 e^{sT(1+\alpha_1\delta)}(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-(1+\frac{1}{\delta})}$$

We know, $g(s,T) = (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))}$. We have to calculate the value of $g(s,T)$ around $s = 0$ and as $T \to \infty$ [i.e. a]. We have;

$$\begin{aligned} g(s,T) =& (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T}|e^{-\alpha_1 sT})}{\hat{\zeta}^*(e^{-\alpha_1 sT}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} \\ =& (2-\alpha_1) + \frac{\alpha_1 e^{-\alpha_1 sT}\alpha_1 e^{sT(1+\alpha_1\delta)}(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-(1+\frac{1}{\delta})}}{(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-\frac{1}{\delta}}} \\ &+ \frac{e^{-sT} f'(F^{-1}(1-e^{-sT}))}{f^2(F^{-1}(1-e^{-sT}))} \end{aligned} \tag{17}$$

$$\begin{aligned} a =& (2-\alpha_1) - \left[1 + \frac{1}{\beta}\right],\ \textit{at } T \to \infty \textit{ and } s \in (0, 0+\epsilon] \\ =& 1 - \alpha_1 - \frac{1}{\beta},\ \textit{at } T \to \infty \textit{ and } s \in (0, 0+\epsilon]. \end{aligned} \tag{18}$$

To satisfy $a > 0$ we need $\alpha_1 + \beta^{-1} \leq 1$. This condition is true. Thus, Watson's lemma implies,

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{\nVdash}{\left(\nVdash - \nsucceq_{\nVdash} - \frac{\nVdash}{\lesssim}\right)},\ \partial\approx\ \approx \to \infty\ \partial\ltimes \sim \in (\nvdash, \nvdash + \ntrianglerighteq]. \tag{19}$$

4. Examples

In our study, we utilize a specific form of Archimedean copula known as the Clayton Copula. By definition, this copula is represented as $\hat{\zeta}^*(u,v) = (u^{-\delta} + v^{-\delta} - 1)^{-\frac{1}{\delta}}$, where $(u,v) \in [0,1]^2$ and $\delta \geq 0$. Subsequently, we apply the Khoudraji non-exchangeable device to transform it into the transformed Clayton Copula, as outlined in Khoudraji et al. (1996) (referred to as KB4). Moreover, in our analysis of non-exchangeability at the tail, we consider three distributions as margins:

Pareto, Weibull, and Exponential. These distributions represent the three primary patterns of tail behavior observed in univariate margins, as derived from extreme value theory [38]. Our primary objective in this paper is to examine the influence of both margins and dependence on tail non-exchangeability.

A bivariate Khoudraji-transformed Clayton copula has been studied in Hofert and Vrins (2013), which takes care about the non-exchangeable structure. In our study we name this as KB4 copula. The CDF of bivariate KB4 copula can be written as;

$$\hat{\zeta}(u, v) = (u^{-\alpha_1\delta} + v^{-\alpha_2\delta} - 1)^{-1/\delta} u^{1-\alpha_1} v^{1-\alpha_2}, \quad \delta \geq 0, \quad (\alpha_1, \alpha_2) \in [0, 1]^2, \tag{20}$$

where the original survival Clayton Copula before Khoudraji transformation is $\hat{\zeta}^*(u, v) = (u^{-\delta} + v^{-\delta} - 1)^{-1/\delta}$. At first, we are interested in checking the behavior of our measure of tail *non-exchangeability* represented by;

$$\Gamma(t) = \frac{\mathbb{E}[\mathbb{Z}_{\not{1}}|\mathbb{Z}_{\neq} > \approx]}{\mathbb{E}[\mathbb{Z}_{\neq}|\mathbb{Z}_{\not{1}} > \approx]},$$

where a general tail behavior of $E[Z_1|Z_2 > t]$ as $t \to \infty$ has been studied earlier in the paper.

4.1. **Pareto Margins.** To begin, let's consider employing the Pareto distribution as univariate margins, characterized by the CDF, $F(z) = 1 - (1 + z)^{-\beta}$, where $\beta > 1$ holds for both variables. This distribution follows a power-law pattern and exhibits relatively heavier tails. Widely utilized in economics, actuarial science, geophysics, and other fields, it's sometimes referred to as the Bradford distribution. Originating from Vilfredo Pareto's investigation into the distribution of wealth in class-based societies, as documented in his seminal work "The New Theories of Economics" published in "The Journal of Political Economy" [20], it reveals a notable skewness towards higher values. Pareto observed that approximately 20% of the population possesses around 80% of the total wealth, unveiling significant disparities in resource allocation, especially evident under extreme circumstances. Consequently, the application of extreme value theory becomes imperative, particularly in sectors like insurance and finance, where risk is inherent. Traditional asymptotic theories often fail to adequately predict extreme events in such scenarios. Utilizing the Pareto distribution with the Clayton Copula transformed by [17], enables us to better anticipate and manage these extreme occurrences.

In our model, we aim to compute a stable conditional expectation at extreme values. Given the heavy right skewness of the Pareto distribution, we adopt this margin alongside a non-exchangeable Clayton Copula [31]. Unlike the previous section where we illustrated examples using general Copula functions with various margins, here we directly employ the KT-transformed Clayton Copula with Pareto margins to derive the g function. We demonstrate that the conditions and outcomes obtained in both the preceding and current sections are identical, which is advantageous as it ensures consistency with the theoretical framework previously established. Employing both Laplace Approximation and Watson's lemma, we derive the form of the tail-order non-exchangeable structure. Through simulation studies, we ascertain that Laplace Approximation outperforms Watson's lemma in this context. Leveraging the propositions and lemmas presented, we compute the form of tail-order conditional expectations.

Lemma 4. *Khoudraji transformed Clayton Copula with Pareto margins the integrand function g satisfies $g(0, T) = 0$ and $g(\infty, T) = -\infty$ as $T \to \infty$.*

Proof. Assume the random variables X_1, X_2 follow Pareto distribution with identical CDF $F(z) = 1-(1+z)^{-\beta}, \beta > 1$. Let $\hat{\zeta}$ be the [17]-transformed Clayton survival Copula of (Z_1, Z_2).

$$F(z) = 1-(1+z)^{-\beta},\ \beta > 1$$
$$\text{Thus,}\ \overline{F}(z) = 1 - F(z) = 1 - 1 + (1+z)^{-\beta}\ = (1+z)^{-\beta}, \tag{21}$$

where $F(z)$ and $\overline{F}(z)$ represent distribution and survival functions respectively.

Following [10] we transform the survival function $\overline{F}(t) \to\ e^{-T}$ $or, T = -\log \overline{F}(t) = \beta \log(1+t)$. Now,

$$\mathbb{E}[\mathbb{Z}_{\not\Bbbk}|\mathbb{Z}_{\not\approx} > \approx] = \frac{\mathbb{T}}{\lesssim}\int_{\not\Bbbk}^{\infty} \mathbb{T}\eth(\sim,\mathbb{T})\sim,\ \forall\ \mathbb{T} \tag{22}$$

where, $g(s, T) = 1 + \frac{s}{\beta} + \frac{1}{T}\log\hat{\zeta}(e^{-sT}, e^{-T})$. Clearly, $t \to \infty \iff T \to \infty$.

Again,

$$\begin{aligned}\widehat{C}(e^{-sT}, e^{-T}) &= (e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-1/\delta} e^{-(1-\alpha_1)sT} e^{-(1-\alpha_2)T}\\ &= (e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1)^{-1/\delta} e^{-sT+\alpha_1 sT - T + \alpha_2 T}\end{aligned} \tag{23}$$

Hence,

$$\begin{aligned}&\frac{1}{T}\log\hat{\zeta}(e^{-sT}, e^{-T}) = -\frac{1}{T}[1/\delta\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + sT - \alpha_1 sT + T - \alpha_2 T]\\ \implies &\frac{1}{T}\log\hat{\zeta}(e^{-sT}, e^{-T}) = -\frac{1}{T}[1/\delta\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]\end{aligned} \tag{24}$$

In order to do either Laplace Approximation or Watson's Lemma we need to check if (i) $g(0; T) = 0$ and (ii) $g(\infty; T) = -\infty$ are true. Lets check the first case; (i) $\implies$

$$\begin{aligned}g(0; T) &= 1 - \frac{1}{T}[\frac{1}{\delta}\log(1 + e^{\alpha_2\delta T} - 1) + (1-\alpha_2)T], as\ e^0 = 1\\ &= 1 - \frac{1}{T}[\frac{1}{\delta}\log e^{\alpha_2\delta T} + (1-\alpha_2)T]\\ &= 1 - \frac{1}{T}[\alpha_2 T + T - \alpha_2 T] = 1 - 1 = 0,\end{aligned} \tag{25}$$

as $T \to \infty$. From the above calculation we conclude that, condition (i) holds. To show (ii) let use write g(s;T) one more time. We know from the previous calculations,

$$\begin{aligned}g(s; T) &= 1 + \frac{s}{\beta} - \frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]\\ \implies g(\infty; T) &= 1 + \infty - \frac{1}{T}[\frac{1}{\delta}\log(e^{\infty} + e^{\alpha_2\delta T} - 1) + \infty + (1-\alpha_2)T]\\ \implies g(\infty; T) &= -\infty,\end{aligned} \tag{26}$$

as $T \to \infty$. From (25) and (26) we get the result. □

Lemma 5. *$g'(0, T) > 0$ if $\alpha_1 + \beta^{-1} > 1$ and $\alpha_2 > \alpha_1$, for all $(\alpha_1, \alpha_2) \in [0, 1]^2$ and $\beta > 1$.*

Proof. Consider the g function,

$$g(s; T) = 1 + \frac{s}{\beta} - \frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T\right] \quad (27)$$

$$\begin{aligned}
\text{Thus, } g'(s; T) &= \frac{1}{\beta} - \frac{1}{T}\left[\frac{\alpha_1 \delta T e^{\alpha_1 \delta s T}}{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1-\alpha_1)T\right] \\
&= \frac{1}{\beta} - \frac{1}{T}\left[\frac{\alpha_1 \delta T}{\delta[1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}]} + (1-\alpha_1)T\right] \\
&= \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}} - (1-\alpha_1) \quad (28)
\end{aligned}$$

$$\implies \lim_{T\to\infty} g'(0, T) = \frac{1}{\beta} + \alpha_1 - 1 \text{ , } if \ \alpha_2 > \alpha_1 \quad (29)$$

where $\alpha_1 \in [0, 1]$ and $\beta^{-1} \in [0, 1)$. To obtain $g'(0, T) > 0$ we need $\beta^{-1} + \alpha_1 - 1 > 0$ or, $\beta^{-1} + \alpha_1 > 1$. If we compare this result with the general case, we can see that, both of the results are same. □

Remark 5. Under Laplace approximation the asymptotic rate is $T^{1/2} e^{T\gamma} h(s_0(T))[-g''(s_0(T), T)]^{-1/2}$, where $h(s_0(T)) = 1$ and $\gamma = \lim_{T\to\infty} \max_s g(s, T)$, $s_0(T) = \arg\max_s g(s, T)$.

Proposition 6. *Under Clayton Copula with Pareto margins, $\gamma \to \alpha_2 \ (\alpha_1)^{-1}$ as $t \to \infty$, which is the ratio of two non-exchangeable parameters.*

Proof. For KB4 copula with Pareto margins, g function is

$$g(s, T) = 1 + s\beta^{-1} - T^{-1}[\delta^{-1}\log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T],$$

which is continuous everywhere through $\mathbb{R}$. Before calculating condition (ii) let us first determine the exact form of $s_0(T) = \arg\max_s \ g(s, T)$. Equation (27) implies,

$$\begin{aligned}
g'(s, T) &= \frac{1}{\beta} - \frac{1}{T}\left[\frac{\alpha_1 \delta T}{\delta(1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T})} + (1-\alpha_1)T\right] \\
&= \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}} - (1-\alpha_1) \quad (30)
\end{aligned}$$

First order condition yields,

$$\begin{aligned}
&\frac{1}{\beta} - \frac{\alpha_1}{1 + e^{\alpha_2 \delta T - \alpha_1 \delta s T} - e^{-\alpha_1 \delta s T}} - (1-\alpha_1) = 0 \\
&\frac{\alpha_1}{1 + e^{-\alpha_1 \delta s T}(e^{\alpha_2 \delta T} - 1)} = \frac{1}{\beta} - (1-\alpha_1) \\
&e^{-\alpha_1 \delta s T}(e^{\alpha_2 \delta T} - 1) = \frac{\alpha_1 \beta}{1 - \beta(1-\alpha_1)} - 1 \\
&e^{-\alpha_1 \delta s T} = \frac{1}{e^{\alpha_2 \delta T} - 1}\left[\frac{\alpha_1 \beta}{1 - \beta(1-\alpha_1)} - 1\right]
\end{aligned}$$

$$
\begin{aligned}
e^{\alpha_1 \delta s T} &= \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1 - \alpha_1)]}{\alpha_1 \beta - (1 - \beta(1 - \alpha_1))} \\
\alpha_1 \delta s T &= \log \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1 - \alpha_1)]}{\alpha_1 \beta - (1 - \beta(1 - \alpha_1))} \\
s_0(T) &= \frac{1}{\alpha_1 \delta T} \log \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1 - \alpha_1)]}{\alpha_1 \beta - (1 - \beta(1 - \alpha_1))} \qquad (31)
\end{aligned}
$$

From (31) we get the exact expression of $s_0(T)$. Now we have to check the behavior of this above function when $T \to \infty$. We know,$\gamma = \lim_{T \to \infty} max_s g(s, T)$. Using the expression of $s_0(T)$ in (31) and taking the limit of T in both the sides yields

$$
\begin{aligned}
\gamma &= \lim_{T \to \infty} s_0(T) \\
&= \lim_{T \to \infty} \left\{ \frac{1}{\alpha_1 \delta T} \log \frac{(e^{\alpha_2 \delta T} - 1)[1 - \beta(1 - \alpha_1)]}{\alpha_1 \beta - (1 - \beta(1 - \alpha_1))} \right\} \\
&= \lim_{T \to \infty} \frac{\log(e^{\alpha_2 \delta T} - 1)}{\alpha_1 \delta T} + \lim_{T \to \infty} \frac{\log(1 - \beta(1 - \alpha_1))}{\alpha_1 \delta T} \\
&\quad - \lim_{T \to \infty} \frac{\alpha_1 \beta - (1 - \beta(1 - \alpha_1))}{\alpha_1 \delta T} \\
&= \lim_{T \to \infty} \frac{\log(e^{\alpha_2 \delta T} - 1)}{\alpha_1 \delta T} [\text{ two right hand side terms go to 0 as} T \to \infty \text{ }] \\
&= \lim_{T \to \infty} \frac{\alpha_2 \delta e^{\alpha_2 \delta T}}{\alpha_1 \delta (e^{\alpha_2 \delta T} - 1)}, \text{ [by L'Hospital Rule]} \\
&= \lim_{T \to \infty} \frac{\alpha_2}{\alpha_1 (1 - e^{-\alpha_2 \delta T})}, \text{ [dividing by } \delta e^{\alpha_2 \delta T} \text{ from numeritor and denominator]} \\
&= \frac{\alpha_2}{\alpha_1}, \ \forall \ (\alpha_1, \alpha_2) \in \ [0, 1]^2 \qquad (32)
\end{aligned}
$$

This completes the proof. □

Remark 6. *If we compare this result with using the theoretical results in the previous section, we have to approach in the following way: In the case of Clayton copula we know that,*

$$
\hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 s T}) = \alpha_1 e^{sT(1 + \alpha_1 \delta)} (e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-(1 + \frac{1}{\delta})}
$$

From our previous result we know that, $-g'(s, T) = (2 - \alpha_1) + \frac{\alpha_1 e^{-\alpha_1 s T} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 s T})}{\hat{\zeta}^*(e^{-\alpha_1 s T}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1 - e^{-sT}))}{f^2(F^{-1}(1 - e^{-sT}))}$. *First order condition implies,*

$$
\begin{aligned}
(2 - \alpha_1) + \frac{\alpha_1 e^{-\alpha_1 s T} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 s T})}{\hat{\zeta}^*(e^{-\alpha_1 s T}, e^{-\alpha_2 T})} + \frac{e^{-sT} f'(F^{-1}(1 - e^{-sT}))}{f^2(F^{-1}(1 - e^{-sT}))} &= 0 \\
(2 - \alpha_1) + \frac{\alpha_1 e^{-\alpha_1 s T} \hat{\zeta}^*_{2|1}(e^{-\alpha_2 T} | e^{-\alpha_1 s T})}{\hat{\zeta}^*(e^{-\alpha_1 s T}, e^{-\alpha_2 T})} &= 0, \ \textit{as } T \to \infty \\
e^{-\alpha_1 \delta s T} = \frac{1}{e^{\alpha_2 \delta T} - 1} \left[\frac{\alpha_1^2 e^{(1 - \alpha_1) s T}}{\alpha_1 - 2} - 1 \right] \qquad (33)
\end{aligned}
$$

Furthermore,

$$\begin{aligned}
e^{-\alpha_1\delta sT} &= \frac{1}{e^{\alpha_2\delta T}-1}\left[\frac{\alpha_1^2 e^{(1-\alpha_1)sT}}{\alpha_1-2}-1\right]\\
e^{\alpha_1\delta sT} &= \frac{(\alpha_1-2)(e^{\alpha_2\delta T}-1)}{\alpha_1^2 e^{(1-\alpha_1)sT}-\alpha_1+2}\\
\alpha_1\delta sT &= \log(e^{\alpha_2\delta T}-1)+\log(\alpha_1-2)-\log[\alpha_1^2 e^{(1-\alpha_1)sT}-\alpha_1+2]\\
s &= \frac{\log(e^{\alpha_2\delta T}-1)}{\alpha_1\delta T}+\frac{\log(\alpha_1-2)}{\alpha_1\delta T}-\frac{\log[\alpha_1^2 e^{(1-\alpha_1)sT}-\alpha_1+2]}{\alpha_1\delta T}\\
&= \frac{\alpha_2 e^{\alpha_2\delta T}}{\alpha_1(e^{\alpha_2\delta T}-1)}-\frac{\alpha_1(1-\alpha_1)se^{(1-\alpha_1)sT}}{\delta[\alpha_1^2 e^{(1-\alpha_1)sT}+2-\alpha_1]},\quad \textit{by L'Hospital Rule}\\
&= \frac{\alpha_2}{\alpha_1}-\frac{\alpha_1 s(1-\alpha_1)}{\delta[\alpha_1^2+(2-\alpha_1)e^{-(1-\alpha_1)sT}]},\quad \textit{as } T\to\infty\\
&= \frac{\alpha_2}{\alpha_1}-\frac{(1-\alpha_1)s}{\alpha_1\delta}, \textit{as } T\to\infty\\
\gamma &= \frac{\alpha_2\delta}{\alpha_1\delta+(1-\alpha_1)},\quad \textit{as } T\to\infty
\end{aligned} \tag{34}$$

From the earlier example we know that, for Clayton Copula with Pareto margin we can use Laplace Approximation if $\alpha_1+\beta^{-1}>1$*. By our assumption we also know that,* $\beta>1$*. Thus,* $\beta^{-1}\in[0,1)$*. To satisfy both the conditions we need* α_1 *is very close to 1. Then the expression* $\gamma\to\ \alpha_2\ (\alpha_1)^{-1}$ *as* $\alpha_1\to 1$*. In the previous proposition we see the expression is exactly same. Thus,* $\gamma\to\frac{\alpha_2}{\alpha_1}$ *as* $T\to\infty$ *and* $\alpha_1\to 1$*.*

Proposition 7. *Second order sufficient condition of the integrand g of Khoudraji transformed Clayton Copula satisfies.*

Proof. From the previous section we know that, $g'(s;T)=\beta^{-1}-T^{-1}[(\alpha_1\delta Te^{\alpha_1\delta sT})\{\delta(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)\}^{-1}+(1-\alpha_1)T]$. In order to satisfy Laplace Approximation we need negative second order value of the integrand. Here we prove that when $\alpha_2=\alpha_1 s$ holds, the second order condition for maximization occurs.

$$\begin{aligned}
As,\ g'(s;T) &= \frac{1}{\beta}-\frac{1}{T}\left[\frac{\alpha_1\delta Te^{\alpha_1\delta sT}}{\delta(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}+(1-\alpha_1)T\right]\\
g''(s;T) &= \frac{\partial g'(s;T)}{\partial s}\\
&= -\left[\frac{\alpha_1^2\delta Te^{\delta sT}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}-\frac{\alpha_1 e^{\delta sT}(\alpha_1\delta Te^{\alpha_1\delta sT})}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}\right]\\
\implies -g''(s;T) &= \left[\frac{\alpha_1^2\delta Te^{\delta sT}(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}-\frac{\alpha_1 e^{\delta sT}(\alpha_1\delta Te^{\alpha_1\delta sT})}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}\right]\\
&= \left[\frac{\alpha_1^2\delta Te^{\delta sT}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}-\frac{\alpha_1 e^{\delta sT}(\alpha_1\delta Te^{\alpha_1\delta sT})}{(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)^2}\right]\\
&= \frac{\alpha_1^2\delta Te^{\delta sT}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\left(1-\frac{e^{\alpha_1\delta sT}}{e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1}\right)
\end{aligned} \tag{35}$$

Taking limit of T in both sides of the (35) we get,

$$\lim_{T\to\infty} -g''(s;T) = \lim_{T\to\infty} \frac{\alpha_1^2 \delta T e^{\delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\left(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right) \tag{36}$$

In order to make $\lim_{T\to\infty} -g''(s;T) > 0$ we need $(1 - e^{\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-1}) > 0$ as

$$\lim_{T\to\infty} \alpha_1^2 \delta T e^{\delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-1} > 0,$$

since the denominator is greater than 0, $0 < \alpha_1 < 1$, and $\delta \geq 0$. Now from the above condition we get, $e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1 - e^{\alpha_1 \delta s T} > 0$ which further implies $e^{\alpha_2 \delta T} > 1$. This is possible only when $s \in [0, \infty)$, $0 < \alpha_1 < 1$ and $\delta \geq 0$.

Let us discuss the property of $g''(s,T)$ in further details. From (36) we know that,

$$\begin{aligned}
\lim_{T\to\infty} -g''(s;T) &= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\left(1 - \frac{e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T e^{\alpha_1 \delta s T}}{e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1}\left(1 - \frac{e^{(\alpha_1 \delta s - \alpha_2 \delta)T}}{e^{(\alpha_1 \delta s - \alpha_2 \delta)T} + 1 - e^{-\alpha_2 \delta T}}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T}{1 + e^{(\alpha_2 \delta - \alpha_1 \delta s)T} - e^{-\alpha_1 \delta s T}}\left(1 - \frac{e^{(\alpha_1 \delta s - \alpha_2 \delta)T}}{e^{(\alpha_1 \delta s - \alpha_2 \delta)T} + 1 - e^{-\alpha_2 \delta T}}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T}{2}\left(1 - \frac{1}{2}\right) \\
&= \lim_{T\to\infty} \frac{\alpha_1^2 \delta T}{4} > 0, \ \ as\ \alpha_1 = \alpha_2.
\end{aligned} \tag{37}$$

This completes the proof. □

Combining the above conditions of function g we get, $g(0;T) = 0$ and $g(\infty, T) = -\infty$. Thus, $g(s;T)$ is strictly increasing for $s \in (0, \gamma]$ and is strictly decreasing for $s \in [\gamma; \infty)$. Now at $t \to \infty$ we can write,

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{\mathbb{T}}{\lnsim} {}^{\mathbb{T}\eth(\sim_{\nvdash}(\mathbb{T});\mathbb{T})} \sqrt{\frac{\nvDash \gnapprox}{-\mathbb{T}\eth''(\sim_{\nvdash}(\mathbb{T});\mathbb{T})}} \sim \frac{\nVdash}{\lnsim} {}^{\mathbb{T}\eth(\sim_{\nvdash}(\mathbb{T});\mathbb{T})} \sqrt{\frac{\nvDash \gnapprox \mathbb{T}}{-\eth''(\sim_{\nvdash}(\mathbb{T});\mathbb{T})}} \tag{38}$$

Again,

$$g(s;T) = 1 + s_0(T)\beta^{-1} - T^{-1}[\delta^{-1} \log(e^{\alpha_1 \delta s_0(T)\ T} + e^{\alpha_2 \delta T} - 1) + (1 - \alpha_1)s_0(T)\ T + (1 - \alpha_2)T],$$

$$-g''(s_0(T);T) = (\alpha_1^2 \delta T e^{\alpha_1 \delta s_0(T)\ T})(e^{\alpha_1 \delta s_0(T)\ T} + e^{\alpha_2 \delta T} - 1)^{-1}(1 - e^{\alpha_1 \delta s T}(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)^{-1}),$$

$$s_0(T) = (\alpha_1 \delta T)^{-1} \log\left[(e^{\alpha_2 \delta T} - 1)[1 - \beta(1 - \alpha_1)](\alpha_1 \beta - (1 - \beta(1 - \alpha_1)))^{-1}\right]$$

and $T = \beta\ \log(1 + t)$.

Corollary 8. *For all $(\alpha_1, \alpha_2) \in [0,1]^2$ and very large value of T, the integrand function g varies slowly and takes the value;*

$$g_a(s,T) = \begin{cases} 1 + (1-\alpha_1)s & \text{if } \alpha_2 > \alpha_1; \\ 1 + s - \alpha_2 & \text{if } \alpha_2 < \alpha_1; \\ (1-\frac{\alpha_2}{2}) + (1-\frac{\alpha_1}{2})s & \text{if } \alpha_2 = \alpha_1, \end{cases}$$

where $g_a(s,T) = T^{-1}[\delta^{-1}\log(e^{\alpha_1\delta sT} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]$.

Again, as we have calculated the optimal s, then we can say that the limit goes to a constant number. Thus, the conditional expectation at $T \to \infty$ becomes,

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{1}{\beta} e^{\beta\log(1+t)g(\alpha_1,\alpha_2,\delta)} \sqrt{\frac{2\pi\beta\log(1+t)}{-g''(\alpha_1,\alpha_2,\delta,\beta\log(1+t))}}, \tag{39}$$

as $T = \beta\log(1+t)$.

Now,

$$\begin{aligned} \mathbb{E}[\mathbb{X}_{\nVdash}|\mathbb{X}_{\nvDash} > \approx] &\sim \frac{1}{\beta} e^{\beta\log(1+t)g(\alpha_1,\alpha_2,\delta)} \sqrt{\frac{8\pi\beta\log(1+t)}{\alpha_1^2\delta\beta\log(1+t)}} \\ &\sim \frac{1}{\beta}(1+t)^{\beta[1+\frac{\alpha_2}{\alpha_1\beta}-F]} \sqrt{\frac{8\pi}{\alpha_1^2\delta}}, \ \forall\ (\alpha_1,\alpha_2) \in [0,1]^2,\ \beta > 1 \text{ and } \delta > 0 \end{aligned} \tag{40}$$

where F is a constant where $F = (1-\frac{\alpha_2}{2}) + (1-\frac{\alpha_1}{2})\frac{\alpha_2}{\alpha_1}$.

Corollary 9. *Let Z_1 and Z_2 be two dependent random variables. For all $(\alpha_1,\alpha_2) \in [0,1]^2$, $\delta \geq 0$, $\beta > 1$ and as $t \to \infty$, we have;*

$$\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx] \sim \frac{1}{\beta}(1+t)^{\beta\left[1+\frac{\alpha_2}{\alpha_1\beta}-\left\{(1-\frac{\alpha_1}{2})+(1-\frac{\alpha_2}{2})\frac{\alpha_2}{\alpha_1}\right\}\right]} \sqrt{\frac{8\pi}{\alpha_2^2\delta}}. \tag{41}$$

To measure the non-exchangeability, we have to take the ratio of two conditional expectations [i.e $\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx]\ \mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx]^{-\nVdash}$]. Using conditions (40) and (41) we get;

$$\frac{\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx]}{\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx]} \sim \frac{(1+t)^{\beta\left[1+\frac{\alpha_2}{\alpha_1\beta}-(1-\frac{\alpha_2}{2})-(1-\frac{\alpha_1}{2})\frac{\alpha_2}{\alpha_1}\right]}}{(1+t)^{\beta\left[1+\frac{\alpha_1}{\alpha_2\beta}-(1-\frac{\alpha_1}{2})-(1-\frac{\alpha_2}{2})\frac{\alpha_1}{\alpha_2}\right]}}$$

$$\forall\ (\alpha_1,\alpha_2) \in [0,1]^2,\ \beta > 1,\ \delta > 0,\ \text{and } s_0 \in [0,\infty). \tag{42}$$

From (42) it is clear that the ratio of two conditional expectations depends on some constant power of $(1+t)$. In order to satisfy Laplace approximation we need just only two conditions, which are $\beta^{-1} + \alpha_1 > 1$ and $\beta^{-1} + \alpha_2 > 1$. By the framework we also know that, $(\alpha_1,\alpha_2) \in [0,1]^2$, $\beta > 1$, $\delta > 0$ and $s_0 \in [0,\infty)$.

Remark 7. In (42) we can see that, if $\alpha_1 = \alpha_2$, the ratio the two conditional expectations becomes 1. Intuitively, $\alpha_1 = \alpha_2$ means the KB4 copula is *exchangeable*. Then, there is no difference between two conditional expectations.

Proposition 10. *Suppose Z_1 and Z_2 are two dependent random variables. If $\alpha_1 + \beta^{-1} < 1$ and $\int_0^\infty e^{g(s,T)} < \infty$, then the integrand function g slowly converges to a constant as $t \to \infty$; where $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta > 0$, $\beta > 1$ and $s \in [0, 0+\epsilon)$. In this case, conditional tail expectation converges to*

$$\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \frac{\nVdash}{\lesssim}\left(\frac{\nVdash}{\nVdash - \frac{\nVdash}{\lesssim} - \nsucceq_{\nVdash}}\right),$$

as $t \to \infty$.

Proof. Now let us consider the other case when Laplace approximation does not work. We use Watson's lemma if $g'(s;T) < 0$. From our previous results we know, $g(0,T) = 0$ and $g(\infty, T) = -\infty$. Again we know, $g(s,T) = 1 + s\beta^{-1} - T^{-1}[\delta^{-1}\log(e^{\alpha_1\delta s T} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T]$. Thus,

$$g(s,T) = 1 + \frac{s}{\beta} - \frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1\delta s T} + e^{\alpha_2\delta T} - 1) + (1-\alpha_1)sT + (1-\alpha_2)T\right]$$

$$\implies g'(s,T) = \frac{1}{\beta} - \frac{1}{T}\left[\frac{\alpha_1\delta T e^{\alpha_1\delta s T}}{\delta(e^{\alpha_1\delta s T} + e^{\alpha_2\delta T} - 1)} + (1-\alpha_1)T\right] \tag{43}$$

Suppose we consider, like in the previous case $\beta^{-1} + \alpha_1 - 1 > 0$ does not hold anymore as $T \to \infty$. Then we no longer can use Laplace approximation. We have to check if $g'(s,T)$ is decreasing in s. In this situation if we divide the numerator and denominator of the second term of the right hand side of (43) by $e^{\alpha_1 s T}$, we get,

$$g'(s,T) = \frac{1}{\beta} - \frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1\delta s T}} - (1-\alpha_1), \ \textit{with } \alpha_2 > \alpha_1$$

$$\implies \lim_{s\to 0^+,\ T\to\infty} g'(s,T) = \frac{1}{\beta} - \lim_{s\to 0^+,\ T\to\infty}\frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1\delta s T}} - (1-\alpha_1)$$

$$= \frac{1}{\beta} + \alpha_1 - 1 \tag{44}$$

In order to show $g'(s,T) < 0$ we have to assume $\beta^{-1} + \alpha_1 \ngtr 1$ where $\beta > 1$ *and* $\alpha_1 \in [0,1]$. Now we can use Watson's lemma. Here we are not using the Watson's lemma directly. Since $g(s,T)$ is a real function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$, this function is continuously differentiable and

$$\sup_{0+\epsilon\le s\le\infty} g(s,T) \le g(0,T) - \psi, \tag{45}$$

with $\psi > 0$.

Now for $g'(s,T)$ we have $g'(s,T) < 0$ as $\beta^{-1} + \alpha_1 < 1$ and $s \to \infty$. We can also write

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \ \forall\ r > 0.$$

Now if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s\to 0^+,\ T\to\infty} g'(s,T) = \beta^{-1} - 1 + \alpha_1$, which is a constant. Thus, $-a = \beta^{-1} + \alpha_1 - 1$ or,

$a = 1 - \beta^{-1} - \alpha_1 > 0$. Assume there is another real and continuous function $h(s, T) \in [0, \infty)$ such that,

$$h(s, T) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. More specifically we assume $h(s, T) = 1$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = 1 \implies b = 1 \tag{46}$$

when $m = 1$.

Finally, as we are assuming $\int_0^\infty e^{g(s,T)} ds < \infty$. Now,

$$\begin{aligned}\int_0^\infty e^{g(s,T)}\, ds &= \int_0^\infty e^{1+\frac{s}{\beta}-\frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T\right]}\, ds\\ &= \int_0^\infty e^{1+\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s-(1-\alpha_2)}\, ds\\ &= e^{\alpha_2}\int_0^\infty e^{\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\, ds\\ &= \vartheta\int_0^\infty e^{\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\, ds,\ \ where\ \vartheta = e^{\alpha_2} < \infty\end{aligned} \tag{47}$$

If we are able to show the integration in (47) is finite, then we are able to use *Watson's* lemma. When T is very large [i.e., $T \to \infty$],

$$\begin{aligned}&\int_0^\infty e^{\frac{s}{\beta}-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\, ds\\ &\le \int_0^{\frac{\alpha_2}{\alpha_1}} e^{\frac{s}{\beta}-\frac{\alpha_1\delta sT}{\delta T}-(1-\alpha_1)s}\, ds + \int_{\frac{\alpha_2}{\alpha_1}}^\infty e^{\frac{s}{\beta}-\frac{\alpha_2\delta T}{\delta T}-(1-\alpha_1)s}\, ds\ \ as\ T \to \infty\\ &= \int_0^{\frac{\alpha_2}{\alpha_1}} e^{\frac{s}{\beta}-\alpha_1 s-(1-\alpha_1)s}\, ds + \int_{\frac{\alpha_2}{\alpha_1}}^\infty e^{\frac{s}{\beta}-\frac{\alpha_2}{\delta}-(1-\alpha_1)s}\, ds\\ &= \int_0^{\frac{\alpha_2}{\alpha_1}} e^{\left[\frac{1}{\beta}-\alpha_1-(1-\alpha_1)\right]s}\, ds + e^{-\alpha_2}\int_{\frac{\alpha_2}{\alpha_1}}^\infty e^{\left[\frac{1}{\beta}-(1-\alpha_1)\right]s}\, ds\\ &= \frac{e^{\left[\frac{1}{\beta}-1\right]\frac{\alpha_2}{\alpha_1}}}{\frac{1}{\beta}-1} + e^{-\alpha_2}\left.\frac{e^{\left[\frac{1}{\beta}-(1-\alpha_1)\right]s}}{\frac{1}{\beta}-(1-\alpha_1)}\right|_{\frac{\alpha_2}{\alpha_1}}^\infty\end{aligned} \tag{48}$$

It is clear that in (48) the first term is always finite . The second term in (48) is finite if $\alpha_1 + \beta^{-1} < 1$ which is obvious as $\beta > 1$. Watson's lemma implies that the approximated value of the integral $I(T) = \int_0^\infty e^{Tg(s,T)} ds$ with $T \ge 1$ are all finite and the asymptotic approximation is

$$I(T) \sim \left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right)\left(T^{-1}\right)e^{Tg(0,T)}\ as,\quad T \to \infty$$

$$\implies I(T) \sim \left(\frac{1}{1-\frac{1}{\beta}-\alpha_1}\right)\left(T^{-1}\right)e^{0}\ as,\ g(0,T) = 0,\ T \to \infty \tag{49}$$

$$\implies I(T) \sim \left(\frac{1}{1 - \frac{1}{\beta} - \alpha_1} \right) \left(T^{-1} \right) \; as, \; g(0,T) = 0, \; T \to \infty \tag{50}$$

Hence,(50) implies,

$$\begin{aligned} \mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] &\sim \left(\frac{1}{1 - \frac{1}{\beta} - \alpha_1} \right) \left(\beta^{-1} \right) \; as, \; g(0,T) = 0, \; T \to \infty \\ &\sim \frac{1}{\beta} \left(\frac{1}{1 - \frac{1}{\beta} - \alpha_1} \right) \; as \; t \to \infty \end{aligned} \tag{51}$$

This completes the proof. □

Corollary 11. *Suppose Z_1 and Z_2 are two random variables. For all $(\alpha_1, \alpha_2) \in [0,1]^2$, $\delta > 0$ and $\beta > 1$ we have;*

$$\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx] \sim \frac{1}{\beta} \left(\frac{1}{1 - \frac{1}{\beta} - \alpha_2} \right), \tag{52}$$

as $t \to \infty$.

Again in this case in order to measure non-exchangeability we have to take the absolute difference between two conditional expectations defined in (51) and (52) respectively. Conditions (51) and (52) yield

$$\frac{\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx]}{\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx]} \sim \left(\frac{1 - \frac{1}{\beta} - \alpha_2}{1 - \frac{1}{\beta} - \alpha_1} \right) \tag{53}$$
$$\forall \; (\alpha_1, \alpha_2) \in [0,1]^2, \; and \; \beta > 1$$

From (53) it is clear that the ratio of two conditional expectations goes to 1 as $t \to \infty$ if $\alpha_1 = \alpha_2$. If we carefully look at (51) and (52), both of the conditional expectations equal $O(\log(1+t)^{-1})$. If we do simulations we always get horizontal lines of $E[Z_1|Z_2 > t]$ and $E[Z_2|Z_1 > t]$ respectively.

Remark 8. *In order to satisfy Watson's lemma, we need just only two conditions, which are $\beta^{-1} + \alpha_1 < 1$ and $\beta^{-1} + \alpha_2 < 1$. By the framework we also know that, $(\alpha_1, \alpha_2) \in [0,1]^2$, $\beta > 1$, $\delta > 0$, and $s \in [0, 0 + \epsilon)$.*

Remark 9. *In (53) we observe that, if $\alpha_1 = \alpha_2$, the ratio of two conditional expectations becomes 1. Intuitively, $\alpha_1 = \alpha_2$ means the KB4 copula becomes exchangeable. Therefore, there is no difference between two conditional expectations.*

4.2. **Weibull Margins.** In this section, we delve into the Weibull distribution. We've opted for this distribution due to its sub-exponential nature. Utilizing this distribution offers the advantage of accommodating small sample sizes effectively. Our focus in this paper lies in developing a model based on extreme value theory, where we are particularly interested in the extremes. Essentially, when dealing with a density function, attention must be paid to its tails. Gathering sufficient data on these tails, especially in extreme value theory, can be prohibitively expensive. For instance, consider the incidence of tornadoes in certain regions of the United States. It's well-known that some Southern regions have a higher tornado probability than others. Tornadoes are categorized based on severity, with EF-5 tornadoes being the most perilous. Within the realm of extreme value theory, we aim to ascertain the probability of EF-5 tornado occurrences. Given their rarity, data availability for EF-5 tornadoes is extremely limited, making acquiring a sizable dataset a costly endeavor in this context.

Consider another scenario, such as that found in economics and finance. Imagine an individual keen on investing in the stock market, a realm known for its high volatility. When entering this market, investors calculate their expected returns based on the probabilities of both gaining and losing money. Before committing funds, investors must bear a cost. By subtracting this cost from the expected returns, one can derive the anticipated profit from investing in the market. Let us illustrate this with a simple model to elucidate the scenario. Consider an investor faces two possibilities: either winning a sum (denoted as W) or losing money (denoted as L). With only two outcomes, the probabilities of winning and losing are equal, each being $\frac{1}{2}$. Consequently, the investor's expected return from the stock market is expressed as $\mathbb{E}(\mathbb{R}) = \frac{⊮}{⊭}\mathbb{W} + \frac{⊮}{⊭}(-\mathbb{L})$ [26]. Such expectations align with Von Neumann-Morgenstern type returns. Assume that the investor incurs a cost of c before investing in the stock market. Thus, the expected profit for the investor becomes $\mathbb{E}(\mathbb{R}) -$. Additionally, consider the extreme cases in this scenario, such as a complete loss of investment. Obtaining data on such losses is exceedingly scarce, and seeking more instances of such data is costly. Therefore, predicting the occurrence of such investor losses necessitates the utilization of small datasets. In such cases, the Weibull distribution proves highly beneficial. Furthermore, this distribution offers visually intuitive graphical representations, facilitating a better understanding of extreme behaviors simply by examining the plots. Furthermore, this distribution is useful in cancer studies [2, 8, 14–16, 41, 42].

In this section, we examine Z_1 and Z_2, which are assumed to adhere to the Weibull distribution characterized by identical cumulative distribution functions $F(z) = 1 - e^{-z^{\gamma}}$ for all z and $\gamma > 0$. Consequently, the survival function becomes $\overline{F}(z) = 1 - F(z) = e^{-z^{\gamma}}$ for all z and $\gamma > 0$. The choice of a positive value for γ is motivated by the desire to emphasize significance at the tail of the distribution. We employ our KB4 copula in conjunction with this margin. Given the Weibull distribution's ability to provide meaningful insights with relatively few observations, we anticipate that conditional expectations derived from it will offer a good fit for the tail. However, as these conditional expectations lack closed-form solutions, we explore the viability of employing Laplace Approximation or Watson's lemma. Prior to delving into these methods, we transform the integrand function into the form $e^{g(s,T)}$ and verify if conditions such as $g(0, T) = 0$ and $g(\infty, T) = -\infty$ hold. Throughout this paper, we denote s as a proxy for Z_1. Subsequently, we discover that Laplace Approximation is not applicable in this case, leaving Watson's lemma as the viable option. To obtain the exact value of conditional expectations when $t \to \infty$. Finally, $\mathbb{E}[\mathbb{Z}_{⊮}|\mathbb{Z}_{⊭} > ≈] \sim ⪈^{-⊮} ≨ (\frac{⊮}{⪈})(⊮ - ⋡_{⊮})^{-⊮}$ as $t \to \infty$ *and* $T = t^{\gamma}$, which is constant. In a similar way by, Watson's lemma yields, $\mathbb{E}[\mathbb{Z}_{\not\neq}|\mathbb{Z}_{\not\Bbbk} > \approx \sim \gtrsim^{-\not\Bbbk} \lesseqgtr (\frac{\not\Bbbk}{\gtrsim})(\not\Bbbk - \ngtr_{\not\neq})^{-\not\Bbbk}$ as $t \to \infty$. Since both of them are constants, their ratio is going to be unity.

Proposition 12. *Integrand of the conditional expectations are multiplicative separative of two monotonic functions.*

Proof. In this example we consider two dependent random variables Z_1 and Z_2 which follow Weibull distribution with identical cumulative distribution functions $F(z) = 1 - e^{-z^{\gamma}}$, $\forall z, \gamma > 0$. Now the survival function should be $\overline{F}(z) = 1 - F(z) = e^{-z^{\gamma}}$ $\forall x, \gamma > 0$. Following [10] we will transform the survival function $\overline{F}(t) \to \ e^{-T}$ $or, T = -\log \overline{F}(t) = t^{\gamma}$, $y = -log\overline{F}(x) = x^{\gamma} \implies z = y^{\frac{1}{\gamma}}$. Differentiating totally both sides of the previous equation we get; $dz = \frac{1}{\gamma} y^{\frac{1}{\gamma}-1} dy$.

Now, by using the method provided by [10] we get;

$$\begin{aligned}
\mathbb{E}(\mathbb{Z}_{\not\Bbbk}|\mathbb{Z}_{\not\neq} > \approx) &= \int_0^{\infty} e^{T} \hat{\zeta}(e^{-y}, e^{-T}) y^{\frac{1}{\gamma}-1} \gamma^{-1} dy \\
&= \gamma^{-1} T^{\frac{1}{\gamma}} \int_0^{\infty} e^{T} \hat{\zeta}(e^{-sT}, e^{-T}) s^{\frac{1}{\gamma}-1} ds, \\
&= \gamma^{-1} T^{\frac{1}{\gamma}} \int_0^{\infty} e^{T[1+\frac{1}{T}\log \hat{C}(e^{-sT}, e^{-T})]} s^{\frac{1}{\gamma}-1} ds \qquad (54)\\
&= \gamma^{-1} T^{\frac{1}{\gamma}} \int_0^{\infty} e^{T g(s;T)} h(s) ds, \ \forall s \in [0, \infty)
\end{aligned}$$

where, $g(s, T) = 1 + T^{-1} \log \hat{\zeta}(e^{-sT}, e^{-T})$, and $h(s) = s^{\frac{1}{\gamma}-1} > 0$. □

Remark 10. Clearly, from the above two equations we can say that, the behavior of the conditional expectation depends on $g(s, T)$ function.

In order to check if Laplace approximation or Watson's lemma is valid, we need to check two conditions first, (i) $g(0, T) = 0$ and, (ii) $g(\infty, T) = -\infty$ first. After using KB4 copula in the g function above we have,

$$g(s, T) = 1 - \frac{1}{T}\left[\frac{1}{\delta}\log(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1) + (1 - \alpha_1)sT + (1 - \alpha_2)T\right] \qquad (55)$$

Thus,

$$g(0; T) = 1 - \frac{1}{T}[\alpha_2 T + T - \alpha_2 T] = 1 - 1 = 0 \qquad (56)$$

and,

$$\begin{aligned}
&g(\infty, T) = 1 - \frac{1}{T}[\frac{1}{\delta}\log(e^{\infty} + e^{\alpha_2 \delta T} - 1) + \infty + (1 - \alpha_2)T] \\
\implies &g(\infty, T) = -\infty, \ as \ T \to \infty. \qquad (57)
\end{aligned}$$

From the above discussion we can see that in the case of KB4 Copula with Weibull margins we can have the g function which satisfies $g(0, T) = 0$ and $g(\infty, T) = -\infty$. Now in order to calculate the conditional tail expectations we need to check if $g'(0, T) > 0$ as $T \to \infty$.

Claim 13. *Khoudraji transformed Clayton Copula does not have any solution of conditional expectations at the tail by Laplace Approximation.*

Proposition 14. *Suppose Z_1 and Z_2 are two dependent random variables. If $\gamma > 1$, conditional tail expectation of KB4 Copula with Weibull margin goes to some constant,*

$$\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\vDash} > \approx] \sim \gtrsim^{-\not\Vdash} \lesseqqgtr \left(\frac{\not\Vdash}{\gtrsim}\right) \frac{\not\Vdash}{\not\Vdash - \ngtr_{\not\Vdash}}$$

as $t \to \infty$.

Proof. In order to satisfy Watson's lemma we have to check if $g(s;T)$ is decreasing in s or if $g'(s,T) < 0$.We know from above;

$$\begin{aligned} g'(s,T) &= -\frac{1}{T}\Big[\frac{\alpha_1 \delta T e^{\alpha_1 \delta s T}}{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1-\alpha_1)T\Big] \\ &= -\Big[\frac{\alpha_1 e^{\alpha_1 \delta s T}}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1-\alpha_1)\Big] \\ &= -\Big[\frac{\alpha_1}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)e^{-\alpha_1 \delta s T}} + (1-\alpha_1)\Big] \\ &= -\Big[\frac{\alpha_1}{e^0 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} + (1-\alpha_1) \\ &= -\Big[\frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} + (1-\alpha_1)\Big] < 0 \\ \implies \lim_{T\to\infty} g'(s,T) &= -[\alpha_1 + 1 - \alpha_1] = -1 < 0, \; if \; \alpha_2 < \alpha_1, \; or \\ &= -(1-\alpha_1) < 0, \; if \; \alpha_2 > \alpha_1 \end{aligned} \tag{58}$$

From (58) we are sure that we need to use Watson's lemma. As $g(s,T)$ is a real valued function on the semi-infinite interval $[0,\infty)$ and in $(0, 0+\epsilon]$ with $\epsilon > 0$ this function is continuously differentiable and

$$\sup_{0+\epsilon \le s \le \infty} g(s,T) \le g(0,T) - \psi, \tag{59}$$

with $\psi > 0$. Now for $g'(s,T)$ we have $g'(s,T) < 0$ as $\frac{1}{\beta} < 1$ and $s^+ \to 0$. We can also write,

$$g'(s,T) = -as^{r-1} + o(s^{r-1}) \; \forall r > 0$$

Now, if we assume $r = 1$ then $g'(s,T) = -a$. From our previous results we know that, $\lim_{s^+\to 0,\; T\to\infty} g'(s,T)$ is either -1 or $-(1-\alpha_1)$, based on the conditions described in (58). As we are concentrating on asymmetric copulas we better choose $-(1-\alpha_1)$ in order to maintain some non-exchangeability. Thus, $-a = -(1-\alpha_1)$ or, $a = 1-\alpha_1 > 0$.

Let us assume there is another continuous real valued function $h(s,T) \in [0,\infty)$ such that,

$$h(s) = bs^{m-1} + o(s^{m-1})$$

with $m > 0$. From (54) we know, $h(s) = s^{\frac{1}{\gamma}-1}$ in our case. Thus,

$$bs^{m-1} + o(s^{m-1}) = s^{\frac{1}{\gamma}-1} \implies b = 1, \; and \; m = \frac{1}{\gamma} \tag{60}$$

Finally, a we are assuming $\int_0^\infty |h(s)|e^{g(s,T)}ds < \infty$. Let us find out the exact condition under which the whole integration becomes finite. After putting the value of $g(s,T)$ in the integrand we have,

$$
\begin{aligned}
&\int_0^\infty |h(s)|e^{g(s,T)}\ ds \\
&= \int_0^\infty |s^{\frac{1}{\gamma}-1}|\ e^{1-\frac{1}{T}[\frac{1}{\delta}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)+(1-\alpha_1)sT+(1-\alpha_2)T]}\ ds \\
&= e^{\alpha_2}\ \int_0^\infty\ s^{\frac{1}{\gamma}-1}\ e^{-\frac{1}{\delta T}\log(e^{\alpha_1\delta sT}+e^{\alpha_2\delta T}-1)-(1-\alpha_1)s}\ ds \\
&\le e^{\alpha_2}\ \left[\int_0^{\frac{\alpha_2}{\alpha_1}}\ s^{\frac{1}{\gamma}-1}\ e^{-\alpha_2-(1-\alpha_1)s}\ ds\ +\int_{\frac{\alpha_2}{\alpha_1}}^\infty\ s^{\frac{1}{\gamma}-1}\ e^{-(\alpha_1+1-\alpha_1)s}\ ds\right],\ at\ T\to\infty \\
&= e^{\alpha_2}\Bigg[e^{-\alpha_2}\ e^{-(1-\alpha_1)s}\sum_{i=0}^{\frac{1}{\gamma}-1}(-1)^{\frac{1}{\gamma}-i-1}\frac{(\frac{1}{\gamma}-1)!}{i!(\alpha_1-1)^{\frac{1}{\gamma}-i}}\ s^i\Bigg|_0^{\frac{\alpha_2}{\alpha_1}} \\
&\qquad\qquad + e^{-s}\sum_{i=0}^{\frac{1}{\gamma}-1}(-1)^{\frac{1}{\gamma}-i-1}\frac{(\frac{1}{\gamma}-1)!}{i!}\ s^i\Bigg|_{\frac{\alpha_2}{\alpha_1}}^{\infty}\Bigg]
\end{aligned}
\tag{61}
$$

If we carefully look at (61) both the terms on the right hand side is always finite. The main reasons are we have e^{-s} and $\gamma > 0$; which leads us three possibilities, $\gamma \in (0,1)$, $\gamma = 1$ and $\gamma > 1$. Let us discuss each of the cases separately. As we have e^{-s} as the first term, it is always finite as $s \to \infty$. Now, only thing matters is the value of γ. When $\gamma \in (0,1)$, $\gamma^{-1} - 1$ takes the highest value when $\gamma \to 0$. By assumption $\gamma > 0$. So $\gamma^{-1} - 1 < \infty$. Under this case we still possibility to have $s^i \to \infty$ as $s \to \infty$. Therefore, we need more restriction on γ. In this bound of $(0,1)$ s^i is not finite. Furthermore, when $\gamma = 1$, $s^i \to \infty$ as $s \to \infty$. Hence, we need $\gamma > 1$ to make $s^i < \infty$ for any large s.

By Watson's lemma we can write the approximated value of the integral $I(T) = \int_0^\infty e^{T g(s,T)} ds$ with $T \ge 1$ are all finite and the asymptotic approximation is;

$$
I(T) \sim \Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1}\left(T^{-\frac{1}{\gamma}}\right)\ e^{T g(0,T)}\ as,\ T\to\infty
$$

$$
I(T) \sim \Gamma\left(\frac{1}{\gamma}\right)\frac{1}{1-\alpha_1}\left(T^{-\frac{1}{\gamma}}\right)\ as,\ T\to\infty\ and\ e^0 = 1 \tag{62}
$$

From equation (54) we know,

$$
\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \gneqq^{-\nVdash}\lneqq\left(\frac{\nVdash}{\gneqq}\right)\frac{\nVdash}{\nVdash - \ntrianglerighteq_{\nVdash}}\ \eth\!\sim\ \approx\ \to \infty\ \eth\!\ltimes\ \mathbb{T} = \approx^{\gtrsim}, \tag{63}
$$

where $\alpha_1 \in [0,1]$, $\gamma > 0$ and $t \to \infty$. This completes the proof. □

Claim 15. *In the similar fashion if X_1 and X_2 are two dependent random variables then,* $\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx] \sim \gneqq^{-\nVdash}\lneqq\left(\frac{\nVdash}{\gneqq}\right)\frac{\nVdash}{\nVdash-\ntrianglerighteq_{\nvDash}}$, *as $t \to \infty$ and $T = t^\gamma$, $\forall\ \gamma > 0$.*

Thus, in our case the measure of tail *non-exchangeability* is going to be;

$$\Gamma(t) \sim \frac{1-\alpha_2}{1-\alpha_1}, \tag{64}$$

as $(\alpha_1, \alpha_2) \in [0,1]^2$, $\gamma > 0$ and $t \to \infty$.

Remark 11. *If we do not consider non-exchangeability* $\lim_{s^+ \to 0, T \to \infty} g'(s,T) = -1$, *which is also a constant . Thus,* $-a = -1$ *or,* $a = 1 > 0$. *If we do Watson's lemma then, the expressions of two conditional expectations should be,* $\mathbb{E}[\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx] \sim \gvertneqq^{-\nVdash} \lneqq \left(\frac{\nVdash}{\gvertneqq}\right)$ *and* $\mathbb{E}[\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx] \sim \gvertneqq^{-\nVdash} \lneqq \left(\frac{\nVdash}{\gvertneqq}\right)$ *respectively as* $t \to \infty$.

4.3. **Exponential Margins.** Finally, we utilize the non-exchangeable transformation of the Clayton Copula with exponential margins as proposed by [17]. This distribution holds significant properties in the literature, such as being part of the exponential family, lacking memory, and crucially, exhibiting characteristics of a Poisson process. Being a Poisson process implies that at extreme values, the distribution stabilizes to a constant. The primary focus of our paper is to explore extreme events and ascertain their probability of occurrence across various domains including nature, actuarial science, econometrics, and finance [37]. Additionally, we aim to identify a slow variation function capable of elucidating conditional tail expectations. The determinant of non-exchangeability represents the ratio of two conditional tail-order expectations. Towards the conclusion of this section, we present simulation results within this framework.

To derive the tail conditional expectations following [10] we are able to show that the integrand function can be decomposed into $g(.)$ and $h(.) > 0$ functions which are multiplicatively separable. Here, $g(.)$ is a monotonically decreasing function which satisfies, $g(0, -\log(\overline{F}(t))) = 0$ and $g(\infty, -\log(\overline{F}(t))) = -\infty$ as $t \to \infty$. As this integration does not have any closed form solution, we try to use different simulation methods. We finally end up with Watson's lemma. In this case Laplace approximation does not work as $g'(0, \log(\overline{F}(t))) \ngtr 0$ but, $g'(s, -\log(\overline{F}(t))) < 0$ as $t \to \infty$; where t is the proxy of X_2 and $\overline{F}(.)$ is the survival function and always in $[0,1]$. After using Watson's lemma we get our tail non-exchageability as

$$\frac{\mathbb{E}(\mathbb{Z}_{\nVdash}|\mathbb{Z}_{\nvDash} > \approx)}{\mathbb{E}(\mathbb{Z}_{\nvDash}|\mathbb{Z}_{\nVdash} > \approx)} \sim \begin{cases} 1 & \text{if } \alpha_2 > \alpha_1; \\ \frac{1-\alpha_2}{1-\alpha_1} & \text{if } \alpha_2 < \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}}{1-\frac{\alpha_1}{2}} & \text{if } \alpha_2 = \alpha_1; \end{cases}$$

where α_1 and α_2 are non-exchangeable components under [17] non-exchangeable transformation of Clayton Copula. In the above expression we can easily see that, at extreme values our measure of non-exchangeability goes to a constant irrespective of the relation between the non-exchangeable components. At the end of this section as we do the simulation we can see that, our simulation-results are consistent with the results corresponding to numerical integrations with certain levels of error.

We know the cumulative distribution function of exponential type is $F(z) = 1 - e^{-\lambda z}, \forall x \in [0, \infty)$. Thus, the survival function becomes, $\overline{F}(x) = 1 - F(x) = e^{-\lambda z}$, $\lambda > 0$, and $\forall\ z \in [0, \infty)$. In the following propositions we are trying to derive a slow variation function of conditional tail expectations.

Proposition 16. *Integrand of the conditional expectations of Khoudraji transformed Clayton Copula with exponential margin is multiplicatively separative of two monotonic functions one of which is strictly positive and other is decreasing.*

Proof. Following [10] we transform the survival function $\overline{F}(t) \to e^{-T}$, *or*, $T = -\log \overline{F}(t) = -\log e^{-\lambda t} = \lambda t$. Now, $y = \log \overline{F}(z) = -\log e^{-\lambda z} = \lambda z$. As, $y = \lambda z$ then after totally differentiate this equation n both sides we get $dy = \lambda dz \implies dz = \lambda^{-1} dy$. Thus, $dy = sdT + Tds = Tds$, as we are assuming T is constant.

Now,

$$\begin{aligned}
\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\models} > \approx] &= \int_0^\infty e^T \widehat{C}(e^{-y}, e^{-T})\lambda^{-1} dy,\ \forall \lambda > 0 \\
&= \lambda^{-1} T \int_0^\infty e^T \widehat{C}(e^{-sT}, e^{-T}) ds,\ \forall s \in [0, \infty) \\
&= \lambda^{-1} T \int_0^\infty e^{T(1 + \frac{1}{T}\log \widehat{C}(e^{-sT}, e^{-T}))} ds,\ \forall s \in [0, \infty) \qquad (65)
\end{aligned}$$

where, $g(s, T) = 1 + T^{-1} \log \hat{\zeta}(e^{-sT}, e^{-T})$ and , $h(s) = 1 > 0$. Clearly, from (65) we can say that, the behavior of the conditional expectation depends on $g(s, T)$ function. □

In this scenario, we observe that $g(0, T) = 0$ and $g(\infty, T) = -\infty$ as T tends to infinity. Consequently, we have the option to employ either Laplace Approximation or Watson's lemma. Should we opt for the former, we must additionally verify if $g'(0, T) > 0$ as T approaches infinity. Upon examination, we ascertain that this condition is not satisfied. Thus, we are compelled to resort to Watson's lemma. Prior to applying Watson's lemma, it is imperative to confirm if $g(s, T)$ exhibits a decreasing trend with respect to s.

Lemma 17. *In the context of the Khoudraji transformed Clayton Copula, the function g demonstrates a decreasing trend with respect to s, with its value contingent upon the non-exchangeable coefficients of the Copula.*

Proof. We know from above;

$$\begin{aligned}
g'(s; T) &= -\frac{1}{T}\left[\frac{\alpha_1 \delta T e^{\alpha_1 \delta s T}}{\delta(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)} + (1 - \alpha_1)T\right] \\
&= -\left[\frac{\alpha_1}{(e^{\alpha_1 \delta s T} + e^{\alpha_2 \delta T} - 1)e^{-\alpha_1 \delta s T}} + (1 - \alpha_1)\right] \\
&= -\left[\frac{\alpha_1}{1 + e^{(\alpha_2 - \alpha_1 s)\delta T} - e^{-\alpha_1 \delta s T}} + (1 - \alpha_1)\right] < 0 \\
\implies \lim_{T \to \infty} g'(s; T) &= -[\alpha_1 + 1 - \alpha_1] = -1 < 0,\ \textit{if}\ \alpha_2 < \alpha_1
\end{aligned}$$

$$\begin{aligned} or\ , &= -(1-\alpha_1) < 0,\ if\ \alpha_2 > \alpha_1 \\ or\ , &= -\left(1-\frac{\alpha_1}{2}\right) < 0,\ if\ \alpha_2 = \alpha_1 \end{aligned} \tag{66}$$

From (66) we are sure that we can use Watson's lemma. Thus, g is decreasing in s and the value depends on the coefficients of non-exchangeability [i.e. α_1, α_2]. □

Claim 18. *Let Z_1 and Z_2 be two dependent random variables. Conditional tail expectation of KB4 Copula with Exponential margins goes to some constant as $t \to \infty$ and the measure of tail non-exchangeability can be written as;*

$$\frac{\mathbb{E}[\mathbb{Z}_{\not\Vdash}|\mathbb{Z}_{\not\models} > \approx]}{\mathbb{E}[\mathbb{Z}_{\not\models}|\mathbb{Z}_{\not\Vdash} > \approx]} \sim \begin{cases} 1 & if\ \alpha_2 > \alpha_1; \\ \frac{1-\alpha_2}{1-\alpha_1} & if\ \alpha_2 < \alpha_1; \\ \frac{1-\frac{\alpha_2}{2}}{1-\frac{\alpha_1}{2}} & if\ \alpha_2 = \alpha_1. \end{cases}$$

where α_1 and α_2 are non-exchangeable components.

5. Concluding remarks

Heavy-tailed distributions and copulas provide a unified methodology for examining crises, significant fluctuations, dependency, and contagion effects in the realms of economics and finance. Our manuscript primarily concentrates on assessing the degree of tail non-exchangeability. We begin by introducing metrics tailored for quantifying the strength of tail non-exchangeability, employing conditional tail expectations [35]. Following this, we present theoretical outcomes for non-exchangeable bivariate copulas generated using Khoudraji's methodology, along with three distinct types of univariate marginals [12]. Our results underscore the heightened importance of tail non-exchangeability particularly when Pareto marginals are employed. Consequently, we advocate for transforming each marginal distribution to conform to a Pareto distribution. In our endeavor to identify tail non-exchangeability, we propose an analytical framework. Throughout our analysis, we examine a specific type of Archimedean copula, namely the Clayton Copula, paired with exponential, Pareto, and Weibull margins.

This type of tail non-exchangeability carries significance in the field of time series analysis, especially when evaluating the evolution of two random variables over time [26]. Such occurrences are apparent when examining the interchangeability of market shares between two separate companies operating within the same industry [24, 29, 38, 40]. If these shares demonstrate interchangeability, it becomes feasible to forecast future share prices for one company based on the information regarding the other. Looking forward, we can investigate the existence of tail interchangeability among different soccer positions across diverse teams. For example, our approach can be utilized to examine the interchangeability of goal dynamics between two strikers from different clubs in the European Football League [39]. Should these dynamics exhibit interchangeability, it suggests that a club can smoothly substitute one striker with another once the contract of the first striker expires.

A similar analysis can be carried out for various batting positions in cricket matches involving different teams [36].

In the domain of infectious disease modeling, the widely utilized susceptibility-infection-recovery (SIR) framework becomes relevant [34]. Our approach facilitates the assessment of exchangeability by analyzing various SIR datasets across two regions [27, 28, 30, 32]. Should exchangeability be confirmed, it indicates that implementing a particular vaccination strategy could result in disease recovery within those regions [31]. This adaptable methodology extends its utility to a wide array of fields, demonstrating its capacity to derive valuable insights from different types of data [25].

References

[1] C. Bernard, C. Czado, Conditional quantiles and tail dependence, J. Multivar. Anal. 138 (2015), 104–126. https://doi.org/10.1016/j.jmva.2015.01.011.

[2] S. Dasgupta, S. Acharya, M.A. Khan, P. Pramanik, S.M. Marbut, F. Yunus, J.N. Galeas, S. Singh, A.P. Singh, S. Dasgupta, Abstract 3318: Frequent loss of CACNA1C, a calcium voltage-gated channel subunit is associated with lung adenocarcinoma progression and poor prognosis, Cancer Res. 83 (2023), 3318–3318. https://doi.org/10.1158/1538-7445.am2023-3318.

[3] F. Durante, E.P. Klement, C. Sempi, M. Úbeda-Flores, Measures of non-exchangeability for bivariate random vectors, Stat. Papers 51 (2008), 687–699. https://doi.org/10.1007/s00362-008-0153-0.

[4] F. Durante, R. Mesiar, L∞-measure of non-exchangeability for bivariate extreme value and Archimax copulas, J. Math. Anal. Appl. 369 (2010), 610–615. https://doi.org/10.1016/j.jmaa.2010.04.005.

[5] E. Furman, J. Su, R. Zitikis, Paths and indices of maximal tail dependence, ASTIN Bull. 45 (2015), 661–678. https://doi.org/10.1017/asb.2015.10.

[6] C. Genest, J. Nešlehová, J.F. Quessy, Tests of symmetry for bivariate copulas, Ann. Inst. Stat. Math. 64 (2011), 811–834. https://doi.org/10.1007/s10463-011-0337-6.

[7] M. Harder, U. Stadtmüller, Maximal non-exchangeability in dimension d, J. Multivar. Anal. 124 (2014), 31–41. https://doi.org/10.1016/j.jmva.2013.10.003.

[8] K.L. Hertweck, K.S. Vikramdeo, J.N. Galeas, S.M. Marbut, P. Pramanik, F. Yunus, S. Singh, A.P. Singh, S. Dasgupta, Clinicopathological significance of unraveling mitochondrial pathway alterations in non-small-cell lung cancer, FASEB J. 37 (2023), e23018. https://doi.org/10.1096/fj.202201724rr.

[9] M. Hofert, F. Vrins, Sibuya copulas, J. Multivar. Anal. 114 (2013), 318–337. https://doi.org/10.1016/j.jmva.2012.08.007.

[10] L. Hua, H. Joe, Strength of tail dependence based on conditional tail expectation, J. Multivar. Anal. 123 (2014), 143–159. https://doi.org/10.1016/j.jmva.2013.09.001.

[11] L. Hua, H. Joe, Multivariate dependence modeling based on comonotonic factors, J. Multivar. Anal. 155 (2017), 317–333. https://doi.org/10.1016/j.jmva.2017.01.008.

[12] L. Hua, A. Polansky, P. Pramanik, Assessing bivariate tail non-exchangeable dependence, Stat. Prob. Lett. 155 (2019), 108556. https://doi.org/10.1016/j.spl.2019.108556.

[13] R. Ibragimov, A. Prokhorov, Heavy tails and copulas: Topics in dependence modelling in economics and finance, World Scientific, 2017. https://doi.org/10.1142/9644.

[14] S. Kakkat, P. Pramanik, S. Singh, A.P. Singh, C. Sarkar, D. Chakroborty, Cardiovascular complications in patients with prostate cancer: Potential molecular connections, Int. J. Mol. Sci. 24 (2023), 6984. https://doi.org/10.3390/ijms24086984.

[15] M.A. Khan, S. Acharya, S. Anand, F. Sameeta, P. Pramanik, C. Keel, S. Singh, J.E. Carter, S. Dasgupta, A.P. Singh, MYB exhibits racially disparate expression, clinicopathologic association, and predictive potential for biochemical recurrence in prostate cancer, iScience 26 (2023), 108487. https://doi.org/10.1016/j.isci.2023.108487.

[16] M.A. Khan, S. Acharya, N. Kreitz, S. Anand, F. Sameeta, P. Pramanik, C. Keel, S. Singh, J. Carter, S. Dasgupta, A. Singh, MP60-05 MYB exhibits racially disparate expression and clinicopathologic association and is a promising predictor of biochemical recurrence in prostate cancer, J. Urol. 211 (2024), e1000. https://doi.org/10.1097/01.ju.0001008804.84010.ec.05.

[17] A. Khoudraji, Contributions à l'étude des copules et à la modélisation de valeurs extrêmes bivariées, Ph.D. Thesis, Université Laval Québec, Canada, 1996. https://elibrary.ru/item.asp?id=5414079.

[18] E.P. Klement, R. Mesiar, How non-symmetric can a copula be? Commentationes Math. Univ. Carolinae, 47 (2006), 141–148. http://dml.cz/dmlcz/119580.

[19] R.B. Nelson, Extremes of nonexchangeability, Stat. Papers 48 (2007), 329–336. https://doi.org/10.1007/s00362-006-0336-5.

[20] V. Pareto, The new theories of economics, J. Polit. Econ. 5 (1897), 485–502. https://doi.org/10.1086/250454.

[21] J. Pickands, Multivariate extreme value distributions, in: Proceedings 43rd Session International Statistical Institute, Volume 2, 859–878, 1981.

[22] A.M. Polansky, P. Pramanik, A motif building process for simulating random networks, Comput. Stat. Data Anal. 162 (2021), 107263. https://doi.org/10.1016/j.csda.2021.107263.

[23] P. Pramanik, Tail non-exchangeability, Thesis, Northern Illinois University, 2016.

[24] P. Pramanik, Optimization of market stochastic dynamics, SN Oper. Res. Forum 1 (2020), 31. https://doi.org/10.1007/s43069-020-00028-x.

[25] P. Pramanik, Effects of water currents on fish migration through a Feynman-type path integral approach under $\sqrt{8/3}$ Liouville-like quantum gravity surfaces, Theory Biosci. 140 (2021), 205–223. https://doi.org/10.1007/s12064-021-00345-7.

[26] P. Pramanik, Optimization of dynamic objective functions using path integrals, PhD thesis, Northern Illinois University, 2021.

[27] P. Pramanik, On lock-down control of a pandemic model, arXiv:2206.04248 [math.OC], (2022). https://doi.org/10.48550/ARXIV.2206.04248.

[28] P. Pramanik, Stochastic control of a sir model with non-linear incidence rate through euclidean path integral, arXiv:2209.13733 [math.OC], (2022). https://doi.org/10.48550/ARXIV.2209.13733.

[29] P. Pramanik, Consensus as a nash equilibrium of a stochastic differential game, Eur. J. Stat. 3 (2023), 10. https://doi.org/10.28924/ada/stat.3.10.

[30] P. Pramanik, Optimal lock-down intensity: A stochastic pandemic control approach of path integral, Comput. Math. Biophys. 11 (2023), 20230110. https://doi.org/10.1515/cmb-2023-0110.

[31] P. Pramanik, Path integral control in infectious disease modeling, arXiv:2311.02113 [physics.soc-ph], (2023). https://doi.org/10.48550/ARXIV.2311.02113.

[32] P. Pramanik, Path integral control of a stochastic multi-risk SIR pandemic model, Theory Biosci. 142 (2023), 107–142. https://doi.org/10.1007/s12064-023-00388-y.

[33] P. Pramanik, Dependence on tail copula, J. 7 (2024), 127–152. https://doi.org/10.3390/j7020008.

[34] P. Pramanik, Estimation of optimal lock-down and vaccination rate of a stochastic SIR model: A mathematical approach, Eur. J. Stat. 4 (2024), 3. https://doi.org/10.28924/ada/stat.4.3.

[35] P. Pramanik, A.K. Maity, Bayes factor of zero inflated models under Jeffereys prior, arXiv:2401.03649 [stat.ME], (2024). https://doi.org/10.48550/ARXIV.2401.03649.

[36] P. Pramanik, A.M. Polansky, Motivation to run in one-day cricket, arXiv:2001.11099 [math.OC], (2020). https://doi.org/10.48550/ARXIV.2001.11099.

[37] P. Pramanik, A.M. Polansky, Optimal estimation of Brownian penalized regression coefficients, arXiv:2107.02291 [stat.ME], (2021). https://doi.org/10.48550/ARXIV.2107.02291.

[38] P. Pramanik, A.M. Polansky, Optimization of a dynamic profit function using Euclidean path integral, SN Bus. Econ. 4 (2023), 8. https://doi.org/10.1007/s43546-023-00602-5.

[39] P. Pramanik, A.M. Polansky, Scoring a goal optimally in a soccer game under Liouville-like quantum gravity action, Oper. Res. Forum 4 (2023), 66. https://doi.org/10.1007/s43069-023-00247-y.

[40] P. Pramanik, A.M. Polansky, Semicooperation under curved strategy spacetime, J. Math. Sociol. 48 (2023), 172–206. https://doi.org/10.1080/0022250x.2023.2180002.

[41] K. Vikramdeo, S. Anand, S. Sudan, P. Pramanik, S. Singh, A. Godwin, A. Singh, S. Dasgupta, Abstract PO3-16-05: Mitochondrial DNA mutation detection in tumors and circulating extracellular vesicles of triple negative breast cancer patients for biomarker development, Cancer Res. 84 (2024), PO3-16-05. https://doi.org/10.1158/1538-7445.sabcs23-po3-16-05.

[42] K.S. Vikramdeo, S. Anand, S.K. Sudan, P. Pramanik, S. Singh, A.K. Godwin, A.P. Singh, S. Dasgupta, Profiling mitochondrial DNA mutations in tumors and circulating extracellular vesicles of triple-negative breast cancer patients for potential biomarker development, FASEB BioAdv. 5 (2023), 412–426. https://doi.org/10.1096/fba.2023-00070.