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A New Family of Distributions Based on Amalgamation of Two Methods with an Application to the Rayleigh Model

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ABSTRACT. In this paper, we present the Alpha power generalized odd generalized exponential-G (APGOGE-G) family of distributions and provides the most common shapes of the hazard rate function: increasing, decreasing, bathtub, and inverted bathtub. We provide some of its structural properties. We estimate the parameters by maximum likelihood estimation method, and perform a simulation study to verify the asymptotic properties of the estimator for the inverse Weibull baseline. The practicality of the new APGOGE-Rayleigh model is shown through application to uncensored real dataset.

1. INTRODUCTION

One of the preferred research fields in the probability distribution submitted is the development of new distributions starting with a baseline distribution with parameters to current distributions for creating groups to exhibit flexibility. Numerous techniques for adding a parameter to distributions have been put forth and utilized to simulate outcomes in a variety of applicable domains, including economics, environmental sciences, architecture, biological research, etc. Over time, statistical distributions have drawn a lot of interest. Because of this, its appeal has evolved throughout time and to make these new families of distributions more adaptive and highly desirable, distribution theory researchers have extended baseline distributions with new parameters. The Alpha power (AP) transformation family [1] has emerged as a useful model in the biological sciences, engineering, medicine, and other areas. The cumulative distribution function (CDF) and probability density function (PDF) of the AP family are

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$$F_{APT}(x)=\begin{cases}\dfrac{\alpha^{G(x;\zeta)}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ G(x;\zeta), & \alpha=1,\end{cases} \tag{1}$$

and

$$f_{APT}(x)=\begin{cases}\dfrac{\log(\alpha)\alpha^{G(x;\zeta)}g(x;\zeta)}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ g(x;\zeta), & \alpha=1.\end{cases} \tag{2}$$

Recent extensions of AP transformation family include: AP Marshall-Olkin-G (APMO-G) family [2], extended AP-G family [3], transmuted AP-G family [4], exponential-AP-G family [5], AP transformed Weibull-G family [6], new-extended AP family [7], generalized AP family [8], Gull AP-G family [9] and so on. [10] introduced the generalized odd generalized exponentiated (GOGE-G). This family was created using the T-X family [11]. The CDF of the GOGE-G family distributions is

$$G_{GOGE}(x;\zeta)=\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}, \tag{3}$$

and the corresponding PDF is

$$g_{GOGE}(x;\zeta)=\frac{\kappa\beta h(x;\zeta)H(x;\zeta)^{\kappa-1}}{\left[1-H(x;\zeta)^{\kappa}\right]^{2}}e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta-1}, \tag{4}$$

For $\kappa,\beta>0$ and $H(x;\zeta)$ is the baseline CDF with vector parameter ζ. The survival function (SF) of the GOGE-E family is

$$SF_{GOGE}(x;\zeta)=1-\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}, \tag{5}$$

where $\bar{H}(x;\zeta)^{\kappa}=1-H(x;\zeta)^{\kappa}$.

The generalisations of the GOGE-G family include the alternative GOGE-G family [12], transmuted-GOGE family [13], and GOGE extended one parameter skew-t distribution [14]. The motivation for this study is based on the increased flexibility attained by combining the AP-transformation family and the GOGE-G family in modelling real-life datasets. This new family can be used with right-or-left skewed and heavy-tailed datasets. In comparison to the GOGE-G family, the APGOGE-G family has multiple shapes for the hazard rate function, such as increasing, decreasing, increasing-decreasing, decreasing-increasing, bathtub and inverted bathtub. The APGOGE-G family can be expressed as an infinite linear combination of the exponentiated-G (Exp-G) distribution and other statistical qualities can be inferred using this property. We develop and study this new family known as the APGOGE-G family by amalgamating the AP-transformation and GOGE-G families of distributions. The rest of the article will be organised as follows. Section 2 comprise the APGOGE-G family, linear representation and structural properties. the maximum likelihood estimation method for estimating the parameters is presented in Section 3. Special cases of the new family are offered in Section 4 and simulation study using the inverse Weibull as the baseline model is presented in Section 5. Section 6 deals with application by fitting a real dataset with the APGOGE-Rayleigh model. Finally, we conclude the paper in Section 7.

2. THE APGOGE-G FAMILY AND ITS PROPERTIES

This section presents the new APGOGE-G family and explicit developments of the structural properties. The CDF and PDF of the new family achieved through amalgamation of the AP transformation family and GOGE-G family are presented as follows

$$F_{APGOGE}(x)=\begin{cases}\dfrac{\alpha^{\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\ x>0,\\ \left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}, & \alpha=1,\ x>0,\end{cases} \tag{6}$$

and

$$f_{APGOGE}(x)=\begin{cases}\dfrac{\kappa\beta\log(\alpha)h(x;\zeta)H(x;\zeta)^{\kappa-1}}{(\alpha-1)\left[1-H(x;\zeta)^{\kappa}\right]^{2}}e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta-1}\alpha^{\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}}, & \alpha>0,\ \alpha\neq1,\ x>0,\\ \dfrac{\kappa\beta h(x;\zeta)H(x;\zeta)^{\kappa-1}}{\left[1-H(x;\zeta)^{\kappa}\right]^{2}}e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta-1}, & \alpha=1,\ x>0,\end{cases} \tag{7}$$

Henceforth, the random variable $X\sim APGOGE(\Phi)$ with $\Phi=(\kappa,\beta,\alpha,\zeta)$ has density function in Eq. (7). The survival and hazard rate functions of the X is specified as

$$SF_{APGOGE}(x;\zeta)=\begin{cases}1-\dfrac{\alpha^{\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq1,\ x>0,\\ 1-\left\{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right\}^{\beta}, & \alpha=1,\ x>0,\end{cases} \tag{8}$$

and

$$h_{APGOGE}(x;\zeta)=\begin{cases}\dfrac{\kappa\beta\log(\alpha)h(x;\zeta)H(x;\zeta)^{\kappa-1}e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\left[1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right]^{\beta-1}\alpha^{\left[1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right]^{\beta}}}{(\alpha-1)\left[1-H(x;\zeta)^{\kappa}\right]^{2}\left(1-\dfrac{\alpha^{\left[1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right]^{\beta}}-1}{\alpha-1}\right)}, & \alpha>0,\ \alpha\neq1,\\ \dfrac{\kappa\beta h(x)H(x)^{\kappa-1}e^{\frac{-H(x)^{\kappa}}{\bar{H}(x)^{\kappa}}}\left\{1-e^{\frac{-H(x)^{\kappa}}{\bar{H}(x)^{\kappa}}}\right\}^{\beta-1}}{\left[1-H(x)^{\kappa}\right]^{2}\left\{1-\left[1-e^{\frac{-H(x)^{\kappa}}{\bar{H}(x)^{\kappa}}}\right]^{\beta}\right\}}, & \alpha=1,\end{cases} \tag{9}$$

2.1 QUANTILE FUNCTION

The quantile function of X, say $Q(u)$ found by inverting Eq. (6) is given by

$$Q(u)=\begin{cases}H^{-1}\left[\dfrac{-\log\left(1-\left\{\dfrac{l\log\left[1+u(\alpha-1)\right]}{l\log\alpha}\right\}^{\frac{1}{\beta}}\right)}{1-\log\left(1-\left\{\dfrac{l\log\left[1+u(\alpha-1)\right]}{l\log\alpha}\right\}^{\frac{1}{\beta}}\right)}\right]^{\frac{1}{\kappa}}, & \alpha>0,\ \alpha\neq1,\ x>0,\\ H^{-1}\left[\dfrac{-\log\left(1-u^{\frac{1}{\beta}}\right)}{1-\log\left(1-u^{\frac{1}{\beta}}\right)}\right]^{\frac{1}{\kappa}}, & \alpha=1,\ x>0.\end{cases} \tag{10}$$

Hence, random numbers can be generated from the APGOGE-G family for specified baseline CDF using Eq. (10).

2.2 LINEAR REPRESENTATION

Here, a useful representation for Eq. (7) of the APGOGE-G family is presented. For $\alpha > 0, \alpha \neq 1$, using the power series expansion

$$\alpha^z = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} z^i, \tag{11}$$

and the generalized Binomial series expansion expressed as

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} z^i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{i!\Gamma(b-1)} z^i \tag{12}$$

Which holds for $|z| < 1$ and $b > 0$ real non-integer, then Eq. (7) is written as

$$f_{APGOGE}(x) = \frac{\kappa\beta h(x;\zeta) H(x;\zeta)^{\kappa-1}}{(\alpha-1)\left[1-H(x;\zeta)^{\kappa}\right]^2} \sum_{a,b=0}^{\infty} \frac{(-1)^a (\log \alpha)^{\alpha+1}}{a!} \binom{\beta(\alpha+1)-1}{b} e^{\frac{-(b+1)H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}, \tag{13}$$

Utilizing Eq. (11), we have

$$e^{\frac{-(b+1)H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}} = \sum_{c=0}^{\infty} \frac{(-1)^c (b+1)^c}{c!} \left[\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right]^c, \tag{14}$$

Hence, Eq. (13) reduces to

$$f_{APGOGE}(x) = \frac{\kappa\beta}{(\alpha-1)} \sum_{a,b,c=0}^{\infty} \frac{(-1)^{a+c} (\log \alpha)^{\alpha+1} (b+1)^c}{a!c!} \binom{\beta(\alpha+1)-1}{b} \frac{h(x;\zeta) H(x;\zeta)^{\kappa(c+1)-1}}{\left[1-H(x;\zeta)^{\kappa}\right]^{c+2}}, \tag{15}$$

Utilizing Eq. (12), the preceding equation takes the form

$$f_{APGOGE}(x) = \sum_{c,d=0}^{\infty} \vartheta_{c,d} h_{\kappa(c+d+1)}(x;\zeta), \tag{16}$$

where $h_{\kappa(c+d+1)}(x;\zeta) = \kappa(c+d+1) H(x;\zeta)^{\kappa(c+d+1)-1} h(x;\zeta)$ is the Exp-G family with power parameter $\kappa(c+d+1)$ and

$$\vartheta_{c,d} = \kappa\beta \sum_{a,b=0}^{\infty} \frac{(-1)^{a+c+d} (\log \alpha)^{\alpha+1} (b+1)^c}{a!c!(\alpha-1)\left[\kappa(c+d+1)\right]} \binom{\beta(\alpha+1)-1}{b} \binom{-(c+2)}{d}.$$

For $\alpha = 1$, using Eq. (11) again, we have

$$f_{APGOGE}(x) = \sum_{b,c=0}^{\infty} \vartheta_{b,c} h_{\kappa(b+c+1)}(x;\zeta), \tag{17}$$

where $h_{\kappa(b+c+1)}(x;\zeta) = \kappa(b+c+1) H(x;\zeta)^{\kappa(b+c+1)-1} h(x;\zeta)$ is the Exp-G family with power parameter $\kappa(b+c+1)$ and

$$\vartheta_{b,c}=\kappa\beta\sum_{a=0}^{\infty}\frac{(-1)^{a+c+d}(a+1)^{b}}{b!\left[\kappa(b+c+1)\right]}\binom{\beta-1}{a}\binom{-(b+2)}{c}.$$

Therefore, the linear representation of the PDF for the APGOGE-G family is specified as

$$f_{APGOGE}(x)=\begin{cases}\sum_{c,d=0}^{\infty}\vartheta_{c,d}h_{\kappa(c+d+1)}(x;\zeta), & \alpha>0,\ \alpha\neq 1,\\ \sum_{b,c=0}^{\infty}\vartheta_{b,c}h_{\kappa(b+c+1)}(x;\zeta), & \alpha=1,\end{cases}\tag{18}$$

By integrating Eq. (18), the linear representation of the CDF for the APGOGE-G family is specified as

$$F_{APGOGE}(x)=\begin{cases}\sum_{c,d=0}^{\infty}\vartheta_{c,d}H_{\kappa(c+d+1)}(x;\zeta), & \alpha>0,\ \alpha\neq 1,\\ \sum_{b,c=0}^{\infty}\vartheta_{b,c}H_{\kappa(b+c+1)}(x;\zeta), & \alpha=1,\end{cases}\tag{19}$$

where $H_{\kappa}(x;\zeta)$ is the CDF of the Exp-G family with power parameter κ.

2.3 RAW AND INCOMPLETE MOMENTS, MOMENT GENERATING FUNCTION

The rth raw moment of X is defined by $\mu_r'=\mathrm{E}(X^r)=\int_{-\infty}^{+\infty}x^r f(x)\,dx.$ The rth moment of the APGOGE-G family obtained using the linear representation is specified as

$$\mu_r'=\begin{cases}\sum_{c,d=0}^{\infty}\vartheta_{c,d}\mathrm{E}\left[Z_{\kappa(c+d+1)}^{r}\right], & \alpha>0,\ \alpha\neq 1,\\ \sum_{b,c=0}^{\infty}\vartheta_{b,c}\mathrm{E}\left[Z_{\kappa(c+d+1)}^{r}\right], & \alpha=1,\end{cases}\tag{20}$$

where Z_{τ} denotes the Exp-G distribution with power parameter τ. By setting $r=1$ in Eq. (20), the mean of X is obtained. For most baseline distributions, the last integral can be numerically computed. Further, the rth incomplete of the APGOGE-G family, say $\upsilon_r^{(t)}$ is specified as

$$\upsilon_r^{(t)}=\begin{cases}\sum_{c,d=0}^{\infty}\vartheta_{c,d}\int_{-\infty}^{t}x^r h_{\kappa(c+d+1)}(x;\zeta)\,dx, & \alpha>0,\ \alpha\neq 1,\\ \sum_{b,c=0}^{\infty}\vartheta_{b,c}h_{\kappa(b+c+1)}(x;\zeta)\,dx, & \alpha=1,\end{cases}\tag{21}$$

The last integral in Eq. (21) denotes the rth incomplete moment of Z_{τ}.

The moment generating function (MGF) of X, say $M_X(t)=\mathrm{E}(e^{tX})$ for the APGOGE-G family is specified as

$$M_X(t)=\begin{cases}\sum_{c,d=0}^{\infty}\vartheta_{c,d}\int_{-\infty}^{t}x^r M_{\kappa(c+d+1)}(x;\zeta)\,dx, & \alpha>0,\ \alpha\neq 1,\\ \sum_{b,c=0}^{\infty}\vartheta_{b,c}M_{\kappa(b+c+1)}(x;\zeta)\,dx, & \alpha=1,\end{cases} \tag{22}$$

2.4 ORDER STATISTICS

Let $x_1,x_2,\ldots,x_n$ be a random sample from the APGOGE.G family, and the sequence $x_{1:n}<x_{2:n}<\ldots<x_{n:n}$ are the corresponding order statistics (O.S) from the sample. The PDF of the j[th] O.S, say $X_{j:n}$ is

$$f_{j:n}(x)=\frac{1}{\mathrm{B}(j,n-j+1)}\sum_{a=0}^{n-j}(-1)^a\binom{n-j}{a}\left[F(x)\right]^{a+j-1}f(x), \tag{23}$$

where $\mathrm{B}(\cdot)$ is the beta function. By inserting Eqs. (6) and (7) into Eq. (23), and expanding using Eqs. (11) and (12). The PDF of $X_{j:n}$ is specified as

$$f_{j:n}(x)=\begin{cases}\sum_{a=0}^{n-j}\frac{\kappa\beta(-1)^a\binom{n-j}{a}}{B(j,n-j+1)}\sum_{f,g=0}^{\infty}\vartheta_{f,g}h_{\kappa(f+g+1)}(x;\zeta), & \alpha>0,\ \alpha\neq 1,\\ \sum_{a=0}^{n-j}\frac{\kappa\beta(-1)^a\binom{n-j}{a}}{B(j,n-j+1)}\sum_{c,d=0}^{\infty}\vartheta_{c,d}h_{\kappa(c+d+1)}(x;\zeta),, & \alpha=1,\end{cases} \tag{24}$$

where

For $\alpha>0,\ \alpha\neq 1,$

$h_{\kappa(f+g+1)}(x;\zeta)=\kappa(f+g+1)H(x;\zeta)^{\kappa(f+g+1)-1}h(x;\zeta)$ is the Exp-G family with power parameter $\kappa(f+g+1)$ and

$$\vartheta_g=\frac{\kappa\beta}{\kappa(f+g+1)}\sum_{a=0}^{n-j}\sum_{b,c,d,e,f=0}^{\infty}\frac{(-1)^{a+b+e+f+g}(\log\alpha)^{c+d+1}(e+1)^f}{B(j,n-j+1)c!d!f!(\alpha-1)^{a+j}}\binom{n-j}{a}\binom{a+j-1}{b}\times\binom{\beta(bc+d+1)-1}{e}\binom{-(f+2)}{g}$$

For $\alpha=1,$

$h_{\kappa(c+d+1)}(x;\zeta)=\kappa(c+d+1)H(x;\zeta)^{\kappa(c+d+1)-1}h(x;\zeta)$ is the Exp-G family with power parameter $\kappa(c+d+1)$ and

$$\vartheta_d=\frac{\kappa\beta}{\kappa(c+d+1)}\sum_{a=0}^{n-j}\sum_{b,c=0}^{\infty}\frac{(-1)^{a+b+c}(b+1)^c}{B(j,n-j+1)c!}\binom{n-j}{a}\binom{\beta(a+j)-1}{b}\binom{-(c+2)}{d}.$$

2.5 ENTROPIES

The Rényi entropy of a random variable X described as the variability of uncertainty is expressed as

$$I_{\lambda}(X)=\frac{1}{1-\lambda}\log\int_{-\infty}^{+\infty} f(x)^{\lambda}\,dx,\quad \lambda>0 \text{ and } \lambda\neq 1. \tag{25}$$

For $\alpha>0,\ \alpha\neq 1,$ in Eq. (7) and expanding using Eqs. (11) and (12), we have

$$f_{APGOGE}(x)^{\lambda}=\sum_{c,d=0}^{\infty}\vartheta_{c,d}\,h_{\kappa(c+d+\lambda)-\lambda+1}(x;\zeta), \tag{26}$$

Therefore, the Rényi entropy of the APGOGE-G family is specified as

$$I_{\lambda}(X)=\frac{1}{1-\lambda}\log\left[\sum_{c,d=0}^{\infty}\vartheta_{c,d}\int_{-\infty}^{+\infty}h_{\kappa(c+d+\lambda)-\lambda+1}(x;\zeta)\,dx\right] \tag{27}$$

where $h_{\kappa(c+d+\lambda)-\lambda+1}(x;\zeta)=\left[\kappa(c+d+\lambda)-\lambda+1\right]H(x;\zeta)^{\kappa(c+d+\lambda)-\lambda}\,h(x;\zeta)^{\lambda}$ is the Exp-G family with power parameter $\kappa(c+d+\lambda)-\lambda+1$ and

$$\vartheta_{c,d}=(\kappa\beta)^{\lambda}\sum_{a,b=0}^{\infty}\frac{(-1)^{b+c+d}(\log\alpha)^{a+\lambda}\,\lambda^{a}\,(b+\lambda)^{c}}{a!\,c!\,(\alpha-1)^{\lambda}\left[\kappa(c+d+\lambda)-\lambda+1\right]}\binom{\beta(c+\lambda)-\lambda}{b}\binom{-(c+2\lambda)}{d}.$$

The q-entropy of the APGOGE-G family is specified as

$$H_{q}(X)=\frac{1}{q-1}\log\left\{1-\left[\sum_{c,d=0}^{\infty}\vartheta^{*}_{c,d}\int_{-\infty}^{+\infty}h_{\kappa(c+d+\lambda)-\lambda+1}(x;\zeta)\,dx\right]\right\} \tag{28}$$

where $h_{\kappa(c+d+\lambda)-\lambda+1}(x;\zeta)=\left[\kappa(c+d+\lambda)-\lambda+1\right]H(x;\zeta)^{\kappa(c+d+\lambda)-\lambda}\,h(x;\zeta)^{\lambda}$ is the Exp-G family with power parameter $\kappa(c+d+\lambda)-\lambda+1$ and

$$\vartheta^{*}_{c,d}=(\kappa\beta)^{q}\sum_{a,b=0}^{\infty}\frac{(-1)^{b+c+d}(\log\alpha)^{a+q}\,q^{a}\,(b+q)^{c}}{a!\,c!\,(\alpha-1)^{q}\left[\kappa(c+d+q)-q+1\right]}\binom{\beta(c+q)-q}{b}\binom{-(c+2q)}{d}.$$

The Shannon entropy (SE) considered as a special case of the Rényi entropy when $\lambda\uparrow 1$ is expressed as $SE=\mathrm{E}\{-[\log f(x)]\}$. Therefore, the SE for the APGOGE-G family can be obtained from Eq. (27).

3. MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimator is utilized to estimate the parameters of the APGOGE-G family for observed samples. Let $X\sim APGOGE-G(\Phi)$, where $\Phi=(\kappa,\beta,\alpha,\zeta)^{T}$ is the vector of unknown parameters. The log-likelihood function $\ell(\Phi)$ is specified as

$$\ell(\Phi)=n\log\kappa+n\log\beta+n\log(\log\alpha)+n\log(\alpha-1)+\sum_{i=1}^{n}\log h(x;\zeta)$$
$$+(\kappa-1)\sum_{i=1}^{n}\log H(x;\zeta)-2\sum_{i=1}^{n}\log\left(1-H(x;\zeta)^{\kappa}\right)-\sum_{i=1}^{n}\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}} \quad (29)$$
$$+(\beta-1)\sum_{i=1}^{n}\log\left(1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right)+\sum_{i=1}^{n}\left(1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}\right)^{\beta}\log(\alpha)$$

The components of the score function $U(\Phi)=\left(\frac{\partial\ell(\Phi)}{\partial\kappa},\frac{\partial\ell(\Phi)}{\partial\beta},\frac{\partial\ell(\Phi)}{\partial\alpha},\frac{\partial\ell(\Phi)}{\partial\zeta}\right)^{T}$ are

$$\frac{\partial\ell(\Phi)}{\partial\kappa}=\frac{n}{\kappa}+nl\,og\,H(x;\zeta)+\frac{nH(x;\zeta)^{\kappa}\log H(x;\zeta)}{\bar{H}(x;\zeta)^{\kappa}}-\frac{n\left(H(x;\zeta)^{\kappa}\right)^{2}\log H(x;\zeta)}{\left(\bar{H}(x;\zeta)^{\kappa}\right)^{2}}$$
$$-\frac{(\beta-1)n\left[-\frac{H(x;\zeta)^{\kappa}\log H(x;\zeta)}{\bar{H}(x;\zeta)^{\kappa}}-\frac{\left(H(x;\zeta)^{\kappa}\right)^{2}\log H(x;\zeta)}{\left(\bar{H}(x;\zeta)^{\kappa}\right)^{2}}\right]e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}}{1-e^{\frac{-H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}}}+\frac{1}{H(x;\zeta)^{\kappa}}$$
$$\times\left\{\mathrm{n}\left[-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right]^{\beta}\frac{\beta\left[-\frac{H(x;\zeta)^{\kappa}\log H(x;\zeta)}{\bar{H}(x;\zeta)^{\kappa}}-\frac{\left(H(x;\zeta)^{\kappa}\right)^{2}\log H(x;\zeta)}{\left(\bar{H}(x;\zeta)^{\kappa}\right)^{2}}\right]\left[1-H(x;\zeta)^{\kappa}\right]}{H(x;\zeta)^{\kappa}} \times e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)^{\beta}}\log(\alpha)\right\}$$

$$\frac{\partial\ell(\Phi)}{\partial\beta}=\frac{n}{\beta}+nl\,og\left(1-e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)}\right)-n\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)^{\beta}\log\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)^{\beta}}\log(\alpha)$$

$$\frac{\partial\ell(\Phi)}{\partial\alpha}=\frac{n}{\alpha\log\alpha}+\frac{n}{\alpha-1}+\frac{n\left(1-e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)^{\beta}}\right)}{\alpha}$$

$$\frac{\partial \ell(\Phi)}{\partial \zeta}=\frac{n\left(\frac{\partial}{\partial \zeta}h(x;\zeta)\right)}{h(x;\zeta)}+\frac{(\kappa-1)n\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)}+\frac{nH(x;\zeta)^{\kappa}\kappa\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)\left(1-H(x;\zeta)^{\kappa}\right)}$$

$$-\frac{n\left(H(x;\zeta)^{\kappa}\right)^{2}\kappa\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)\left(1-H(x;\zeta)^{\kappa}\right)^{2}}$$

$$-\frac{1}{1-e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)}}\left\{(\beta-1)n\left[-\frac{H(x;\zeta)^{\kappa}\kappa\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)\left(1-H(x;\zeta)^{\kappa}\right)}-\frac{\left(H(x;\zeta)^{\kappa}\right)^{2}\kappa\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)\left(1-H(x;\zeta)^{\kappa}\right)^{2}}\right]e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)}\right\}$$

$$+\frac{1}{H(x;\zeta)^{\kappa}}\left\{\begin{array}{l} n\left[-\frac{H(x;\zeta)^{\kappa}}{1-H(x;\zeta)^{\kappa}}\right]^{\beta}\beta\left[-\frac{H(x;\zeta)^{\kappa}\kappa\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)\left(1-H(x;\zeta)^{\kappa}\right)}-\frac{\left(H(x;\zeta)^{\kappa}\right)^{2}\kappa\left(\frac{\partial}{\partial \zeta}H(x;\zeta)\right)}{H(x;\zeta)\left(1-H(x;\zeta)^{\kappa}\right)^{2}}\right] \\ \times\left[1-H(x;\zeta)^{\kappa}\right]e^{\left(-\frac{H(x;\zeta)^{\kappa}}{\bar{H}(x;\zeta)^{\kappa}}\right)^{\beta}}\log(\alpha) \end{array}\right\}$$

The nonlinear system of equations can be solved numerically utilizing R-programming, Maple, Mathematical and SAS due to its intricacy. In this work, R-programming software will be utilized.

4. SPECIAL CASES OF APGOGE-G MODEL

We introduce three models generated by the APGOGE-G family.

4.1. APGOGE-INVERSE WEIBULL (APGOGE$_{IW}$) MODEL

Consider the parent distribution to be the inverse Weibull with CDF and PDF given by $\mathrm{H}(x;\eta,\gamma)=e^{-\eta x^{-\gamma}}$ and $h(x;\eta,\gamma)=\eta\gamma x^{-(\gamma+1)}e^{-\eta x^{-\gamma}}, \eta,\gamma>0$. The CDF and PDF of the $\mathrm{APGOGE}_{IW}(\Phi)$ model, $\Phi=(\kappa,\beta,\alpha,\eta,\gamma)$ are specified as

$$F(x;\Phi)=\begin{cases}\dfrac{\alpha^{\left\{1-e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}\right\}^{\beta}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\ x>0, \\ \left\{1-e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}\right\}^{\beta}, & \alpha=1,\ x>0,\end{cases} \tag{30}$$

and

$$f(x;\Phi)=\begin{cases}\dfrac{\kappa\beta\eta\gamma x^{-(\gamma+1)}\log(\alpha)\left(e^{-\eta x^{-\gamma}}\right)^{\kappa} e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}\left\{1-e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}\right\}^{\beta-1}}{(\alpha-1)\left[1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}\right]^{2}}\alpha^{\left\{1-e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}\right\}^{\beta}}, & \alpha>0,\ \alpha\neq 1,\\[2ex] \dfrac{\kappa\beta\eta\gamma x^{-(\gamma+1)}\left(e^{-\eta x^{-\gamma}}\right)^{\kappa} e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}}{\left[1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}\right]^{2}}\left\{1-e^{\frac{-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}{1-\left(e^{-\eta x^{-\gamma}}\right)^{\kappa}}}\right\}^{\beta-1}, & \alpha=1.\end{cases} \tag{31}$$

Figure 1 depicts the PDF and HRF plots of the $\text{APGOGE}_{\text{IW}}$ model. The HRF can be increasing, decreasing, increasing-decreasing, decreasing-increasing, therefore indicating that flexibility is provided by the extra shape parameters.

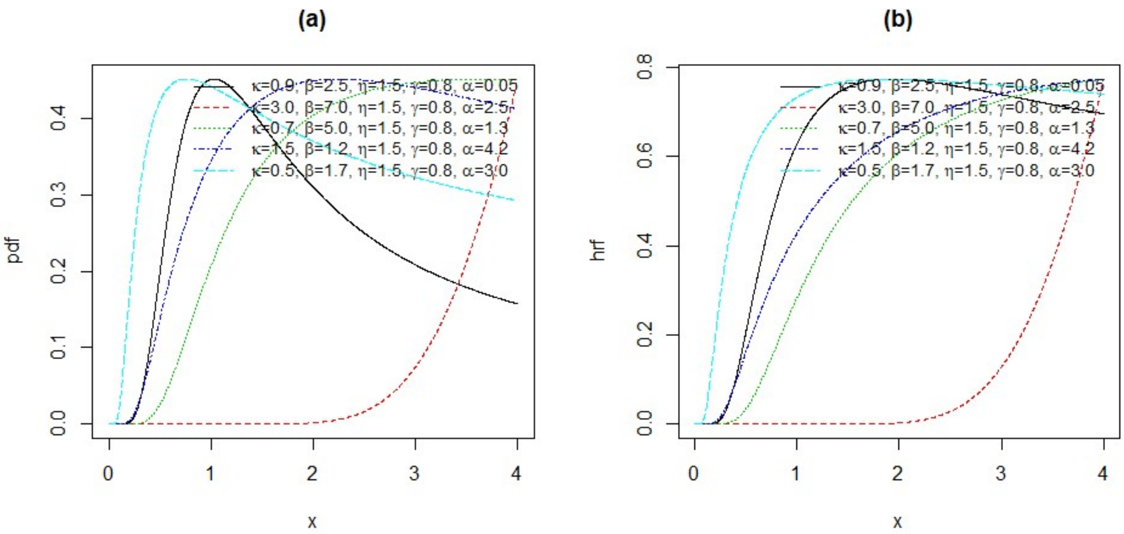

Figure 1: The PDF (a) and HRF (b) plots of the $\text{APGOGE}_{\text{IW}}$ model with selected parameters values.

4.2. APGOGE-RAYLEIGH (APGOGE_{R}) MODEL

The CDF and PDF of the Rayleigh is given by $\mathrm{H}(x;\eta)=1-e^{-\frac{\eta}{2}x^{2}}$ and $h(x;\eta)=\eta x e^{-\frac{\eta}{2}x^{2}}, \eta>0$. The CDF and PDF of the $\text{APGOGE}_{R}(\Phi)$ model, $\Phi=(\kappa,\beta,\alpha,\eta)$ are specified as

$$F(x;\Phi)=\begin{cases}\dfrac{\alpha^{\left\{1-e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\right\}^{\beta}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\ \left\{1-e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\right\}^{\beta}, & \alpha=1,\end{cases} \tag{32}$$

and

$$f(x;\Phi)=\begin{cases}\dfrac{\kappa\beta\log(\alpha)\eta x e^{-\frac{\eta}{2}x^2}\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa-1} e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\left\{1-e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\right\}^{\beta-1}\alpha^{\left\{1-e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\right\}^{\beta}}}{(\alpha-1)\left[1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}\right]^2}, & \alpha>0,\ \alpha\neq 1,\\ \dfrac{\kappa\beta\eta x e^{-\frac{\eta}{2}x^2}\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa-1} e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\left\{1-e^{-\frac{\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}{1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}}}\right\}^{\beta-1}}{\left[1-\left(1-e^{-\frac{\eta}{2}x^2}\right)^{\kappa}\right]^2}, & \alpha=1.\end{cases} \tag{33}$$

Figure 2 depicts the PDF and HRF plots of the APGOGE$_R$ model. The HRF can be increasing-decreasing and upside-down bathtub, indicating that flexibility is provided by the extra shape parameters.

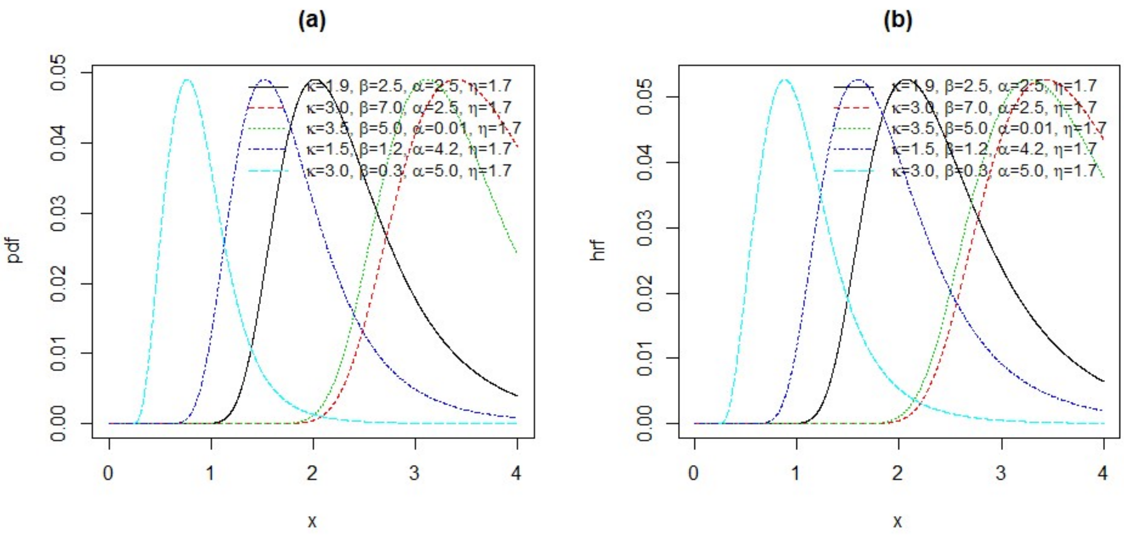

Figure 2: The PDF (a) and HRF (b) plots of the APGOGE$_R$ model with selected parameters values.

4.3. APGOGE-BURR-HARKE EXPONENTIAL (APGOGE$_{BHE}$) MODEL

Consider the CDF and PDF of the Burr-Harke exponential as the parent distribution given by $\mathrm{H}(x;\eta)=1-\frac{e^{-\eta x}}{1+\eta x}$ and $h(x;\eta)=\eta e^{-\eta x}\frac{2+\eta x}{(1+\eta x)^2}, \eta>0$. The CDF and PDF of the $\mathrm{APGOGE}_{BHE}(\Phi)$ model, $\Phi=(\kappa,\beta,\alpha,\eta)$ are specified as

$$F(x;\Phi)=\begin{cases}\dfrac{\alpha^{\left\{1-e^{-\frac{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\right\}^{\beta}}-1}{\alpha-1}, & \alpha>0,\ \alpha\neq 1,\\[2ex] \left\{1-e^{-\frac{\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\right\}^{\beta}, & \alpha=1.\end{cases} \tag{34}$$

and

$$f(x;\Phi)=\begin{cases}\dfrac{\kappa\beta\eta\log(\alpha)e^{-\eta x}\dfrac{2+\eta x}{(1+\eta x)^{2}}\left(1-\dfrac{e^{-\eta x}}{1+\eta x}\right)^{\kappa-1}e^{\frac{-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\left\{1-e^{\frac{-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\right\}^{\beta-1}\alpha^{\left\{1-e^{\frac{-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\right\}^{\beta}}}{(\alpha-1)\left[1-\left(1-\dfrac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}\right]^{2}}, & \alpha>0,\ \alpha\neq 1,\\[2ex] \dfrac{\kappa\beta\eta e^{-\eta x}\dfrac{2+\eta x}{(1+\eta x)^{2}}\left(1-\dfrac{e^{-\eta x}}{1+\eta x}\right)^{\kappa-1}e^{\frac{-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\left\{1-e^{\frac{-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}{1-\left(1-\frac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}}}\right\}^{\beta-1}}{\left[1-\left(1-\dfrac{e^{-\eta x}}{1+\eta x}\right)^{\kappa}\right]^{2}}, & \alpha=1.\end{cases}\tag{35}$$

Figure 3 depicts the PDF and HRF plots of the $APGOGE_{BHE}$ model. The HRF can be decreasing, and decreasing-increasing, indicating that flexibility is provided by the extra shape parameters.

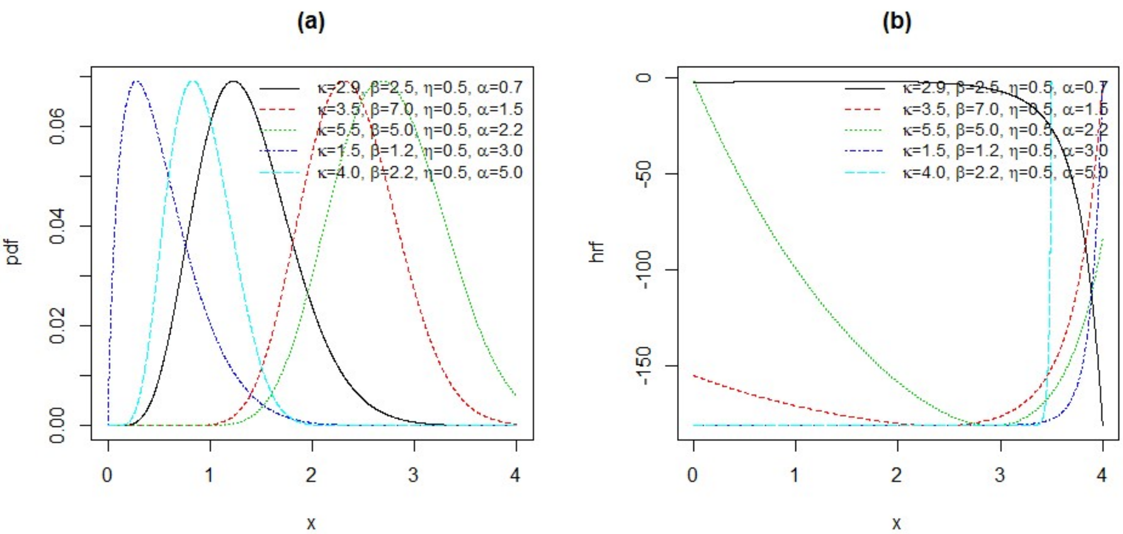

Figure 3: The PDF (a) and HRF (b) plots of the $APGOGE_{BHE}$ model with selected parameters values.

5. SIMULATION

A simulation study is executed by using the inverse Weibull (IW) for baseline to examine the accuracy of the MI estimator of the parameters (pa.). The procedure entails generating samples from the $\mathrm{APGOGE}_{IW}(\kappa,\beta,\alpha,\eta,\gamma)$ model utilizing the inversion method for different parameter groupings (Gps). The number of replicates is 1000, the sample size is n = 50,100,250, with three groupings: $\kappa=1.0,\beta=1.1,\alpha=0.03,\eta=1.0,\gamma=0.3$ (Gp1); $\kappa=1.0,\beta=1.5,\alpha=0.05,\eta=1.6,\gamma=1.3$ (Gp2); and $\kappa=1.0,\beta=2.0,\alpha=0.1,\eta=1.9,\gamma=2.0$ (Gp3). The BFGS algorithm is adopted in the R-programming to maximize Eq. (29) and we compute the ML estimates (MLEst), average biases (AVBs), mean square errors (MSEs) and mean relative error (MREs) from each generated dataset. The simulation results are reported in Table 1. The estimates are quite stable and approach the true parameter values with minimal bias as the sample size increases.

Table 1: Simulation results for APGOGE_{IW} model.

Groupings	n	Pa.	MLEst	AVB	MSE	MRE
Gp1	$n=50$	κ	0.989	-0.011	0.000	0.011
		β	1.181	0.081	0.007	0.074
		α	0.000	-0.030	0.001	0.998
		η	0.974	-0.026	0.001	0.026
		γ	0.369	0.069	0.005	0.232
	$n=100$	κ	0.989	-0.011	0.000	0.011
		β	1.185	0.085	0.007	0.077
		α	0.000	-0.030	0.001	0.998
		η	0.972	-0.028	0.001	0.028
		γ	0.371	0.071	0.005	0.236
	$n=250$	κ	0.987	-0.013	0.000	0.013
		β	1.189	0.089	0.008	0.081
		α	0.000	-0.030	0.001	0.998
		η	0.969	-0.031	0.001	0.031
		γ	0.372	0.072	0.005	0.240
Gp2	$n=50$	κ	1.031	0.031	0.001	0.031
		β	1.615	0.115	0.015	0.077
		α	0.000	-0.050	0.002	0.996
		η	1.525	-0.075	0.006	0.047
		γ	1.344	0.044	0.002	0.034

	$n=100$	κ	1.035	0.035	0.001	0.035
		β	1.608	0.108	0.013	0.072
		α	0.000	-0.050	0.002	0.996
		η	1.520	-0.080	0.007	0.050
		γ	1.349	0.049	0.003	0.038
	$n=250$	κ	1.037	0.037	0.001	0.037
		β	1.601	0.101	0.012	0.111
		α	0.000	-0.050	0.002	0.050
		η	1.517	-0.083	0.007	0.083
		γ	1.354	0.054	0.003	0.055
Gp3	$n=50$	κ	0.876	-0.124	0.015	0.124
		β	2.115	0.115	0.014	0.058
		α	0.000	-0.100	0.010	0.996
		η	1.805	-0.095	0.009	0.050
		γ	2.103	0.103	0.012	0.051
	$n=100$	κ	0.872	-0.128	0.016	0.128
		β	2.110	0.110	0.012	0.055
		α	0.000	-0.100	0.010	0.996
		η	1.811	-0.089	0.008	0.047
		γ	2.094	0.094	0.010	0.047
	$n=250$	κ	0.869	-0.131	0.017	0.131
		β	2.109	0.109	0.012	0.054
		α	0.000	-0.100	0.010	0.996
		η	1.817	-0.083	0.007	0.044
		γ	2.082	0.082	0.007	0.041

6. NUMERICAL APPLICATION

Here, we provide an application of the proposed APGOGE-Rayleigh (APGOGER) model to a lifetime data. The real data is the glass fibre strengths of 1.5 cm collected by employees at the UK National Physical Laboratory and analysed by [2]. The observations are

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25,
1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52,
1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64,
1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81,
2.00, 2.01, 2.24

The performance of the suggested model is checked by the goodness of fit criteria (AIC, CAIC, BIC, HQIC), and the P-value. For more details of the goodness of fit criteria, we refer to see [14-15]. Overall, the probability-model with least values of these statistics would be said to perform better than others. Hence, the proposed APGOGE$_R$ model is compared with the Exponential (E), Gamma exponential (GE), Beta exponential (BE), Beta gamma exponential (BGE), Weibull gamma exponential (WGE), Beta Burr xii (BBXII), Weibull Burr xii (WBXII), Kumaraswamy Burr xii (KBXII), generalized odd generalized Rayleigh (GOGER) and Rayleigh (R) distributions. Table 2 reports the estimated parameter values of the models and the goodness of fit measures. Thus, it is apparent that the proposed model has the least values for the goodness of fit measures which suggest that fits better than the other competing models.

Table 2: The MLEs and information criteria.

Model	MLE					CAIC	AIC	BIC
$APGOGE_R(\kappa,\beta,\alpha,\eta)$	1.583	1.329	7.658	1.079	-	34.924	33.497	38.296
$KBXII(\kappa,\beta,\alpha,\eta,\lambda)$	0.397	0.685	1.753	2.115	12.329	36.973	35.920	46.636
$BBXII(\kappa,\beta,\alpha,\eta,\lambda)$	0.603	3.963	2.414	3.518	8.118	39.591	38.538	49.254
$WBXII(\kappa,\beta,\alpha,\eta)$	0.036	1.489	1.269	3.436	0.036	38.229	37.540	46.112
$GOGER(\kappa,\beta,\eta)$	1.832	1.762	-	1.057	-	37.463	37.056	43.486
$BE(\kappa,\beta,\eta)$	17.779	22.722	-	0.390	-	54.661	54.254	60.683
$GE(\kappa,\beta)$	2.610	31.303	-	-	-	179.726	179.660	181.803
$BGE(\kappa,\beta,\alpha,\eta)$	0.4125	93.465	0.923	22.612	-	39.889	39.199	47.772
$WGE(\kappa,\beta,\eta)$	56.881	4.893	-	0.222	-	36.063	35.656	42.085
$R(\eta)$	-	-	-	0.842	-	101.582	101.647	103.725
$E(\eta)$	-	-	-	0.664	-	36.063	35.656	42.085

Figure 4 shows the fitted density and distribution plots of the APGOGE$_R$ model and some competitive models to the dataset. It is clear from the plots that the APGOGE$_R$ model provides close fit to the real-life dataset.

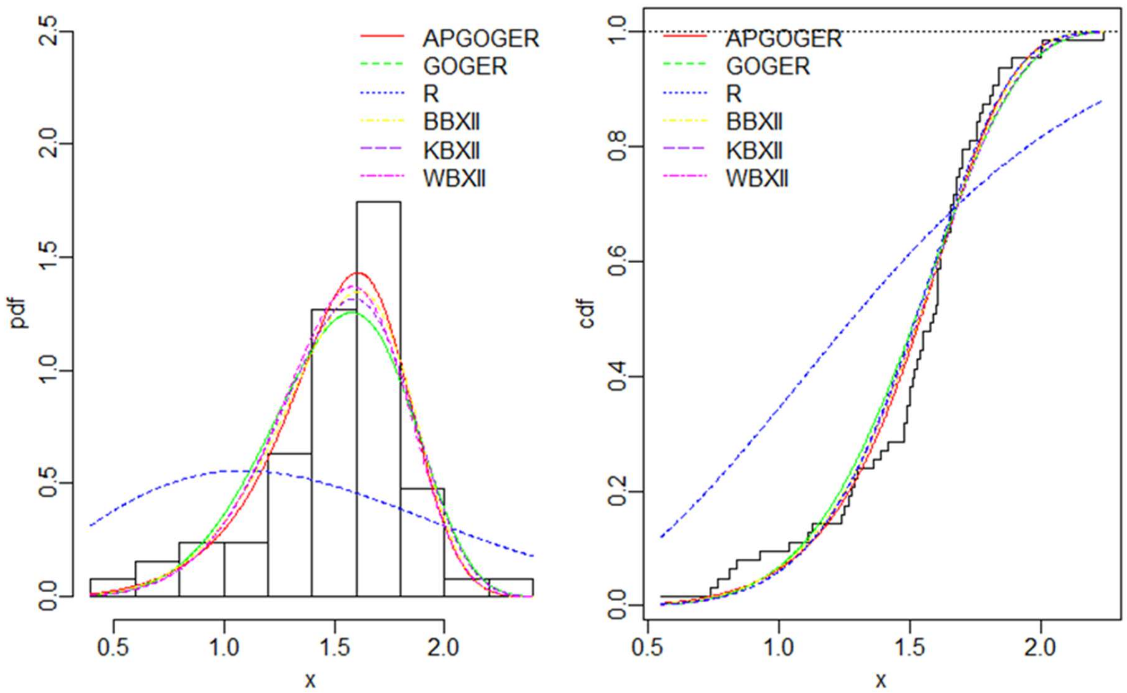

Figure 4. Empirical density and distribution plots for $APGOGE_R$ and some competitive models

7. CONCLUSION

The paper presents a new family of distributions called Alpha Power generalized odd generalized (APGOGE-G) family. The desirable properties of the new family are derived and three special models are introduced. In other to estimate the parameters of the new family, the maximum likelihood estimation procedure is utilized and assessed through simulation study. Additionally, to appraise the performance of the new family, the APGOGER was fitted to a real dataset. The empirical results showed that the new $APGOGE_R$ model provides a better fit to the dataset as compared to other models. Future researchers may propose new flexible models by using the new family and existing baseline distributions.

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