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Proposal of a Modified Clayton Copula: Theory, Properties and Examples

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Abstract. The Clayton copula is a mathematical tool used in copula theory to model dependence between random variables. It is a notable member of the Archimedean copula family and is best known for its ability to capture tail dependence. In this article, we present a new modified variant of the Clayton copula that aims to improve its flexibility. The proposed modification scheme perturbs its Archimedean nature by integrating a bivariate product of logarithmic functions and an additional tuning parameter. The elaborated copula benefits from a more nuanced representation of the copula density, and negative dependence can be obtained in a regular manner. We study its properties, including limit results showing some connection with the Gumbel-Barnett copula, important related functions, modifications and extensions, simulation of random couples of values, various lower and upper bounds, various tail dependences, and the correlation properties through the medial correlation and the Kendall tau. As an example of probability application, a new modified bivariate Gaussian distribution is presented via equations and graphics. Finally, two special cases of copula are discussed, including a simple single-parameter copula, which is intended to be a practical alternative to the Clayton copula. A brief analysis on simulated data shows that it may be preferable to the Clayton copula according to the Akaike information criterion. The overall result contributes to the advancement of the theoretical foundations of copula-based modeling techniques.

1. Introduction

1.1. **Context.** In the field of statistical modeling, the concept of copula has emerged as a powerful tool for characterizing dependence structures between random variables. It was introduced by Sklar in 1959 (see [27]), and over time has gained importance in fields as diverse as finance, insurance, hydrology, genetics, neuroscience, environmental science, telecommunications, and reliability engineering. Copula theory is based on the so-called Sklar theorem, which states that any multivariate distribution can be decomposed into its marginal distributions and a copula function. This separation allows researchers and practitioners to analyze specific dependence behaviors independently, facilitating a more nuanced understanding of complex data sets. The established copulas are numerous, as evidenced by the books by [17], [21] and [11]. From the basic Gaussian copula, which

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is often used in financial modeling because of its simplicity, to the more complex copulas of the Archimedean family, such as the Ali-Mikhail-Haq, Clayton, Gumbel and Frank copulas, each has different properties for capturing dependence structures and tail dependence characteristics. They provide a versatile toolbox for modeling different degrees of association, from linear to non-linear and asymmetric relationships. We can refer to [21] and, for a modern and global study, to [18]. Recent copula creation work includes the various extended FGM copula families (see [2], [1] and [9]) and the variable-power copula families (see [4] and [5]). In addition, the introduction of vine copulas further extends this diversity by allowing the combination of simpler copulas to represent complex dependence structures in high-dimensional data (see [19]).

1.2. **On the Clayton copula.** To understand the motivation behind this article, a retrospective examination of the Clayton copula is essential. To begin with, it was created by Clayton in 1978 (see [6]) and, as already mentioned, it is one of the most famous Archimedean copulas (see again [17], [21] and [11]). Considering a parameter $a > 0$, the standard version of the Clayton copula is derived from the following strict generator function:

$$\phi_{\dagger}(t) = t^{-a} - 1, \quad t > 0,$$

and can be expressed as

$$C_{\dagger}(u, v) = \phi_{\dagger}^{-1}\left[\phi_{\dagger}(u) + \phi_{\dagger}(v)\right] = \left(u^{-a} + v^{-a} - 1\right)^{-1/a}, \quad (u, v) \in [0, 1]^2.$$

The coefficients of lower and upper dependence are $\lambda_l = 2^{-1/a}$ and $\lambda_u = 0$, respectively, implying that the Clayton copula is lower tail dependent but not upper tail dependent. Furthermore, the Clayton copula converges to the independence copula when $a \to 0^+$, and to the minimum copula $C_\circ(u, v) = \min(u, v)$ when $a \to +\infty$. The Kendall tau has the unit range $[0, 1]$. Therefore, it can only account for positive dependence. On the other hand, a well-known modified Clayton copula can be presented, which allows for negative dependence. Considering $a \in [-1, +\infty)/\{0\}$, it is derived from the following (non-strict) generator function:

$$\phi_{\ddagger}(t) = \frac{t^{-a} - 1}{a}, \quad t > 0,$$

and the related pseudo-inverse function $\phi_{\ddagger}^{[-1]}(t)$, i.e., $\phi_{\ddagger}^{[-1]}(t) = \phi_{\ddagger}^{-1}(t) = (at + 1)^{-1/a}$ for $t \in [0, \phi_{\ddagger}(0)]$, and $\phi_{\ddagger}^{[-1]}(t) = 0$ for $t > \phi_{\ddagger}(0)$ (with $\phi_{\ddagger}(0) = -1/a$ for $a \in [-1, 0)$), and can be expressed as

$$\begin{aligned} C_{\ddagger}(u, v) &= \phi_{\ddagger}^{[-1]}\left[\phi_{\ddagger}(u) + \phi_{\ddagger}(v)\right] \\ &= \left\{\max\left[\left(u^{-a} + v^{-a} - 1\right), 0\right]\right\}^{-1/a} \\ &= \begin{cases} \left(u^{-a} + v^{-a} - 1\right)^{-1/a}, & (u, v) \in [0, 1]^2 \text{ such that } u^{-a} + v^{-a} > 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

When $a \to -1^+$, it tends to the maximum copula $C_\star(u,v) = \max(u+v-1,0)$. The negative dependence is achieved for $a \in [-1,0)$. In this case, this modified Clayton copula is non-regular; its support depends on a and corresponds to a restricted region, breaking the symmetry about the diagonal from the vertices $(0,1)$ to $(1,0)$. Furthermore, it is also proved that it cannot properly capture higher negative dependence (see [17], [21] and [11]). The Clayton copula, in its standard or maximum modified form, has been at the heart of several important applied studies, including [8], [12], [22], [24] and [20]. It has also served as the main example for testing an artificial intelligence system in [14], mainly because of its simple expression and all the deep computational knowledge around it.

In order to provide suitable alternatives with the same mathematical ingredients, several articles have focused on extensions of the Clayton copula by modifying its strict generator function. In particular, in [7], the following weighted Clayton strict generator function is proposed:

$$\phi_\triangleleft(t) = \frac{t^{-a}-1}{a}(1-bt^a), \quad t > 0,$$

under the following parameter conditions: $b \in [0,1]$, $a \geq 0$, or $b < 0$ and $b(a-1)+a+1 \geq 0$. The associated Archimedean copula has a sophisticated expression (see [7, Remark 2.2]), but it has the advantages of being regular and of achieving negative dependence thanks to the combined action of the parameters a and b. It is shown in [7, Subsection 3.4] that the corresponding Kendall tau has the optimal range $[-1,1]$. In the same vein, a more simple approach was studied in [3]. It consists of considering the following parameter-extended Clayton strict generator function:

$$\phi_\triangleright(t) = (at^{-b}-1)^c - (a-1)^c, \quad t > 0,$$

under the following parameter conditions: $b > 0$, $c > 0$ and $a \geq \max[1, (b+1)/(bc+1)]$. The corresponding Archimedean copula is expressed as

$$\begin{aligned} C_\triangleright(u,v) &= \phi_\triangleright^{-1}\left[\phi_\triangleright(u) + \phi_\triangleright(v)\right] \\ &= a^{1/b}\left\{\left[(au^{-b}-1)^c + (av^{-b}-1)^c - (a-1)^c\right]^{1/c} + 1\right\}^{-1/b}, \quad (u,v) \in [0,1]^2. \end{aligned}$$

This copula is regular, and it can reach the negative dependence thanks to the combined action of the parameters a, b, and c. For some parameter-value tests, it is shown in [3, Tables 1 and 2] that the corresponding Kendall tau has the range $[-0.31, 0.75]$, but a larger numerical analysis suggests the range $[-0.31, 1]$. The two copulas above thus provide interesting alternatives to the Clayton copula because of their regularity and possible negative dependence. However, this is achieved by directly modifying the generator function and manipulating several parameters.

1.3. **Contributions.** In this article, we develop an original approach to extend the standard Clayton copula beyond the modification of the corresponding strict generator function. We aim to determine a simple bivariate function $\psi(u, v)$ such that

$$C(u, v) = \left[u^{-a} + v^{-a} + \psi(u, v) - 1\right]^{-1/a}, \quad (u, v) \in [0, 1]^2, \tag{1}$$

is a valid copula that is regular, covers both negative and positive dependence, and depends on a reasonable number of parameters. It can also be expressed as

$$C(u, v) = \phi_{\dagger}^{-1}\left[\phi_{\dagger}(u) + \phi_{\dagger}(v) + \psi(u, v)\right]$$

or

$$C(u, v) = \left\{\left[C_{\dagger}(u, v)\right]^{-a} + \psi(u, v)\right\}^{-1/a}.$$

In this way, we deliberately perturb the Archimedean structure inherent in the standard Clayton copula, while retaining its basic mathematical components in an effort to extend its modeling capabilities. A similar idea can be found implicitly in the BB4 copula created in [16, Example 5.3], which is expressed as Equation (1) with $\psi(u, v) = -[(u^{-a} - 1)^{-b} + (v^{-a} - 1)^{-b}]^{-1/b}$ and $b > 0$. However, this complex functional perturbation was mainly aimed at flexibilizing the (positive) tail dependence of the Clayton copula; the negative dependence detected by the correlation measures was not achieved. Our approach is different. We consider $\psi(u, v)$ as a product of logarithmic functions modulated by a single tuning parameter, i.e., $\psi(u, v) = b \log(u) \log(v)$ with $b \geq 0$, strategically designed to establish a link with the Gumbel-Barnett copula, a well-known copula that exhibits negative dependence (see [21]). We first establish the conditions on the parameters that make the new copula valid. We then conduct a comprehensive study of its properties, including limit results, associated functions, simulation of random pairs of values, natural extensions, exhaustive bounds, tail dependences, correlation measures that clarify both negative and positive dependences, and bivariate distribution generation with an example related to the bivariate Gaussian distribution. We supplement the analysis with relevant figures and numerical tables. We also present two practical versions of the copula. A brief analysis on simulated data shows that it may be preferable to the Clayton copula on the basis of the Akaike information criterion. Given the widespread applications of the Clayton copula, we expect a similar perspective on our proposed modification. However, the applied aspect is left to future work.

1.4. **Article organization.** The rest of the article is as follows: Section 2 describes the proposed Clayton copula. Section 3 is devoted to its main properties. Some special cases are discussed in Section 4. A conclusion is provided in Section 5.

2. Modified Clayton Copula

Let us present the copula concept in the standard bivariate absolutely continuous setting, as recalled below (see [21]).

Definition 2.1. *Under the bivariate absolutely continuous setting, the following definition of a copula is adopted: A copula is a function defined on* $[0,1]^2$*, say* $A(u,v)$*,* $(u,v) \in [0,1]^2$*, differentiable on* $(0,1)^2$*, and satisfying the two assumptions below.*

(a): $A(u,1) = u$, $A(1,v) = v$, $A(u,0) = A(0,u) = 0$,

(b): $\frac{\partial^2}{\partial u \partial v} A(u,v) \geq 0$.

Throughout the article, the bivariate absolutely continuous setting will be taken into account. Mathematically speaking, proving assumption **(b)** can be more difficult than proving assumption **(a)**; it may require extensive mathematical developments (tedious differentiations, factorizations, inequalities, etc.).

In light of Definition 2.1 and the copula construction elucidated in Equation (1) with a judicious function $\psi(u,v)$, the proposition below introduces our modified Clayton copula.

Proposition 2.2. *Let us consider the following bivariate function:*

$$C(u,v) = \left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-1/a}, \quad (u,v) \in [0,1]^2. \tag{2}$$

Then it is a copula for $a > 0$ *and* $a(a+1) \geq b \geq 0$*.*

Proof. The proof is based on the verification of **(a)** and **(b)** in Definition 2.1. Let us concentrate first on **(a)**. For any $u \in [0,1]$, since $\log(1) = 0$, we immediately have

$$C(u,1) = \left[u^{-a} + 1^{-a} + b\log(u)\log(1) - 1\right]^{-1/a} = (u^{-a})^{-1/a} = u.$$

Using a similar approach, for any $v \in [0,1]$, we also obtain $C(1,v) = v$.

On the other hand, for any $u \in [0,1]$, since $a > 0$, we have $\lim_{v \to 0^+} v^a \log(v) = 0$. This implies that

$$\begin{aligned} C(u,0) &= \lim_{v \to 0^+} \left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-1/a} \\ &= \lim_{v \to 0^+} v\left[u^{-a}v^a + 1 + bv^a\log(u)\log(v) - v^a\right]^{-1/a} \\ &= \lim_{v \to 0^+} v \times (0 + 1 + 0 - 0)^{-1/a} = 0. \end{aligned}$$

Similarly, for any $v \in [0,1]$, we obtain $C(0,v) = 0$. As a result of the above developments, **(a)** is fulfilled.

Let us now consider **(b)**. For any $(u,v) \in (0,1)^2$, by using several differentiation rules and appropriate factorizations, we establish that

$$\frac{\partial^2}{\partial u \partial v} C(u,v) = \frac{1}{a^2\left[bu^a v^a \log(u)\log(v) + u^a(1 - v^a) + v^a\right]^2} \times$$
$$u^{a-1}v^{a-1}\left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-1/a}\left[aJ(u,v) + K(u,v)\right],$$

where

$$J(u,v) = a^2 + bu^a(v^a - 1) - (a+1)bu^a\log(v) - bv^a + a$$

and

$$K(u,v) = -bv^a \log(u)\left[a^2 - bu^a \log(v) + a\right].$$

Therefore, given $a > 0$, proving $J(u,v) \geq 0$ and $K(u,v) \geq 0$ are sufficient to establish $\partial^2 C(u,v)/(\partial u \partial v) \geq 0$, i.e., **(b)**.

Since $a > 0$ and $b \geq 0$, we have $-bv^a \log(u) \geq 0$, $a^2 \geq 0$ and $-bu^a \log(v) \geq 0$, which imply that $K(u,v) \geq 0$. On the other hand, under the assumptions $a > 0$ and $a(a+1) \geq b \geq 0$, since $b(1-u^a)(1-v^a) \geq 0$ and $-(a+1)bu^a \log(v) \geq 0$, we have

$$\begin{aligned} J(u,v) &= a^2 + bu^a v^a - bu^a - bv^a + a - (a+1)bu^a \log(v) \\ &= a(a+1) - b + b(1-u^a)(1-v^a) - (a+1)bu^a \log(v) \\ &\geq a(a+1) - b \geq 0. \end{aligned}$$

Hence, **(b)** is satisfied, ending the proof. □

Let us call the copula in Equation (2) the logarithmic-modified Clayton (LMC) copula. It is the copula described in Equation (1) with

$$\psi(u,v) = b \log(u) \log(v).$$

Obviously, by taking $b = 0$, it corresponds to the standard Clayton copula and, for any $b \geq 0$, the following copula ordering holds: For any $(u,v) \in [0,1]^2$, since $b \log(u) \log(v) \geq 0$, we have

$$C(u,v) \leq C_{\dagger}(u,v).$$

This inequality implies, among other things, that certain correlation measures associated with the LMC copula are smaller than those associated with the Clayton copula. Further exploration of this claim will be done in Subsection 3.5. While the conditions $a > 0$ and $a(a+1) \geq b \geq 0$ may seem stringent due to the interdependence of a and b, there are practical alternatives, such as $b = ac$ with $c \in [0,1]$, or $b = a(a+1)$. These specific configurations will be discussed in Section 4.

To get a representative idea of the LMC copula, Figure 1 plots it for some values of a and b that satisfy the required assumptions. The package `plotly` of the R software was used for this purpose (see [23] and [25]).

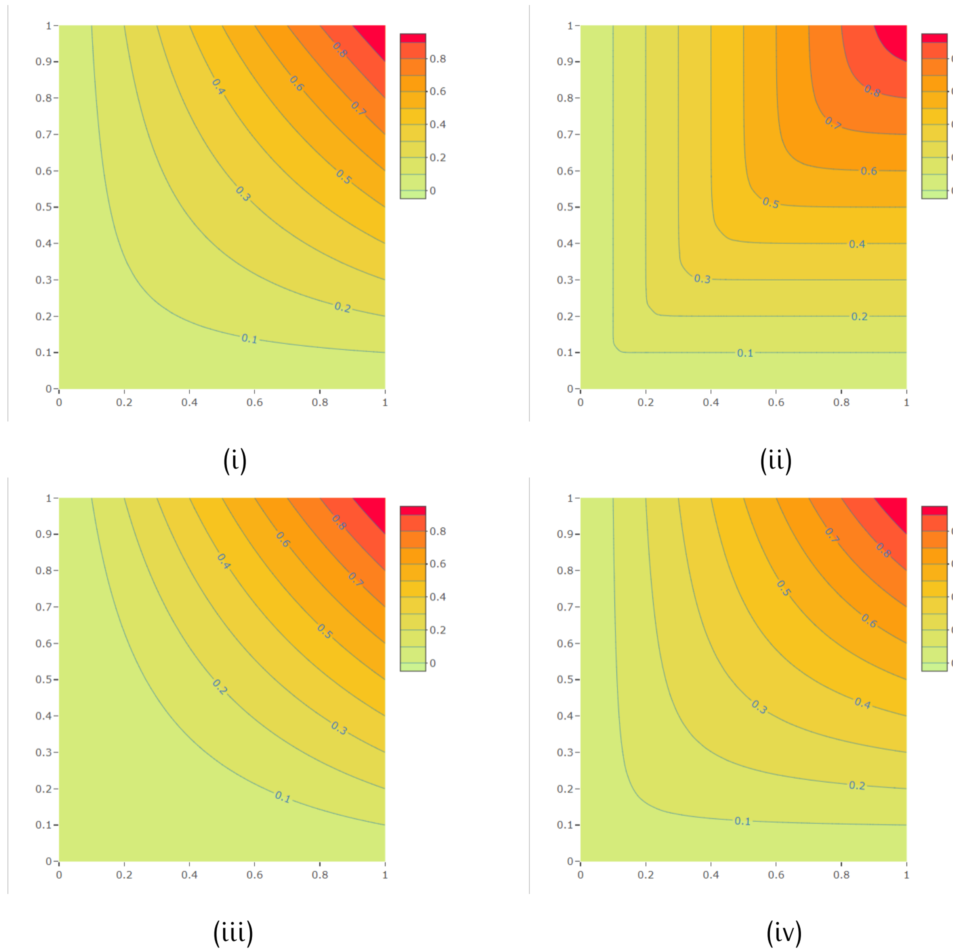


Figure 1. Contour plots of the LMC copula for (i) $a = 1$ and $b = 2$, (ii) $a = 10$ and $b = 1$, (iii) $a = 0.1$ and $b = 0.05$, and (iv) $a = 1.5$ and $b = 2$.

The contours of the LMC copula in this figure are more or less round or square, at different levels and increasing in value (in a sense), depending on the values of a and b. This illustrates both the validity of the copula and its versatility.

In the remainder of the study we will examine the main properties of the LMC copula and discuss how it contributes to developments in modified Clayton copulas.

3. Properties

This section focuses on limit results, related functions, simulation of random couples of values, copula bounds, tail dependence, certain correlation measures, and a new bivariate Gaussian distribution, all derived from the LMC copula.

3.1. Limit results.

Some copula limits of the LMC copula are now examined. The next proposition investigates the limit of the LMC copula when $a \to 0^+$ and an additional assumption on b.

Proposition 3.1. *Let* $(u, v) \in [0, 1]^2$ *and* $C(u, v)$ *be the LMC copula with* $a > 0$ *and* $a(a+1) \geq b \geq 0$*. Suppose that* $\lim_{a \to 0^+} b = 0$ *and* $\lim_{a \to 0^+} b/a = c$ *with* $c \in [0, 1]$*, then we have*

$$\lim_{a \to 0^+} C(u, v) = C_{\triangle}(u, v),$$

where $C_{\triangle}(u, v) = uv \exp\left[-c \log(u) \log(v)\right]$ *is the Gumbel-Barnett copula with parameter* c *(see [21]). When* $c = 0$*, it is reduced to the independence copula.*

Proof. By the basic properties of a copula, the result is immediate for the corner points $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, so let us consider $(u, v) \in (0, 1)^2$. Owing to the equivalence $\exp(t) \sim 1+t$ when $t \to 0$, we get

$$\begin{aligned}
\lim_{a \to 0^+} C(u, v) &= \lim_{a \to 0^+} \left[u^{-a} + v^{-a} + b \log(u) \log(v) - 1\right]^{-1/a} \\
&= \lim_{a \to 0^+} \exp\left[-\frac{1}{a} \log\left\{\exp[-a \log(u)] + \exp[-a \log(v)] + b \log(u) \log(v) - 1\right\}\right] \\
&= \lim_{a \to 0^+} \exp\left\{-\frac{1}{a} \log\left[1 - a \log(u) - a \log(v) + b \log(u) \log(v)\right]\right\}.
\end{aligned}$$

Using the limit assumptions $\lim_{a \to 0^+} b = 0$, it is clear that $\lim_{a \to 0^+} a \log(u) + a \log(v) - b \log(u) \log(v) = 0$. This, combined with the equivalence $\log(1+t) \sim t$ when $t \to 0$, gives

$$\begin{aligned}
\lim_{a \to 0^+} C(u, v) &= \lim_{a \to 0^+} \exp\left[\log(u) + \log(v) - \frac{b}{a} \log(u) \log(v)\right] \\
&= uv \lim_{a \to 0^+} \exp\left[-\frac{b}{a} \log(u) \log(v)\right].
\end{aligned}$$

The assumption $\lim_{a \to 0^+} b/a = c$ with $c \in [0, 1]$ implies that

$$\lim_{a \to 0^+} C(u, v) = uv \exp\left[-c \log(u) \log(v)\right] = C_{\triangle}(u, v),$$

which ends the proof. □

These limit results highlight the role of the parameters a and b and why the LMC copula differs from the Clayton copula. Indeed, it means that for a and b/a closed to 0, the LMC copula can have the behavior of the Gumbel-Barnett copula, and it is known that the Gumbel-Barnett copula can reach negative dependence (with a Kendall tau in the range $[-0.361, 0]$). Therefore, for some values of the parameters, the LMC copula is expected to reach both negative and positive dependence, while maintaining regularity, unlike the (standard or modified) Clayton copula. This comment will be supported numerically in Subsection 3.5.

The proposition below shows that the limit property of the standard Clayton copula for $a \to +\infty$ still holds for the LMC copula.

Proposition 3.2. *Let* $(u, v) \in [0, 1]^2$ *and* $C(u, v)$ *be the LMC copula with* $a > 0$ *and* $a(a+1) \geq b \geq 0$. *Then we have*

$$\lim_{a \to +\infty} C(u, v) = C_\circ(u, v).$$

We recall that $C_\circ(u, v) = \min(u, v)$.

Proof. By the basic properties of a copula, the result is immediate for the corner points $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, so let us consider $(u, v) \in (0, 1)^2$. We have

$$\lim_{a \to +\infty} C(u, v) = \lim_{a \to +\infty} \left[u^{-a} + v^{-a} + b \log(u) \log(v) - 1\right]^{-1/a}$$
$$= \lim_{a \to +\infty} \exp\left[-\frac{1}{a} \log\left\{\exp[-a \log(u)] + \exp[-a \log(v)] + b \log(u) \log(v) - 1\right\}\right].$$

For any $a(a+1) \geq b \geq 0$, even if b depends on a (such as $b = a(a+1)$ at the maximum order with respect to a), for $a \to +\infty$, since $u^{-a} = \exp[-a \log(u)] \to +\infty$ and $v^{-a} = \exp[-a \log(v)] \to +\infty$ with an exponential rate, we have

$$u^{-a} + v^{-a} + b \log(u) \log(v) - 1 \sim u^{-a} + v^{-a} \sim [\min(u, v)]^{-a} = [C_\circ(u, v)]^{-a}.$$

Therefore, we have

$$\lim_{a \to +\infty} C(u, v) = \lim_{a \to +\infty} \exp\left[-\frac{1}{a} \log\left\{[C_\circ(u, v)]^{-a}\right\}\right] = C_\circ(u, v).$$

The proof is achieved. □

Thus, the important min-copula limit property of the Clayton copula is preserved for the LMC copula.

These comprehensive results are advantages for the LMC copula, among other properties to be examined in the next sections.

3.2. **Related functions.** A number of important functions that are derived from the LMC copula are now described.

3.2.1. *Useful functions.* The LMC copula density is a hidden ingredient of the proof of **(b)** in Proposition 2.2; it is given by

$$c(u, v) = \frac{\partial^2}{\partial u \partial v} C(u, v) = \frac{1}{a^2 \left[b u^a v^a \log(u) \log(v) + u^a(1 - v^a) + v^a\right]^2} \times$$
$$u^{a-1} v^{a-1} \left[u^{-a} + v^{-a} + b \log(u) \log(v) - 1\right]^{-1/a} \times$$
$$\left\{a\left[a^2 + b u^a (v^a - 1) - (a+1) b u^a \log(v) - b v^a + a\right] - b v^a \log(u) \left[a^2 - b u^a \log(v) + a\right]\right\},$$
$$(u, v) \in [0, 1]^2,$$

with the limits at the vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. In full generality, the copula density is of interest because the more flexible the copula density, the more the associated copula model is adaptable to versatile dependence structures. Due to the functional complexity of $c(u, v)$, a graphical study is preferable to an analytical one. Thus, Figure 2 shows it for some values of a and b that satisfy the necessary assumptions.

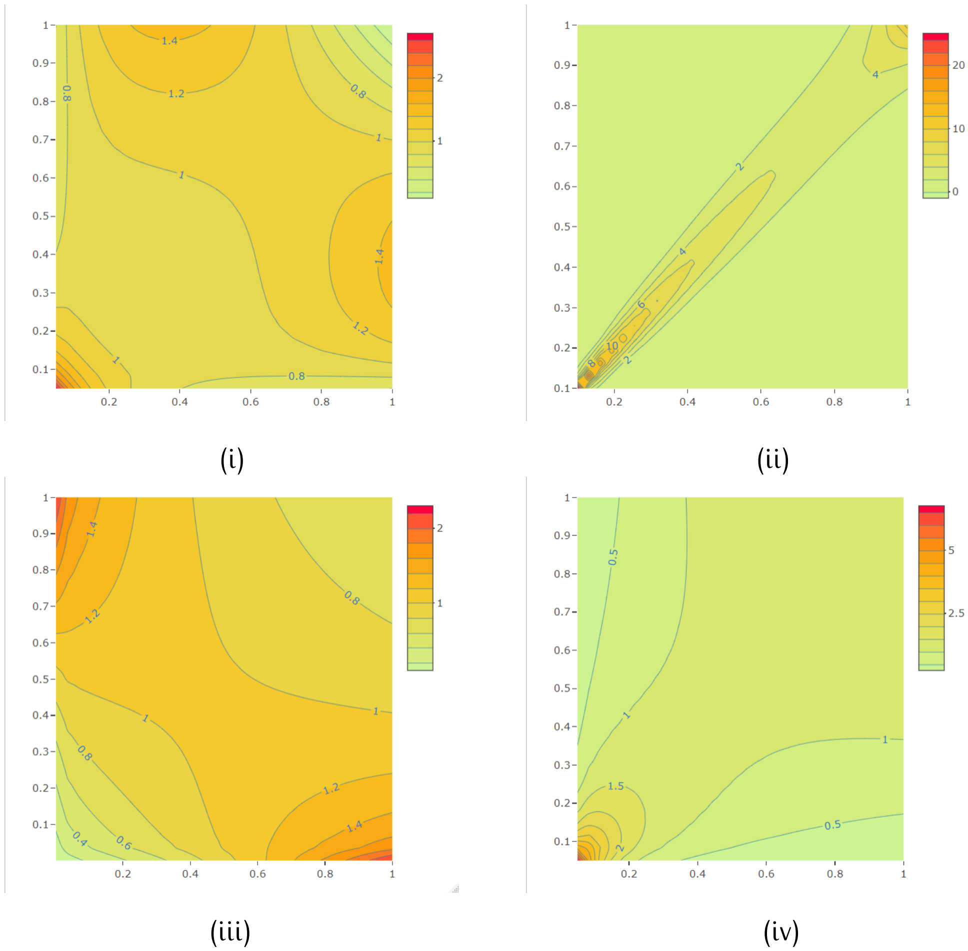


FIGURE 2. Contour plots of the LMC copula density for (i) $a = 1$ and $b = 2$, (ii) $a = 10$ and $b = 1$, (iii) $a = 0.1$ and $b = 0.05$, and (iv) $a = 1.5$ and $b = 2$.

The contours of the LMC copula density are very different for the considered values of a and b. This demonstrates a high degree of adaptability for the underlying dependence model.

On the other hand, the two conditional LMC copulas are given by

$$\begin{aligned} C_1(u, v) &= \frac{\partial}{\partial u} C(u, v) \\ &= -\frac{1}{au}\left[b\log(v) - au^{-a}\right]\left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-1-1/a} \end{aligned} \tag{3}$$

and

$$\begin{aligned} C_2(u, v) &= \frac{\partial}{\partial v} C(u, v) = C_1(v, u) \\ &= -\frac{1}{av}\left[b\log(u) - av^{-a}\right]\left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-1-1/a}, \quad (u, v) \in [0, 1]^2. \end{aligned} \tag{4}$$

The first conditional copula is useful for simulating a couple of values (u_*, v_*) from a random vector (U, V) having the LMC copula as a cumulative distribution function. The standard simulation scheme is as follows:

- Simulate a couple of independent values (u_*, w), each from the uniform distribution over $[0, 1]$.
- Determine numerically the value v_* satisfying the following non-linear equation: $w = C_1(u_*, v_*)$, where $C_1(u, v)$ is given by Equation (3).
- Consider (u_*, v_*) as the generated couple of values.

For any positive integer n, we can repeat the process n times to have n couples of values from (U, V). As an illustration, for four different parameter configurations on a and b, Figure 3 depicts $n = 100$ such simulated couples of values. The package `rootSolve` of the R software was used for the second step of the process (see [28]).

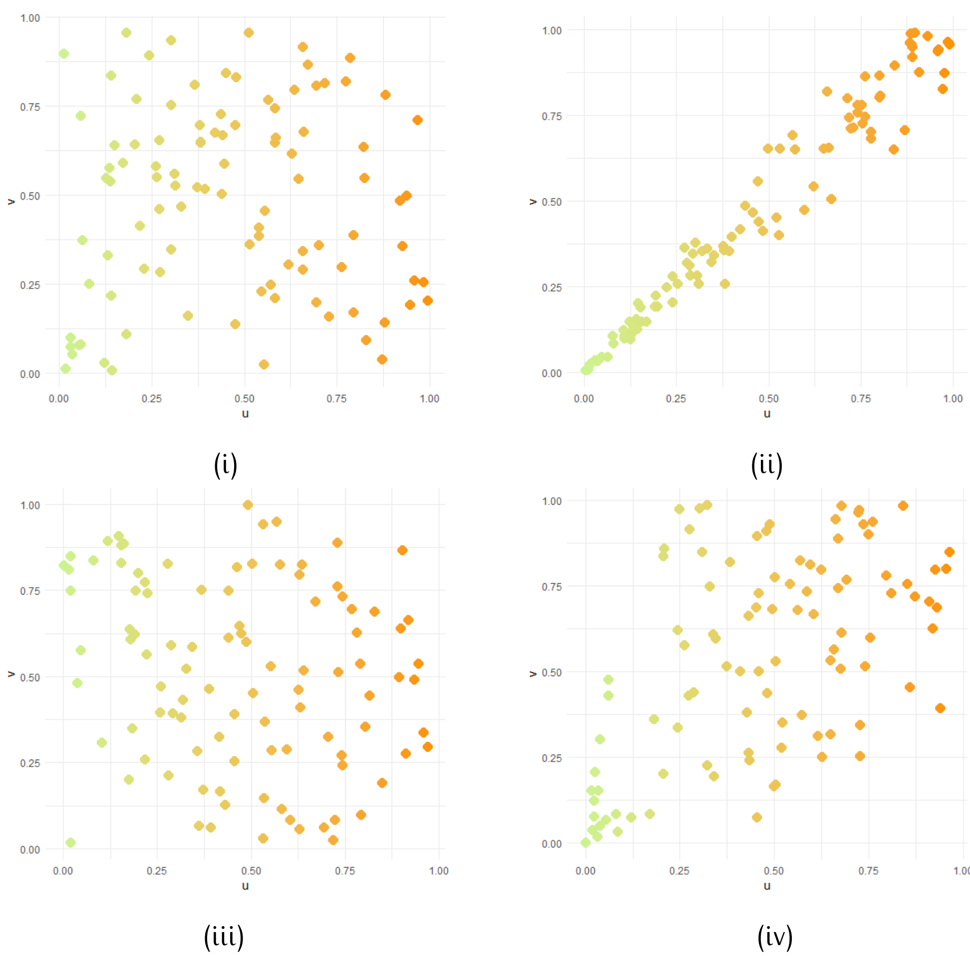


FIGURE 3. Plots of $n = 100$ simulated couples of values from a random vector (U, V) having the LMC copula as cumulative distribution function for (i) $a = 1$ and $b = 2$, (ii) $a = 10$ and $b = 1$, (iii) $a = 0.1$ and $b = 0.05$, and (iv) $a = 1.5$ and $b = 2$.

From this figure, we can see the different types of dependence trends, with a more or less clear structure and with some specific clusters of points.

Simulated couples of values can also be used to test the behavior of different parametric estimation procedures by varying the number n; the larger n is, the more the procedures have to prove their efficiency.

The simulation scheme will be used in Subsection 4.2, to compare the fits of a special LMC copula and the Clayton copula.

On the other hand, the conditional copulas are also involved in the definition of correlation measures, such as the Kendall tau, as will be shown later.

3.2.2. *Derived copulas.* The LMC copula can be used to derive other new copulas with little effort. Based on some schemes presented in [21], still under the assumptions $a > 0$ and $a(a+1) \geq b \geq 0$, a short list of them is given below.

- We define the u-flipping LMC copula by

$$\begin{aligned} C_{\daleth}(u, v) &= v - C(1-u, v) \\ &= v - \left[(1-u)^{-a} + v^{-a} + b\log(1-u)\log(v) - 1\right]^{-1/a}, \quad (u, v) \in [0,1]^2. \end{aligned}$$

- We introduce the v-flipping LMC copula by

$$\begin{aligned} C_{\neg}(u, v) &= u - C(u, 1-v) \\ &= u - \left[u^{-a} + (1-v)^{-a} + b\log(u)\log(1-v) - 1\right]^{-1/a}, \quad (u, v) \in [0,1]^2. \end{aligned}$$

- We define the survival LMC copula by

$$\begin{aligned} C_{\beth}(u, v) &= u + v - 1 + C(1-u, 1-v) \\ &= u + v - 1 + \left[(1-u)^{-a} + (1-v)^{-a} + b\log(1-u)\log(1-v) - 1\right]^{-1/a}, \\ &\quad (u, v) \in [0,1]^2. \end{aligned}$$

All of them offer a new dependence model based on the Clayton copula.

Another interesting scheme is the one elaborated in [10]. It allows for flexibility and asymmetry of a given copula. In particular, with the use of power functions as in [10, Corollary 4], the following result holds: For any $a > 0$, $a(a+1) \geq b \geq 0$, $c \in [0,1]$ and $d \in [0,1]$, we define a new copula by

$$\begin{aligned} C_{\diamond}(u, v) &= u^c v^d C(u^{1-c}, v^{1-d}) \\ &= u^c v^d \left[u^{-a(1-c)} + v^{-a(1-d)} + b(1-c)(1-d)\log(u)\log(v) - 1\right]^{-1/a}, \quad (u, v) \in [0,1]^2. \end{aligned}$$

Clearly, for $c \neq d$, it is not diagonally symmetric. The parameters c and d add versatility to the LMC copula, but complicate the mathematical scheme; there is a risk of over-parameterisation from a practical point of view.

Other copula-generated schemes are possible; the LMC copula may offer new opportunities to advance research in this direction.

3.3. **Copula bounds.** As for any copula, the Fréchet-Hoeffding result holds (see [21]). It ensures that the LMC copula satisfies the following copula inequalities: $C_\star(u,v) \leq C(u,v) \leq C_\circ(u,v)$, i.e.,

$$\max(u+v-1,0) \leq \left[u^{-a}+v^{-a}+b\log(u)\log(v)-1\right]^{-1/a} \leq \min(u,v).$$

In fact, for the LMC copula, these general bounds can be improved under some additional assumptions on a and b. A better lower bound, which has a quadrant dependence interpretation, is described below.

Proposition 3.3. *Let $(u,v) \in [0,1]^2$ and $C(u,v)$ be the LMC copula with $a>0$ and $a^2 \geq b \geq 0$. Then we have*

$$C(u,v) \geq uv.$$

Proof. For $a>0$ and $b \leq a^2$, we have

$$\begin{aligned} u^{-a}+v^{-a}+b\log(u)\log(v)-1 &\leq u^{-a}+v^{-a}+a^2\log(u)\log(v)-1 \\ &= u^{-a}+v^{-a}+\log(u^{-a})\log(v^{-a})-1. \end{aligned}$$

Now, by using the inequality $\log(t) \leq t-1$ for $t>0$, since $u^{-a} \geq 1$ and $v^{-a} \geq 1$, we have $0 \leq \log(u^{-a}) \leq u^{-a}-1$ and $0 \leq \log(v^{-a}) \leq v^{-a}-1$, which imply that $\log(u^{-a})\log(v^{-a}) \leq (u^{-a}-1)(v^{-a}-1)$. Therefore, we have

$$u^{-a}+v^{-a}+b\log(u)\log(v)-1 \leq u^{-a}+v^{-a}+(u^{-a}-1)(v^{-a}-1)-1 = u^{-a}v^{-a}.$$

Raising the two sides at the negative power $-1/a$, we get

$$C(u,v) = \left[u^{-a}+v^{-a}+b\log(u)\log(v)-1\right]^{-1/a} \geq (u^{-a}v^{-a})^{-1/a} = uv.$$

The desired limit result is demonstrated. □

It follows from Proposition 3.3 that, under the assumptions $a>0$ and $a^2 \geq b \geq 0$, the lower bound $C_\star(u,v)$ is improved and that the LMC copula has the positive quadrant dependence property. This was known for the Clayton copula, i.e., $b=0$, and now extends to the LMC copula with $a^2 \geq b \geq 0$.

Under a complementary assumption, the next result suggests an improved upper bound for the LMC copula.

Proposition 3.4. *Let $(u,v) \in [0,1]^2$ and $C(u,v)$ be the LMC copula with $a>0$ and $a^2 \leq b \leq a(a+1)$. Then we have*

$$C(u,v) \leq \left\{[C_\circ(u,v)]^{-\alpha}+(1-u^a)(1-v^a)\right\}^{-1/a},$$

upper bound which is clearly smaller than $C_\circ(u,v)$.

Proof. For $a > 0$ and $a^2 \leq b \leq a(a+1)$, we have

$$\begin{aligned} u^{-a} + v^{-a} + b\log(u)\log(v) - 1 &\geq u^{-a} + v^{-a} + a^2 \log(u)\log(v) - 1 \\ &= u^{-a} + v^{-a} + \log(u^{-a})\log(v^{-a}) - 1. \end{aligned}$$

Now, by using the inequality $\log(t) \geq (t-1)/t$ for $t > 0$, since $u^{-a} \geq 1$ and $v^{-a} \geq 1$, we have $\log(u^{-a}) \geq (u^{-a}-1)/u^{-a} \geq 0$, $\log(v^{-a}) \geq (v^{-a}-1)/v^{-a} \geq 0$, implying that $\log(u^{-a})\log(v^{-a}) \geq (u^{-a}-1)(v^{-a}-1)/(u^{-a}v^{-a})$. Therefore, we have

$$\begin{aligned} u^{-a} + v^{-a} + b\log(u)\log(v) - 1 &\geq u^{-a} + v^{-a} + \frac{1}{u^{-a}v^{-a}}(u^{-a}-1)(v^{-a}-1) - 1 \\ &= u^{-a} + v^{-a} - u^a - v^a + (uv)^a \\ &\geq [\min(u,v)]^{-\alpha} + 1 - u^a - v^a + (uv)^a \\ &= [C_\circ(u,v)]^{-\alpha} + (1-u^a)(1-v^a) \end{aligned}$$

Raising the two sides at the negative power $-1/a$, we get

$$\begin{aligned} C(u,v) &= \left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-1/a} \\ &\leq \left\{[C_\circ(u,v)]^{-\alpha} + (1-u^a)(1-v^a)\right\}^{-1/a}. \end{aligned}$$

The stated upper bound is obtained. □

The interest in the obtained bounds is mainly theoretical but contributes to the understanding of the LMC copula.

3.4. **Tail dependence.** Several types of tail dependence in the LMC copula are now investigated, based on the formulas in [21]. First, the lower left tail dependence parameter is given by

$$\lambda_1 = \lambda_l = \lim_{u\to 0^+} \frac{C(u,u)}{u} = \lim_{u\to 0^+} \frac{\left\{2u^{-a} + b[\log(u)]^2 - 1\right\}^{-1/a}}{u} = 2^{-1/a}.$$

It is the same as the standard Clayton copula; the parameter b plays no role in it. Thus, the LMC copula is lower left tail dependent. In fact, it is the only tail dependence it has. Indeed, the lower right tail dependence parameter is obtained as

$$\begin{aligned} \lambda_2 &= \lim_{u\to 0^+} \frac{u - C(1-u,u)}{u} \\ &= \lim_{u\to 0^+} \frac{u - \left[(1-u)^{-a} + u^{-a} + b\log(1-u)\log(u) - 1\right]^{-1/a}}{u} = 0, \end{aligned}$$

since $C(u,v)$ is symmetric, the corresponding upper left tail dependence parameter is determined by

$$\lambda_3 = \lim_{u\to 0^+} \frac{u - C(u,1-u)}{u} = \lambda_2 = 0$$

and the upper right tail dependence parameter is given by

$$\lambda_4 = \lambda_u = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}$$
$$= \lim_{u \to 1^-} \frac{1 - 2u + \left\{2u^{-a} + b[\log(u)]^2 - 1\right\}^{-1/a}}{1 - u} = 0.$$

Thus, we have $\lambda_2 = \lambda_3 = \lambda_4 = 0$, supporting the claim.

3.5. **Correlation measures.** Let us now examine some correlation measures associated with the LMC copula, namely the medial correlation and the Kendall tau. We refer again to [21] for the details of these measures.

3.5.1. *Medial correlation.* First, the medial correlation of the LMC copula is indicated as follows:

$$M = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1 = 4\left\{2^{a+1} + b[\log(2)]^2 - 1\right\}^{-1/a} - 1,$$

with $\log(2) \approx 0.693$. The effects of a and b on it is clear. In particular, for a fixed value of $a > 0$, M is a decreasing function with respect to b, and for a fixed value of $b > 0$, M is a decreasing function with respect to a. Furthermore, M can be non-positive. It is the case for $a > 0$ and

$$a(a+1) \geq b \geq \max\left\{\frac{1}{[\log(2)]^2}\left[1 + 2^{2a} - 2^{a+1}\right], 0\right\}.$$

These conditions happen when a is small enough, which is consistent with the limit copula result described in Proposition 3.1. We illustrate this claim in Tables 1 and 2 by calculating M for selected values of a and b satisfying $a > 0$ and $a(a+1) \geq b \geq 0$. In fact, Table 2 focuses on the special case $b = a(a+1)$ and small values for a.

Table 1. Values of the medial correlation of the LMC copula for some values of a and b satisfying $a > 0$ and $a(a+1) \geq b \geq 0$ (when it is not true, the cross symbol $\times$ is put).

b	0.0	0.05	0.1	0.2	0.5	0.9	1.1	1.3	1.5	2	2.5	3	4	5
$a = 7$	0.812	0.812	0.812	0.812	0.812	0.812	0.812	0.812	0.812	0.811	0.811	0.811	0.811	0.81
$a = 2$	0.512	0.509	0.507	0.502	0.487	0.467	0.458	0.449	0.44	0.418	0.397	0.377	0.339	0.304
$a = 1$	0.333	0.323	0.312	0.292	0.234	0.165	0.134	0.104	0.075	0.01	$\times$	$\times$	$\times$	$\times$
$a = 0.5$	0.196	0.166	0.136	0.08	-0.065	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
$a = 0.1$	0.046	-0.15	-0.307	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

TABLE 2. Values of the medial correlation of the LMC copula for some small values of $a > 0$ and $b = a(a+1)$.

a	0.01	0.11	0.21	0.31	0.41	0.51	0.61	0.71	0.81	0.91
	−0.377	−0.33	−0.285	−0.242	−0.201	−0.161	−0.123	−0.087	−0.052	−0.019

From these tables, based only on the selected values of the parameters, we find that $M \in [-0.377, 0.812]$, but further numerical tests with large a extend this interval to $[-0.377, 1]$. This shows that the LMC copula can achieve both negative and positive dependences, which is very different from the standard Clayton copula. This is particularly true for the single-parameter LMC copula defined with $b = a(a+1)$ and $a > 0$, as observed in Table 2.

3.5.2. *Kendall tau.* Based on $C_1(u,v)$ and $C_2(u,v)$ specified in Equations (3) and (4), respectively, the Kendall tau of the LMC copula has the following integral expression:

$$\begin{aligned}\tau &= 1 - 4\int_0^1\int_0^1 C_1(u,v)C_2(u,v)dudv\\ &= 1 - 4\int_0^1\int_0^1 \frac{1}{a^2uv}\left[b\log(v) - au^{-a}\right]\left[b\log(u) - av^{-a}\right]\times\\ &\left[u^{-a} + v^{-a} + b\log(u)\log(v) - 1\right]^{-2-2/a}dudv.\end{aligned}$$

The integrated term is too sophisticated from an analytical point of view to expect a nice expression. Nevertheless, based on the behavior of the medial correlation and the limit result in Proposition 3.1, it is plausible that this measure is negative for small values of a. We support this claim in Tables 3 and 4 with a numerical work; we determine τ for selected values of a and b satisfying $a > 0$ and $a(a+1) \geq b \geq 0$. Table 4 focuses on the special case $b = a(a+1)$ and small values of a.

TABLE 3. Values of the Kendall tau of the LMC copula for some values of a and b satisfying $a > 0$ and $a(a+1) \geq b \geq 0$.

b	0.0	0.05	0.1	0.2	0.5	0.9	1.1	1.3	1.5	2	2.5	3	4	5
$a = 7$	0.778	0.778	0.778	0.778	0.777	0.777	0.776	0.776	0.776	0.775	0.775	0.774	0.773	0.772
$a = 2$	0.5	0.498	0.496	0.491	0.478	0.462	0.453	0.445	0.438	0.418	0.4	0.382	0.349	0.317
$a = 1$	0.333	0.324	0.314	0.296	0.244	0.183	0.154	0.127	0.101	0.041	×	×	×	×
$a = 0.5$	0.2	0.17	0.142	0.09	−0.043	×	×	×	×	×	×	×	×	×
$a = 0.1$	0.048	−0.142	−0.286	×	×	×	×	×	×	×	×	×	×	×

Table 4. Values of the Kendall tau of the LMC copula for some small values of $a > 0$ and $b = a(a+1)$.

a	0.01	0.11	0.21	0.31	0.41	0.51	0.61	0.71	0.81	0.91
	−0.356	−0.306	−0.259	−0.214	−0.171	−0.131	−0.092	−0.056	−0.021	0.013

Some numerical monotonic patterns with respect to a and b are observed, but they cannot be validated by rigorous analysis. For the considered parameter scenarios, we find that $\tau \in [-0.356, 0.778]$, but further numerical tests with large a extend it to $\tau \in [-0.356, 1]$. Thus, the analysis of the Kendall tau also confirms that the LMC copula can reach both negative and positive dependences. Again, this is particularly true for the single-parameter LMC copula defined with $b = a(a+1)$ and $a > 0$, as seen in Table 4.

3.6. **The LMC Gaussian distribution.** The LMC copula can be used to generate a variety of bivariate distributions. In particular, by considering the LMC copula and the standard Gaussian distribution for the marginal distributions, we define a new bivariate Gaussian distribution by the following cumulative distribution function:

$$\begin{aligned} F(x,y) &= C[\Phi(x),\Phi(y)] \\ &= \left\{[\Phi(x)]^{-a} + [\Phi(y)]^{-a} + b\log[\Phi(x)]\log[\Phi(y)] - 1\right\}^{-1/a}, \quad (x,y)\in\mathbb{R}^2, \end{aligned}$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard Gaussian distribution, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{u^2}{2}\right)du, \quad x\in\mathbb{R}.$$

Let us call the corresponding bivariate distribution the LMC Gaussian distribution. Based on the LMC copula density, the corresponding probability density function is obtained as

$$\begin{aligned} f(x,y) &= \varphi(x)\varphi(y)c[\Phi(x),\Phi(y)] \\ &= \frac{\varphi(x)\varphi(y)}{a^2\left\{b[\Phi(x)]^a[\Phi(y)]^a\log[\Phi(x)]\log[\Phi(y)] + [\Phi(x)]^a\{1-[\Phi(y)]^a\} + [\Phi(y)]^a\right\}^2}\times \\ &\quad [\Phi(x)]^{a-1}[\Phi(y)]^{a-1}\left\{[\Phi(x)]^{-a} + [\Phi(y)]^{-a} + b\log[\Phi(x)]\log[\Phi(y)] - 1\right\}^{-1/a}\times \\ &\quad \Big\{a\left[a^2 + b[\Phi(x)]^a\{[\Phi(y)]^a - 1\} - (a+1)b[\Phi(x)]^a\log[\Phi(y)] - b[\Phi(y)]^a + a\right] \\ &\quad - b[\Phi(y)]^a\log[\Phi(x)]\left[a^2 - b[\Phi(x)]^a\log[\Phi(y)] + a\right]\Big\}, \quad (x,y)\in\mathbb{R}^2, \end{aligned}$$

where $\varphi(x)$ denotes the probability density function of the standard Gaussian distribution, i.e.,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), \quad x\in\mathbb{R}.$$

In order to see the shape ability of this bivariate probability density function, Figure 4 depicts it for some values of a and b satisfying the required assumptions.

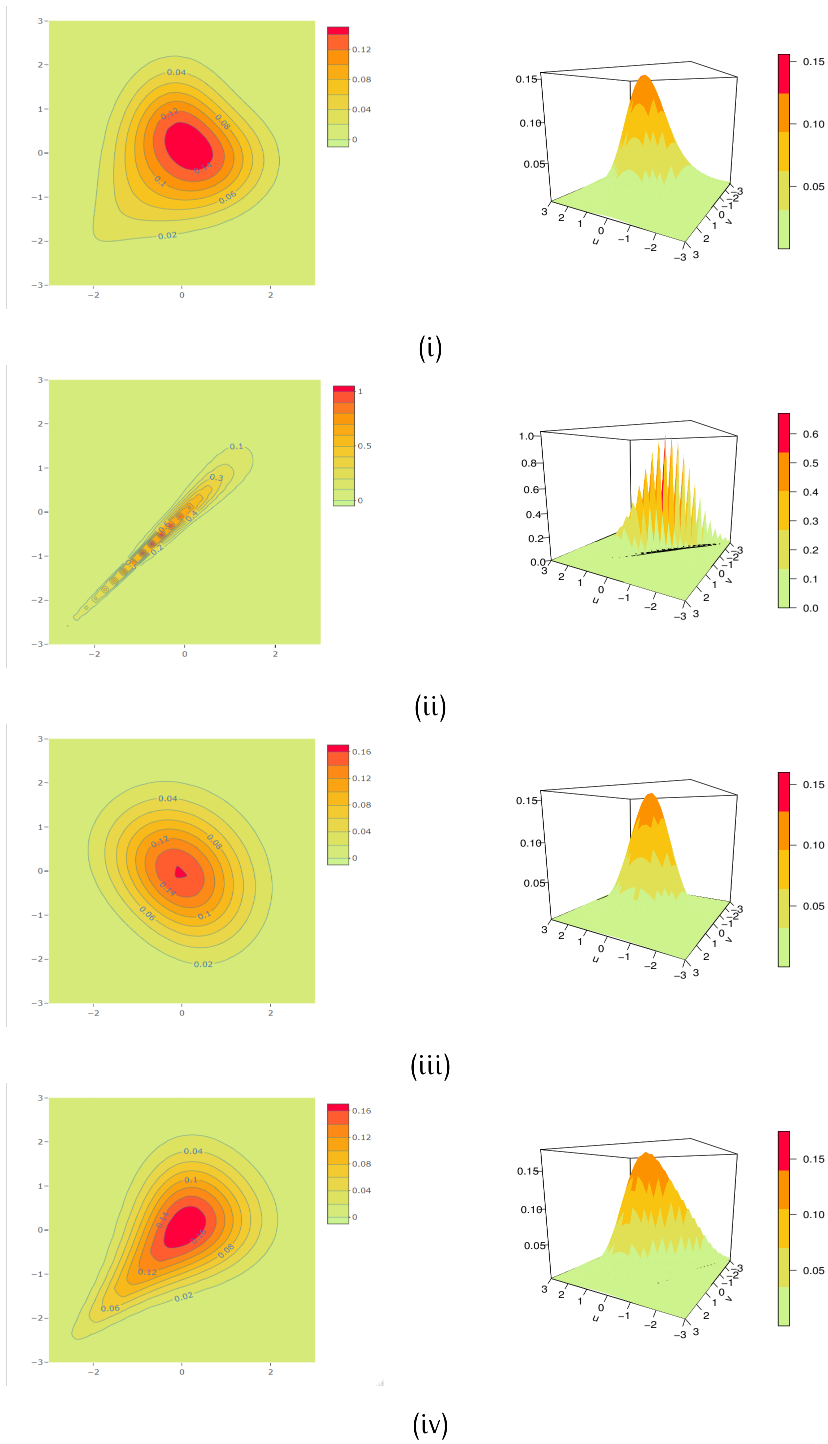


Figure 4. Contour plots (left column) and shape plots (right column) of the LMC Gaussian probability density function for (i) $a = 1$ and $b = 2$, (ii) $a = 10$ and $b = 1$, (iii) $a = 0.1$ and $b = 0.05$, and (iv) $a = 1.5$ and $b = 2$.

From this figure, we can see how the classical Gaussian bell shape is simultaneously skewed and deformed at its base, with various weights on the tails. For the configuration (ii), we see a multimodal phenomenon, certainly caused by the large value of a, the reasonable value of b, and the combined presence of power and logarithmic functions. This versatility of shape makes the LMC Gaussian distribution interesting for further study in various bivariate modeling scenarios (bivariate noise, regression modeling, bivariate data analysis beyond the standard Gaussian case, etc.).

4. Special Copulas

Two special cases of the LMC copula are now highlighted because of their practical design.

4.1. Two-parameter practical LMC copula. Choosing $b = ac$ with $c \in [0, 1]$, which is possible since $ac \leq a \leq a(a + 1)$, the LMC copula can be expressed as

$$C(u, v) = \left[u^{-a} + v^{-a} + ac \log(u) \log(v) - 1\right]^{-1/a}, \quad (u, v) \in [0, 1]^2.$$

It is of particular interest because a and c are completely independent, since $\lim_{a \to 0^+} b = 0$ and $\lim_{a \to 0^+} b/a = c$ with $c \in [0, 1]$, according to Proposition 3.1, the Gumbel-Barnett copula with parameter c is obtained as a limit when $a \to 0^+$, the Clayton copula is recovered with $c = 0$, and it can reach negative and positive dependences. This last claim is supported by Table 5.

Table 5. Values of the Kendall tau of the LMC copula for some some values of a and b such that $b = ac$ with $a > 0$ and $c \in [0, 1]$.

c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$a = 0.1$	0.048	0.004	−0.036	−0.073	−0.108	−0.142	−0.173	−0.203	−0.232	−0.259	−0.286
$a = 0.4$	0.167	0.134	0.103	0.074	0.047	0.020	−0.005	−0.029	−0.052	−0.075	−0.096
$a = 1$	0.333	0.314	0.296	0.278	0.261	0.244	0.228	0.213	0.197	0.183	0.168
$a = 6$	0.750	0.749	0.748	0.746	0.745	0.744	0.743	0.742	0.741	0.739	0.738

As expected, negative values appear in this table for small values of a. Despite a slight restriction of the parameter domain, this variant can be considered as a practical version of the LMC copula because of the independence of the parameters and the fact that the main properties are conserved. Therefore, it is sufficiently motivated for various applications where the Clayton copula model is not the optimal choice.

4.2. **Single-parameter LMC copula.** A special single-parameter LMC copula has been sketched in several parts of the article. It is defined by choosing $b = a(a+1)$ with $a > 0$, and is expressed as

$$C(u, v) = \left[u^{-a} + v^{-a} + a(a+1)\log(u)\log(v) - 1\right]^{-1/a}, \quad (u, v) \in [0, 1]^2.$$

This version is interesting because, with the tune of only one parameter, since $\lim_{a\to 0^+} b = 0$ and $\lim_{a\to 0^+} b/a = 1$, according to Proposition 3.1, the Gumbel-Barnett copula with parameter 1 is obtained as a limit when $a \to 0^+$, and it can reach the negative and positive dependences as already shown in Tables 2 and 4. However, the Clayton copula cannot be recovered. Therefore, it must be considered as a real alternative with similar mathematical ingredients.

In this case, the parameter a can simply be estimated by the omnibus estimation method, also called the canonical maximum likelihood method, as described in [13] and [26]. As a basic description, let us consider observations drawn from a continuous random vector, such as (X, Y). Thus, let n be the number of observations of this vector, and $(x_1, y_1), \ldots, (x_n, y_n)$ be the observations representing the data. Then, the omnibus estimate of a is obtained as

$$\tilde{a} = \text{argmax}_{a>0} \sum_{i=1}^{n} \log\left\{c\left[\tilde{F}(x_i), \tilde{G}(y_i); a\right]\right\},$$

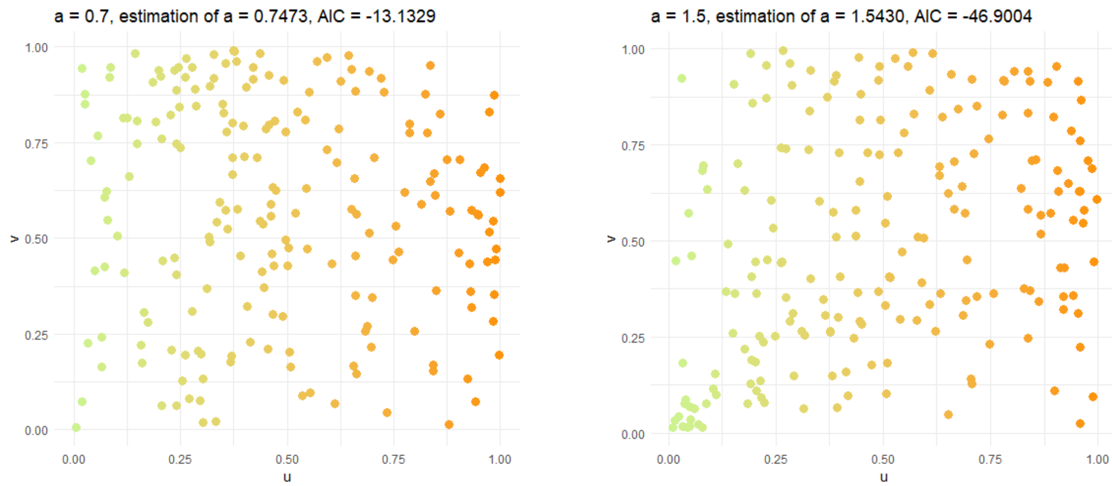
where $c(u, v; a)$ refers to the associated copula density with the indication of the dependence with respect to a,

$$\tilde{F}(x) = \frac{1}{n+1}\sum_{j=1}^{n} \mathfrak{I}\{x_j \leq x\}, \quad \tilde{G}(y) = \frac{1}{n+1}\sum_{j=1}^{n} \mathfrak{I}\{y_j \leq y\},$$

and $\mathfrak{I}\{S\}$ denotes the indicator function with respect to a given set S (see [13] and [26]). For single-parameter copula model comparisons, we can define the Akaike information criterion as

$$AIC = -2\sum_{i=1}^{n} \log\left\{c\left[\tilde{F}(x_i), \tilde{G}(y_i); \tilde{a}\right]\right\} + 2.$$

Basically, the smaller the value of the AIC associated with a (single-parameter) copula, the better the copula is at capturing the underlying dependence behind the data. Illustrative examples are performed. In Figure 5, we simulate $n = 200$ couples of values from a random vector (X, Y) having the single-parameter LMC copula as cumulative distribution function for two different values of a.



(i) (ii)

Figure 5. Plots of $n = 200$ simulated couples of values from a random vector (X, Y) having the single-parameter LMC copula as cumulative distribution function for (i) $a = 0.7$ and (ii) $a = 1.5$, with the omnibus estimates of a and associated AICs.

For comparison purposes, under the scenario $a = 0.7$, we have $\tilde{a} = 0.7473$ and the AIC of the single-parameter LMC copula is -13.1329, while that obtained for the Clayton copula is 1.0905; the LMC copula is the best, which is not surprising since the data have been generated from it. Under the scenario $a = 1.5$, we have $\tilde{a} = 1.5430$ and the AIC of the single-parameter LMC copula is -46.9004, while the AIC obtained for the Clayton copula is -44.7007; the LMC copula is again the best as expected. This brief numerical study shows that, for some data sets, the single-parameter LMC copula can be more efficient in capturing the dependence structure than the Clayton copula.

Remark 4.1. *The choice $b = a^2$ in the LMC copula is really different from the choice $b = a(a+1)$ because it does not permit the negative dependence. This can be deduced from Proposition 3.3, the inequality $C(u, v) \geq uv$ implying that $M \geq 0$ and $\tau \geq 0$.*

5. Conclusion

In conclusion, this article introduces a novel modification of the Clayton copula, which departs from its Archimedean nature in order to increase flexibility. The proposed variant is characterized by an additional bivariate product of logarithmic functions and a tuning parameter. It provides a nuanced representation of the corresponding copula density. Regularity and negative dependence can be achieved simultaneously. The study explores the properties of the new copula, relates it to the Gumbel-Barnett copula, elucidates the simulation of random pairs of values, various bounds and tail dependencies, and examines correlation properties. In addition, the article gives a probability application with a modified bivariate Gaussian distribution. Two special copula cases are also discussed. A brief numerical work is performed to demonstrate the practical alternative to the Clayton copula. The results contribute to the advancement of the theoretical foundations of copula-based modeling techniques. Some possible perspectives of this work are formulated below.

- One can consider other choices for the function $\psi(u, v)$ in Equation (1), perhaps of non-constant sign for a higher level of perturbation of the Clayton copula.
- One can investigate a higher-dimensional version of the proposed copula, such as the trivariate variant, defined as

$$C(u, v, w) = \left[u^{-a} + v^{-a} + w^{-a} + b \log(u) \log(v) \log(w) - 1\right]^{-1/a}, \quad (u, v, w) \in [0, 1]^3,$$

  where a and b are two parameters in ranges of values to be determined (even if the values of b are logically negative to make sense to the power $-1/a$).
- More generally, beyond the Clayton copula setting, based on a well-established strict generator function $\phi_{⋒}(t)$ and a well-chosen perturbation function $\psi(u, v)$, one can explore the modeling horizons opened by copulas of the following form:

$$C_{⋒}(u, v) = \phi_{⋒}^{-1}\left[\phi_{⋒}(u) + \phi_{⋒}(v) + \psi(u, v)\right], \quad (u, v) \in [0, 1]^2.$$

These ideas need further examination, which we leave for future work.

Competing interests

The author declares that there is no conflict of interest regarding the publication of this paper.

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