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Generalized Arctan-Laplace Distribution: Properties and Applications

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Abstract. A new asymmetric counterpart of the Laplace distribution is introduced, the Arctan-Laplace distribution. Its mathematical properties are studied. Flexibility of the proposed distribution family is demonstrated using a real data set.

1. Introduction

The Laplace distribution dates back to the 18th century, it was introduced by Pierre-Simon de Laplace in [21]. There exist many modifications of the Laplace distribution nowadays. One such well-known generalization of the Laplace distribution is the asymmetric Laplace distribution of Hinkley and Revankar (see [14]), its properties are described in detail in [18]. The pdf of this distribution is

$$p(x;\mu,\sigma,\kappa)=\begin{cases}\dfrac{\sqrt{2}}{\sigma}\dfrac{\kappa}{1+\kappa^2}\exp\left\{-\dfrac{\sqrt{2}}{\sigma\kappa}|x-\mu|\right\}, & \text{if } x<\mu;\\[2mm] \dfrac{\sqrt{2}}{\sigma}\dfrac{\kappa}{1+\kappa^2}\exp\left\{-\dfrac{\sqrt{2}\kappa}{\sigma}|x-\mu|\right\}, & \text{if } x\geq\mu\end{cases} \tag{1}$$

(the pdf is given in a slightly reparametrized form).

Other families based on the Laplace distribution include the alpha-skew-Laplace distribution (see [13]), the Balakrishnan-alpha-beta-skew-Laplace Distribution (see [36]), the beta-Laplace distribution (see [4]), the flexible skew Laplace distribution of Yilmaz (see [42]), the Kumaraswamy Laplace distribution (see [27]), the Marshall-Olkin Esscher transformed Laplace distribution (see [10]), a modification based on taking a difference of exponentiated exponentially distributed random variables (see [38]), the skew-symmetric-Laplace distribution (see [29]), the three-parameter asymmetric Laplace distribution of Yu and Zhang (see [43]), to name but a few. Kozubowski and Nadarajah gave in [19] a good (but rather limited) review of Laplace distribution variations.

The Laplace distribution along with its numerous modifications was applied, in particular, in such areas as finance and economics (see [3], [12], [31], [33], [39], [41]) engineering and technology

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(see [15], [16], [22] [23], [28]), the natural sciences ([11], [17], [32], [34], [37], [40]). A wide range of Laplace distribution applications is also discussed in [18].

We propose a new generalization of the Laplace distribution by transforming its cdf. Our approach is based on a modification of a method of generating new distributions described in [2]. The Arctan-X family is defined in [2] as a distribution family with the cdf

$$F_A(x) = \frac{4}{\pi} \arctan(F(x)),$$

where $F(x)$ is the cdf of the parent distribution.

We derive a new distribution from a parent distribution as follows: the cdf of the generalized Arctan-X distribution is defined as

$$F_{GA}(x; \gamma) = \frac{1}{\arctan(\gamma)} \arctan(\gamma F(x)),$$

where $F(x)$ is the cdf of the original distribution ($\gamma = 1$ yields the Arctan-X family of Alkhairy et al.). The case when X is the Laplace distribution corresponds to the generalized Arctan-Laplace (GATL) distribution family.

The generalized Arctan-Laplace distribution is a skewed one, the skewness is regulated by the parameter γ. Therefore it is suitable for modeling asymmetric data such as returns of a financial time series. The paper is organized as follows. We investigate theoretical properties of the new distribution in Section 2: expressions for the cdf, the pdf and the quantile functions are provided, unimodality is proved and the formula for the mode is obtained; we also derive the moments, investigate behavior of the skewness and the kurtosis and obtain the Rényi entropy. The GATL distribution is applied to a real data set in Section 3 in order to illustrate its usefulness. Lastly, concluding remarks are given in Section 4.

2. Properties

Denote by $\text{Lapl}(\theta, \phi)$ the Laplace distribution with the parameters θ and ϕ, its cdf is

$$F_L(x) = \begin{cases} (1/2)\exp\{(x-\theta)/\phi\}, & x \le \theta; \\ 1 - (1/2)\exp\{-(x-\theta)/\phi\}, & x > \theta. \end{cases}$$

Definition. The generalized Arctan-Laplace distribution with the parameters γ, θ and ϕ (where $\gamma \in (0; +\infty)$, $\theta \in \mathbb{R}$, $\phi \in (0; +\infty)$) is defined as a distribution with the cdf

$$F(x; \gamma, \theta, \phi) = \frac{1}{\arctan(\gamma)} \arctan(\gamma F_L(x)).$$

We will use notation $\text{GATL}(\gamma, \theta, \phi)$ for the generalized Arctan-Laplace distribution with the parameters γ, θ and ϕ.

The generalized Arctan-Laplace distribution cdf can be rewritten as

$$F(x;\gamma,\theta,\phi)=\begin{cases}\dfrac{1}{\arctan(\gamma)}\cdot\arctan\Big((\gamma/2)\exp\{(x-\theta)/\phi\}\Big), & x\le\theta;\\ \dfrac{1}{\arctan(\gamma)}\cdot\arctan\Big(\gamma-(\gamma/2)\exp\{-(x-\theta)/\phi\}\Big), & x>\theta.\end{cases}\tag{2}$$

The pdf of GATL(γ,θ,ϕ) distribution is

$$p(x;\gamma,\theta,\phi)=\begin{cases}\dfrac{\gamma}{2\phi\arctan\gamma}\cdot\dfrac{\exp\{(x-\theta)/\phi\}}{1+(\gamma^2/4)\exp\{2(x-\theta)/\phi\}}, & x\le\theta;\\ \dfrac{\gamma}{2\phi\arctan\gamma}\cdot\dfrac{\exp\{-(x-\theta)/\phi\}}{1+\gamma^2(1-(1/2)\exp\{-(x-\theta)/\phi\})^2}, & x>\theta.\end{cases}$$

Remark. We will use notation $p(x)$ and $F(x)$ instead of $p(x;\gamma,\theta,\phi)$ and $F(x;\gamma,\theta,\phi)$ correspondingly when this is unambiguous.

Plots of the pdfs for GATL$(\gamma,0,1)$ distributions for $\gamma=1.6$, $\gamma=2.5$ and $\gamma=5$ are shown in Fig. 1, Fig. 2 and Fig. 3 correspondingly.

Let us mention some special and limiting cases of the GATL distribution.

- The GATL$(1,\theta,\phi)$ distribution is the Arctan-Laplace distribution in the sense of Alkhairy et al.
- It is easy to see that GATL(γ,θ,ϕ) distribution converges to the Laplace distribution with the parameters θ and ϕ as $\gamma\to0+$. It can also be proved that

$$\lim_{\gamma\to0+}p(x;\gamma,\theta,\phi)=p_L(x;\theta,\phi),$$

  where $p_L(x;\theta,\phi)$ is the pdf of Lapl(θ,ϕ).

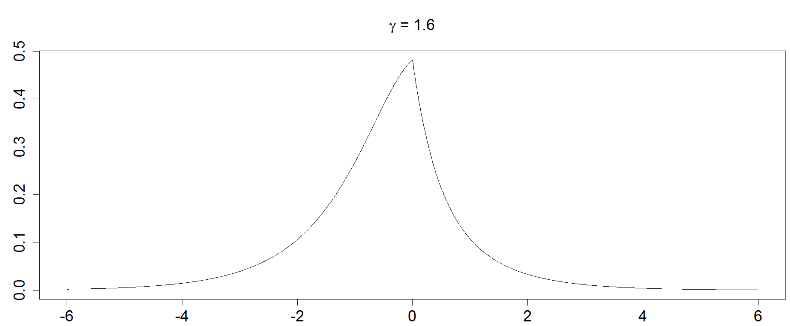

FIGURE 1. The pdf of GATL for $\gamma=1.6$, $\theta=0$, $\phi=1$

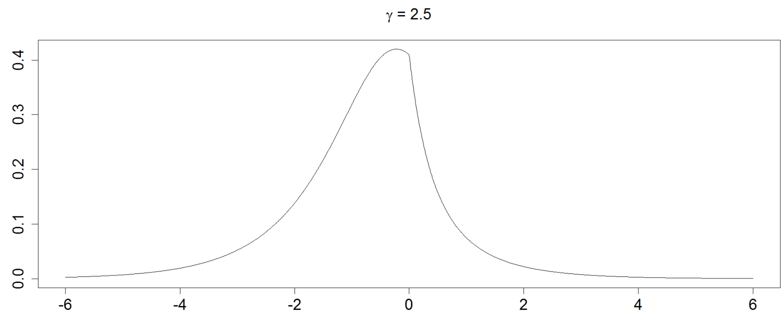

FIGURE 2. The pdf of GATL for $\gamma = 2.5$, $\theta = 0$, $\phi = 1$

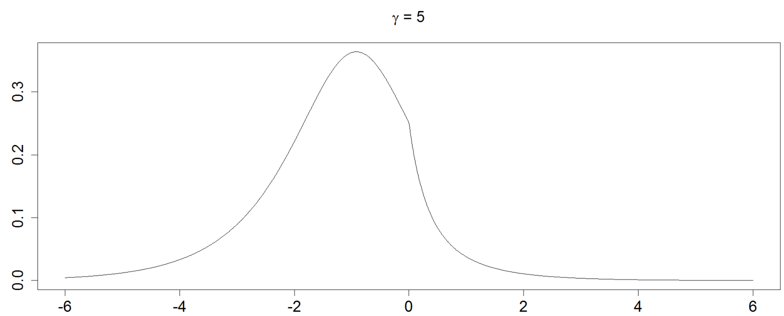

FIGURE 3. The pdf of GATL for $\gamma = 5$, $\theta = 0$, $\phi = 1$

2.1. **Quantiles and Modes.** It is easy to verify that the quantile function of GATL(γ, θ, ϕ) distribution is

$$Q(u) = \begin{cases} \theta + \phi \ln\Big((2/\gamma) \tan(u \arctan \gamma) \Big), & u \in (0; u_0]; \\ \theta - \phi \ln\Big(2 - (2/\gamma) \tan(u \arctan \gamma) \Big), & u \in (u_0; 1), \end{cases} \tag{3}$$

where

$$u_0 = \frac{\arctan(\gamma/2)}{\arctan \gamma}.$$

Theorem 2.1.1. GATL(γ, θ, ϕ) distribution is unimodal. Its mode is

1) $x_0 = \theta + (1/2) \ln(4/\gamma^2)$, if $\gamma \in (2; +\infty)$;

2) $x_0 = \theta$, if $\gamma \in (0; 2]$.

Proof. It is enough to prove this fact for $\theta = 0$ and $\phi = 1$.

Obviously

$$p'_x(x; \gamma, 0, 1) = c'_\gamma \frac{p'_L(x) \Big(1 + \gamma^2 (F_L(x))^2 \Big) - 2\gamma^2 (p_L(x))^2 F_L(x)}{\Big(1 + \gamma^2 (F_L(x))^2 \Big)^2} \tag{4}$$

for any $x \neq 0$, where $c'_\gamma = \gamma / \arctan \gamma$, $p_L(x)$ and $F_L(x)$ are correspondingly the pdf and the cdf of the Laplace distribution with the parameters $\theta = 0$ and $\phi = 1$. It follows from (4) that

$$\operatorname{sgn}\left(p'_x(x; \gamma, 0, 1) \right) = \operatorname{sgn}\left(1 - (\gamma^2/4) e^{2x} \right), \ x \leq 0,$$

and

$$\operatorname{sgn}\left(p'_x(x;\gamma,0,1)\right) = -\operatorname{sgn}\left(1+\gamma^2-(\gamma^2/4)e^{-2x}\right), \ x>0.$$

But

$$1+\gamma^2-(\gamma^2/4)e^{-2x} \geq 1+\frac{3}{4}\gamma^2>0, \ x>0.$$

Therefore $p(x)$ decreases on $(0;\infty)$.

It is evident now that the point of the global maximum for $p(x)$ is $x_0=0$ when $\gamma\in(0;2]$ and $x_0=(1/2)\log(4/\gamma^2)$ when $\gamma\in(2;+\infty)$. □

2.2. **Moments.** Denote by $\mu'_{n;\gamma}$ and $\mu_{n;\gamma}$ correspondingly the n-th moment and the n-th central moment of $\mathrm{GATL}(\gamma,0,1)$.

Let us find the n-th moment $\mu'_{n;\gamma}$ of $\mathrm{GATL}(\gamma,0,1)$.

Theorem 2.2.1. The n-th moment of $\mathrm{GATL}(\gamma,0,1)$ is

$$\mu'_{n;\gamma}=\frac{n!}{2i\arctan\gamma}\left((-1)^n\left(\mathrm{Li}_{n+1}(i\gamma/2)-\mathrm{Li}_{n+1}(-i\gamma/2)\right)+\mathrm{Li}_{n+1}\left(\frac{\gamma}{2(-i+\gamma)}\right)-\mathrm{Li}_{n+1}\left(\frac{\gamma}{2(i+\gamma)}\right)\right), \tag{5}$$

where $\mathrm{Li}_n(z)$ is the polylogarithm.

Proof. Let us represent $\mu'_{n;\gamma}$ as

$$\mu'_{n;\gamma}=\int_{-\infty}^{0}x^np(x;\gamma,0,1)dx+\int_0^\infty x^np(x;\gamma,0,1)dx=\frac{\gamma}{\arctan\gamma}I_L^{(n)}+\frac{\gamma}{\arctan\gamma}I_R^{(n)},$$

where

$$I_L^{(n)}=\frac{\arctan\gamma}{\gamma}\int_{-\infty}^{0}x^np(x;\gamma,0,1)dx=\int_{-\infty}^{0}\frac{(1/2)x^ne^x}{1+(\gamma^2/4)e^{2x}}dx, \tag{6}$$

$$I_R^{(n)}=\frac{\arctan\gamma}{\gamma}\int_0^\infty x^np(x;\gamma,0,1)dx=\int_0^\infty\frac{(1/2)x^ne^{-x}}{1+\gamma^2\left(1-(1/2)e^{-x}\right)^2}dx. \tag{7}$$

Let us transform the integral $I_L^{(n)}$. We have:

$$I_L^{(n)}=\frac{1}{2}(-1)^n\int_0^\infty\frac{y^ne^y}{(\gamma^2/4)+e^{2y}}dy=\frac{1}{2}(-1)^nI'_L, \tag{8}$$

where

$$\begin{aligned}I'_L=\int_0^\infty\frac{y^ne^y}{(\gamma^2/4)+e^{2y}}dy&=\frac{1}{2}\int_0^\infty\frac{y^n}{e^y-i\gamma/2}dy+\frac{1}{2}\int_0^\infty\frac{y^n}{e^y+i\gamma/2}dy\\&=\frac{n!}{i\gamma}\left(\mathrm{Li}_{n+1}(i\gamma/2)-\mathrm{Li}_{n+1}(-i\gamma/2)\right).\end{aligned} \tag{9}$$

The formula

$$\mathrm{Li}_s(z)=\frac{z}{\Gamma(s)}\int_0^\infty\frac{y^{s-1}}{e^y-z}dy$$

was used (see [30], p. 611).

Let us rewrite the integral $I_R^{(n)}$. Since

$$\frac{e^{-x}}{1+\gamma^2\left(1-(1/2)e^{-x}\right)^2}=-\frac{1}{i\gamma}\left(\frac{z_2}{e^x-z_2}-\frac{z_1}{e^x-z_1}\right),$$

where

$$z_1 = \frac{\gamma}{2(-i+\gamma)}, \quad z_2 = \frac{\gamma}{2(i+\gamma)},$$

we have

$$I_R^{(n)} = -\frac{1}{2i\gamma}\left(z_2\int_0^\infty \frac{x^n dx}{e^x - z_2} - z_1\int_0^\infty \frac{x^n dx}{e^x - z_1}\right) = -\frac{n!}{2i\gamma}\left(\mathrm{Li}_{n+1}(z_2) - \mathrm{Li}_{n+1}(z_1)\right). \tag{10}$$

Now (5) follows from (8), (9) and (10). □

Remark. The n-th moment of $\mathrm{GATL}(\gamma,\theta,\phi)$ (denote it by $\mu'_{n;(\gamma,\theta,\phi)}$) can be expressed as

$$\mu'_{n;(\gamma,\theta,\phi)} = \sum_{k=0}^{n}\binom{n}{k}\phi^k\theta^{n-k}\mu'_{k;\gamma},$$

where $\mu'_{0;\gamma} = 1$.

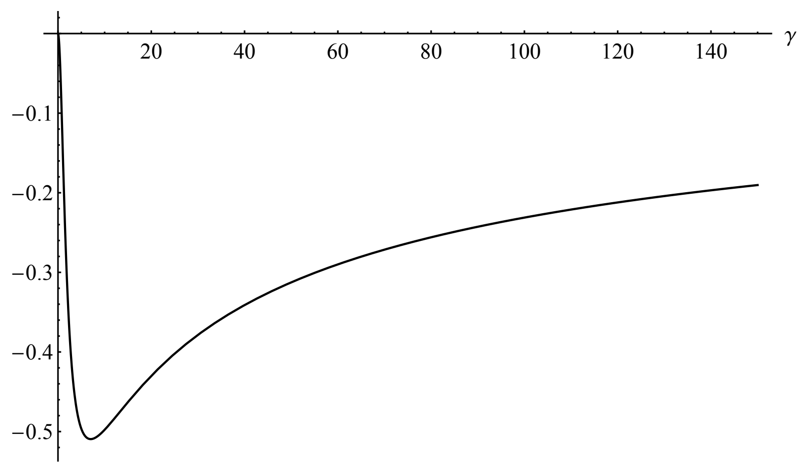

FIGURE 4. The skewness of $\mathrm{GATL}(\gamma, 0, 1)$ as a function of γ

Skewness

Figure 4 shows the skewness of $\mathrm{GATL}(\gamma, 0, 1)$,

$$\eta_{3;\gamma} = \frac{\mu_{3;\gamma}}{\mu_{2;\gamma}^{3/2}}.$$

Kurtosis

Figure 5 displays the excess kurtosis of $\mathrm{GATL}(\gamma, 0, 1)$ distribution,

$$\eta_{4;\gamma} = \frac{\mu_{4;\gamma}}{\mu_{2;\gamma}^{2}} - 3.$$

Theorem 2.2.2.

$$\lim_{\gamma\to 0+}\eta_{3;\gamma} = 0, \quad \lim_{\gamma\to 0+}\eta_{4;\gamma} = 3.$$

Proof. Suppose that $\{\gamma_n\}$ is such a sequence that $\lim_n \gamma_n = 0$ (we can assume that $\sup_n \gamma_n < 2$) and $\xi_n \sim \mathrm{GATL}(\gamma_n, 0, 1)$. And let $\zeta \sim \mathrm{Lapl}(0, 1)$.

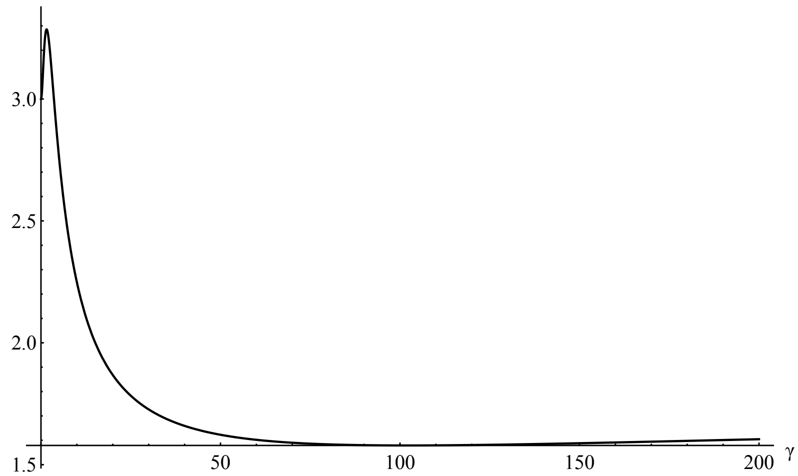

FIGURE 5. The excess kurtosis of GATL$(\gamma, 0, 1)$ as a function of γ

Let us prove that the sequence $\{|\xi_n|^q\}$ is uniformly integrable for $q = 1, 2, 3, 4$. In order to do this it is enough to show that

$$\sup_n \mathrm{E}\xi_n^{2q} < \infty, \quad q = 1, 2, 3, 4. \tag{11}$$

First of all, let us verify that

$$\mathrm{Li}_s(z) = z + o(z), \quad z \to 0, \tag{12}$$

for $s > 1$. Indeed (let $|z| < 1$),

$$\mathrm{Li}_s(z) = z + z^2 \sum_{n=2}^{\infty} \frac{z^{n-2}}{n^s},$$

where

$$\left| \sum_{n=2}^{\infty} \frac{z^{n-2}}{n^s} \right| \le \frac{1}{2^s} + \sum_{k=1}^{\infty} \frac{|z|^k}{k^s} = \frac{1}{2^s} + \mathrm{Li}_s(|z|)$$

is bounded in $\{z : |z| < 1\}$ since $\mathrm{Li}_s(z)$ is an analytic function in this ball.

Let us continue the proof of (11). We obtain using Theorem 2.2.1 and (12):

$$\mathrm{E}\xi_n^{2q} = \mu'_{\gamma_n;2q} \le \frac{q!}{2|\arctan\gamma_n|} \Bigg(|\mathrm{Li}_{q+1}(i\gamma_n/2) - \mathrm{Li}_{q+1}(-i\gamma_n/2)|$$

$$+ \left| \mathrm{Li}_{q+1}\left(\frac{\gamma_n}{2(-i+\gamma_n)}\right) - \mathrm{Li}_{q+1}\left(\frac{\gamma_n}{2(i+\gamma_n)}\right) \right| \Bigg) = \frac{q!}{2|\arctan\gamma_n|}(A_n|\gamma_n| + o(\gamma_n)), \tag{13}$$

where $\sup_n |A_n| < \infty$, and now it is evident that $\{\mathrm{E}\xi_n^{2q}\}$ is bounded.

Now $\xi_n \xrightarrow{d} \zeta$ and uniform integrability of $\{|\xi_n|^q\}$ imply that

$$\mathrm{E}\xi_n^q \to \mathrm{E}\zeta^q, \quad q = 1, 2, 3, 4 \tag{14}$$

(see [35], p. 14, Theorem A). It follows from (14) that

$$\eta_{3;\gamma} \to \eta_{3;0} = 0, \quad \gamma \to 0+,$$

and

$$\eta_{4;\gamma} \to \eta_{4;0} = 3, \quad \gamma \to 0+,$$

where $\eta_{3;0}$ and $\eta_{4;0}$ are the skewness and the excess kurtosis of $\mathrm{Lapl}(0;1)$ distribution correspondingly.

2.3. **Entropy.** Let us obtain the Rényi entropy of the GATL distribution.

Theorem 2.3.1. The Rényi entropy H_λ of GATL(γ, θ, ϕ) distribution is

$$H_\lambda = \ln\phi + \frac{1}{1-\lambda}\left(-\ln\lambda + \lambda \ln \frac{\gamma}{2\arctan\gamma} + \ln\left({}_3F_2\left(\lambda, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+1}{2}, \frac{\lambda+2}{2}; -\frac{\gamma^2}{4}\right) \right.\right.$$
$$\left.\left. + \frac{1}{(\gamma^2+1)^\lambda} F_1\left(\lambda; \lambda, \lambda; \lambda+1; \frac{\gamma}{2\gamma+2i}, \frac{\gamma}{2\gamma-2i}\right)\right)\right), \tag{15}$$

where ${}_3F_2$ is the generalized hypergeometric function and F_1 is the Appell function.

Proof. It suffices to prove (15) for $\theta = 0$ and $\phi = 1$. We have:

$$H_\lambda = \frac{1}{1-\lambda}\ln\left(\int_{\mathbb{R}^1} p^\lambda(x;\gamma,0,1)dx\right) = \frac{1}{1-\lambda}\ln I_\lambda,$$

where

$$I_\lambda = \int_{\mathbb{R}^1} p^\lambda(x;\gamma,0,1)dx = I_\lambda^{(1)} + I_\lambda^{(2)},$$

$$I_\lambda^{(1)} = \int_{-\infty}^0 p^\lambda(x;\gamma,0,1)dx, \tag{16}$$

$$I_\lambda^{(2)} = \int_0^\infty p^\lambda(x;\gamma,0,1)dx. \tag{17}$$

The integral in (16) can be rewritten as follows:

$$\int_{-\infty}^0 p^\lambda(x;\gamma,0,1)dx = 2^\lambda \left(\frac{1}{\gamma\arctan\gamma}\right)^\lambda \tilde{I}_\lambda^{(1)}, \tag{18}$$

where

$$\tilde{I}_\lambda^{(1)} = \int_{-\infty}^0 \frac{\exp\{\lambda x\}dx}{(4/\gamma^2 + \exp\{2x\})^\lambda} = \int_0^1 \frac{y^{\lambda-1}dy}{(y^2+(2/\gamma)^2)^\lambda}$$
$$= (\gamma^2/4)^\lambda \cdot \frac{1}{\lambda} \cdot {}_3F_2\left(\lambda, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+1}{2}, \frac{\lambda+2}{2}; -\frac{\gamma^2}{4}\right) \tag{19}$$

(formula 3.254 from [44] was used).

The integral in (17) can be transformed this way:

$$\int_0^\infty p^\lambda(x;\gamma,0,1)dx = \left(\frac{\gamma}{2\arctan\gamma}\right)^\lambda \tilde{I}_\lambda^{(2)}, \tag{20}$$

where

$$\tilde{I}^{(2)}_\lambda = \int_0^\infty \left(\frac{\exp\{-x\}}{1+\gamma^2(1-(1/2)\exp\{-x\})^2} \right)^\lambda dx$$

$$= \int_0^1 \frac{y^{\lambda-1}dy}{(1+\gamma^2(1-y/2)^2)^\lambda} = \frac{1}{(\gamma^2+1)^\lambda} \int_0^1 \frac{y^{\lambda-1}dy}{\left(1-\frac{\gamma y}{2\gamma+2i}\right)^\lambda \left(1-\frac{\gamma y}{2\gamma-2i}\right)^\lambda}$$

$$= \frac{1}{\lambda(\gamma^2+1)^\lambda} F_1 \left(\lambda; \lambda, \lambda; \lambda+1; \frac{\gamma}{2\gamma+2i}, \frac{\gamma}{2\gamma-2i} \right) \tag{21}$$

(formula 16.15.1 from [30] was applied).

Now (15) follows from (18)–(21). □

3. Applications

We will demonstrate now the usefulness of the GATL distribution family. We will fit the GATL distribution to a financial dataset and compare the quality of fit with that of competing models.

Our dataset consists of the daily returns $\xi_k = \eta_{k+1} - \eta_k$ (where η_k is the stock price on day k) for FCX stock from July 26, 2016 to November 2, 2016 (see [9]). The alternative distribution families are the following modifications of the Laplace distribution: the asymmetric Laplace distribution (ASL) (see [18], this parametrization is denoted by $\mathcal{AL}$ in the book); the truncated-exponential skew-symmetric Laplace distribution (TESSL), see [25]; the exponentiated generalized Laplace distribution (EGL), see [5], and the geometric exponential Poisson Laplace distribution (GEPL), see [24], [26].

The pdfs of these distributions are as follows:

$$p_{\text{ASL}}(x;\theta,\phi,\kappa) = \begin{cases} \frac{\sqrt{2}}{\phi}\frac{\kappa}{1+\kappa^2}\exp\left\{-\frac{\sqrt{2}}{\phi\kappa}|x-\theta|\right\}, & \text{if } x<\theta; \\ \frac{\sqrt{2}}{\phi}\frac{\kappa}{1+\kappa^2}\exp\left\{-\frac{\sqrt{2}\kappa}{\phi}|x-\theta|\right\}, & \text{if } x\geq\theta. \end{cases}$$

$\kappa > 0$;

$$p_{\text{TESSL}}(x;\theta,\phi,\lambda) = \frac{\lambda}{1-\exp\{-\lambda\}} p_L(x)\exp\{-\lambda F_L(x)\},$$

$\lambda \in \mathbb{R}$;

$$p_{\text{EGL}}(x;\theta,\phi,a,b) = ab p_L(x)(1-F_L(x))^{a-1}\left(1-(1-F_L(x))^a\right)^{b-1},$$

$a>0, b>0$;

$$p_{\text{GEPL}}(x;\theta,\phi,\nu,\rho) = \nu(1-\rho)(1-\exp\{-\nu\})\frac{p_L(x)\exp\{-\nu+\nu F_L(x)\}}{(1-\exp\{-\nu\}-\rho+\rho\exp\{-\nu+\nu F_L(x)\})^2},$$

$\nu>0, \rho\in(0;1)$, for the ASL, the TESSL, the EGL and the GEPL distributions correspondingly, where $p_L(x)$ and $F_L(x)$ are respectively the pdf and the cdf of the $\text{Lapl}(\theta,\phi)$ distribution.

All calculations were performed using the R software (including packages `AdequacyModel` [1], `dfoptim` [7] and `fitdistrplus` [8]). The values of the maximum likelihood estimates for these distributions are given in Table 1.

The following goodness-of-fit criteria were used for comparison of the models: the log-likelihood l, the AIC, the BIC and the HQIC. Table 2 contains the values of these criteria. The fitted GATL distribution corresponded to the best results according to all goodness-of-fit statistics.

The histogram and the density of the fitted GATL distribution for FCX dataset are shown in Figure 6.

TABLE 1. MLEs

Model	Estimates
GATL	$\hat{\theta} = 0.0200$, $\hat{\phi} = 0.2402$, $\hat{\gamma} = 0.8559$
ASL	$\hat{\theta} = 0.0200$, $\hat{\phi} = 0.3265$, $\hat{\kappa} = 1.1128$
TESSL	$\hat{\theta} = 0.1818$, $\hat{\phi} = 0.2454$, $\hat{\lambda} = 2.0939$
EGL	$\hat{\theta} = 0.0961$, $\hat{\phi} = 1.4215$, $\hat{a} = 6.7056$, $\hat{b} = 43.2157$
GEPL	$\hat{\theta} = -0.0368$, $\hat{\phi} = 0.2330$, $\hat{\nu} = 1.4324$, $\hat{\rho} = 0.3857$

TABLE 2. Model comparison criteria

Model	l	AIC	BIC	HQIC
GATL	-16.220	38.439	45.185	41.118
ASL	-16.318	38.635	45.380	41.314
TESSL	-16.876	39.752	46.498	42.432
EGL	-16.704	41.408	50.402	44.980
GEPL	-16.932	41.864	50.858	45.437

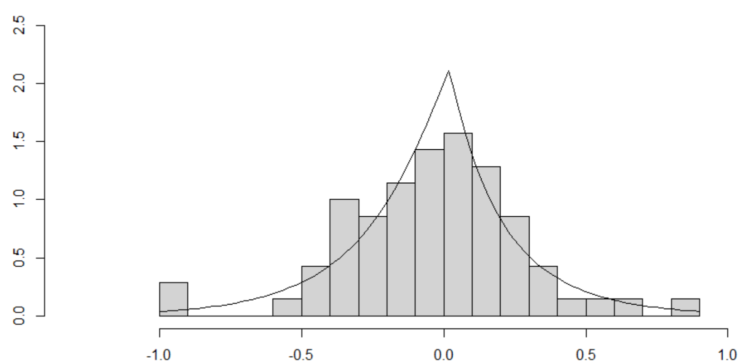

FIGURE 6. The histogram and the fitted GATL pdf for FCX dataset

4. Conclusions

A new generalization of the Laplace distribution is proposed, the Arctan-Laplace distribution. Various properties of the new distribution are studied including modes, the moments, the behavior of the skewness and the kurtosis, the Rényi entropy. A real data set is fitted to the Arctan-Laplace model. The fit results show that the Arctan-Laplace distribution is superior to competing families.

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