Eur. J. Stat. 5 (2025) 4
doi: 10.28924/ada/stat.5.4

On Poisson Sampling for Estimation in Sub-Fractional Levy Stochastic Volatility Models

Jaya P.N. Bishwal

Department of Mathematics and Statistics, University of North Carolina at Charlotte, 376 Fretwell Building, 9201 University City Blvd. Charlotte, NC 28223, USA

J.Bishwal@charlotte.edu

Abstract. The paper studies quasi-maximum likelihood and generalized method of moments estimators of the parameter in the sub-fractional Levy inverse-Gaussian Ornstein-Uhlenbeck stochastic volatility model based on Poissonly sampled data.

1. Introduction and Preliminaries

Non-Gaussian Ornstein-Uhlenbeck stochastic volatility model has been paid recent attention in finance, see Barndorff-Neilsen and Shephard [1]. On the other hand, processes with long-memory has also been recent attention due to volatility clustering. The parameters of the volatility processes are unknown. In view of this, it becomes necessary to estimate the parameters in the unobserved volatility process in the model from irregular discretely sampled data. We consider data from random sampling intervals, specifically the when inter arrival times are exponentially distributed. Random sampling also removes the alishing problem. We consider quasi-likelihood method and generalized method of moments (GMM) for estimating the unknown parameters.

Levy driven processes of Ornstein-Uhlenbeck type have been extensively studied over the last few years and widely used in finance, see Barndorff-Neilsen and Shephard [1]. Parameter estimation in Itô diffusions models with constant volatility from both continuous and discrete observations was studied in Bishwal [7]. Fukasawa [22, 23] and Fukasawa and Rosenbaum [24] studied random sampling for volatility estimation in continuous-time models. Bishwal [11] studied sufficiency and optimal discretization problem in Vasicek model. FLOU process generalizes FOU process to include jumps. Maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [12]. Berry-Esseen inequalities for the discretely observed Ornstein-Uhlenbeck-Gamma process was studied in Bishwal [13]. Minimum contrast estimation in fractional Ornstein-Uhlenbeck process based on both continuous and discrete observations was studied in Bishwal [14]. Berry-Esseen inequalities for the fractional Black-Karasinski model of term structure of interest rates was studied in Bishwal [16]. Parameter estimation in stochastic volatility models from both

Received: 29 Nov 2024.

Key words and phrases. Itô stochastic differential equation; sub-fractional Levy process; Ornstein-Uhlenbeck process; jumps; long memory; weak correlation; discrete observations; random sampling; quasi likelihood estimator; method of moments estimator.

continuous and discrete observations was studied in Bishwal [17]. Quasi-likelihood Estimation in fractional Levy SPDEs from Poisson sampling was studied in Bishwal [18]. Paramater estimation for SPDEs driven by cylindrical stable processes was studied in Bishwal [19].

Recently, long memory processes have received attention in finance. A normalized fractional Brownian motion $\{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s, t \geq 0. \tag{1.1}$$

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a Dirichlet process. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for $H > \frac{1}{2}$ and in this case they display long-range dependence. The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence. The ARIMA(p, d, q) process with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0,\ 0.5)$ converge in Donsker sense to fBm, see Mishura [27].

As a generalization of fBm we have the weighted fBm. A weighted fBm (wfBm) ξ_t has the covariance function

$$q(s,t) = \int_0^{s \wedge t} u^a[(t-u)^b + (s-u)^b]du, \quad s, t \geq 0 \tag{1.2}$$

where $a > -1,\ -1 < b \leq 1,\ |b| \leq 1 + a$. When $a = 0$, it is the usual fBm with Hurst parameter $(b+1)/2$ up to a multiplicative constant. For $b = 0$ it is a time-inhomogeneous Bm.

The function u^a is called the weight function of wfBm. For $a = 0$, this process is usual fBm with Hurst parameter $(b+1)/2$. For the case $b = 1$, this process has the covariance of the process $\int_0^t W_{r^a} dr$ where W is standard Brownian motion. For $b = 0$, this process is time-inhomogeneous Brownian motion. The finite dimensional distributions of the process $(T^{-a/2}(\xi_{t+T} - \xi_T)), t \geq 0$ converge as $T \to \infty$ to those of fBm with Hurst parameter $(1+b)/2$ multiplied by $(2/(1+b)))^{1/2}$. The process has *asymptotically stationary increments* for long time intervals, but not for short time intervals. For $b \neq 0$, the process is neither a semimartingale nor a Markov process.

This process occurs as the limit of occupation time fluctuations of a particle system of independent particles moving in $\mathbb{R}^d$ with symmetric α-stable Levy process, $0 < \alpha \leq 2$, started from an inhomogeneous Poisson configuration with intensity measure $dx/(1+|x|^\gamma), 0 < \gamma \leq d = 1 < \alpha, a = -\gamma/\alpha,\ b = 1 - 1/\alpha, -1 < a < 0, 0 < b \leq 1 + a$. The homogeneous case $\gamma = 0$ gives fBm.

As another generalization, we have the bi-fractional Brownian motion. A bi-fractional Brownian motion (bfBm) has covariance

$$\frac{1}{2}(s^{2H} + t^{2H})^k - |t-s|^{2Hk}), \quad s, t \geq 0, \quad 0 < k \leq 1. \tag{1.3}$$

For $k = 1$, it reduces to fBm. For $H = 1/2$, bfBm can be extended for $1 < k < 2$.

As a further generalization of fractional Brownian motion, we get the Hermite process of order k with Hurst parameter $H \in (\frac{1}{2}, 1)$ which is defined as a multiple Wiener-Itô integral of order k with respect to standard Brownian motion $(B(t))_{t\in\mathbb{R}}$

$$Z_t^{H,k} := c(H,k) \int_{\mathbb{R}} \int_0^t \Pi_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{H-1}{2})} ds\; dB(y_1)dB(y_2)\cdots dB(y_k) \tag{1.4}$$

where $x_+ = \max(x,0)$ and the constant $c(H,k)$ is a normalizing constant that ensures $E(Z_t^{H,k})^2 = 1$.

For $k = 1$ the process is fractional Brownian motion W_t^H with Hurst parameter $H \in (0,1)$. For $k = 2$ the process is Rosenblatt process which is non-Gaussian. For $k \geq 2$, the process is non-Gaussian.

The Rosenblatt process is not a semimartingale and for $H > 1/2$, the quadratic variation is 0. The distribution of the process is infinitely divisible. It is unknown yet whether the process is Markov or not.

The covariance kernel $R(t,s)$ is given by

$$R(t,s) := E[Z_t^{H,k} Z_s^{H,k}] = c(H,k)^2 \int_0^t \int_0^s \left[(u-s)_+^{-(\frac{1}{2}+\frac{H-1}{2})} ds (v-y)_+^{-(\frac{1}{2}+\frac{H-1}{2})} dy\right]^k dudv. \tag{1.5}$$

Let

$$\beta(p,q) := \int_0^1 z^{p-1}(1-z)^{q-1} dz,\;\; p,q > 0 \tag{1.6}$$

be the beta function. Using the identity

$$\int_0^1 \int_{\mathbb{R}} (u-s)_+^{a-1} ds (v-y)_+^{a-1} dy = \beta(a, 2a-1)|u-v|^{2a-1}, \tag{1.7}$$

we have

$$\begin{aligned} R(t,s) &= c(H,k)^2 \beta\left(\frac{1}{2} - \frac{1-H}{k}, \frac{2H-2}{k}\right)^k \int_0^t \int_0^s \left(|u-v|^{\frac{2H-2}{k}}\right)^k dvdu \\ &= c(H,k)^2 \frac{\beta(\frac{1}{2}-\frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \end{aligned} \tag{1.8}$$

In order to obtain $E(Z_t^{(H,k)})^2 = 1$, choose

$$c(H,k)^2 = \left(\frac{\beta(\frac{1}{2}-\frac{1-H}{k}, \frac{2H-2}{k})^k}{H(2H-1)}\right)^{-1} \tag{1.9}$$

and we have

$$R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{1.10}$$

Thus the covariance structure of the Hermite process and the fractional Brownian motion are the same. The process $Z_t^{(H,k)}$ is H-self similar with stationary increments and all moments are finite.

For any $p \geq 1$,

$$E|Z_t^{(H,k)} - Z_s^{(H,k)}|^p \leq c(p,H,k)|t-s|^{pH}. \tag{1.11}$$

Thus the Hermite process has Hölder continuous paths of order $\delta < H$.

Consider the Gaussian process with the covariance function

$$K_H(s,t) = (2-2H)\left(s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right]\right), \quad s,t > 0 \tag{1.12}$$

for $1 < 2H \leq 2$. The case $H = 1/2$ corresponds to standard Brownian motion.

This process occurs as the limit of occupation time fluctuations of a particle system undergoing a critical branching, i.e., each particle independently, at an exponentially distributed lifetime, disappears with probability $1/2$ or is replaced with two particles at the same site with probability $1/2$. More generally, it is a branching particle system with Poisson initial condition, where the particle motion is symmetric α stable Levy process, $\alpha \in (0,2]$. For $\alpha = 2$, which corresponds to Brownian motion, one reaches super-processes.

Recently, sub-fractional Brownian (sub-FBM) motion ζ_t which is a centered Gaussian process with covariance function

$$\text{cov}(\zeta_t, \zeta_s) = C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0 \tag{1.13}$$

for $0 < H < 1$ introduced by Bojdecki *et al.* [20] has received some attention recently in finite dimensional models. For $s \leq t$,

$$E(\zeta_t - \zeta_s)^2 = -2^{2H-1}(t^{2H} + s^{2H}) + (t+s)^{2H} + (t-s)^{2H}.$$

For $H = 1/2$, sfBm is standard Brownian motion. For $H > 1/2$, this covariance is less than that of fBm and for $H < 1/2$, this covariance is more than that of fBm. The interesting feature of this process is that this process has some of the main properties of FBM, but the increments of the process are nonstationary, more weakly correlated on non-overlapping time intervals than that of FBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. We generalize sub-fBM to sub-fractional Levy process (SFLP) as the driving terms in our model.

The nonstationarity of increments of SFLP distinguishes this process from fractional Brownian motion. The sub-fractional Levy Ornstein-Uhlenbeck (sfOU) process, is an extension of sub-fractional Ornstein-Uhlenbeck process with sub-fractional Levy motion (SFLM) driving term. In finance, it could be useful as a generalization of fractional Vasicek model, as an one-factor short-term interest rate model or a stochastic volatility model or a stochastic intensity based credit-risk model which could take into account the long memory effect and jump of the interest rate or the stochastic volatility. The model parameters are usually unknown and must be estimated from data.

2. Quasi Likelihood Method

Maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [15] where the driving term was fBm. Recall that the increments of the fBM are stationary. On the other hand, the increments of sfBM are nonstationary, more weakly correlated on non-overlapping time intervals than that of fBM, and its covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. Weaker correlation seems to fit the financial data well.

The sub-fractional Levy Process (SFLP) is defined as

$$M_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[s^{2H} + t^{2H} + (t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dM_s, \quad t \in \mathbb{R} \tag{2.1}$$

where $\{M_s, s \in \mathbb{R}\}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0$, $E(M_1^2) < \infty$ and without Brownian component.

Here are some properties of the sub-fractional Levy process: 1) the covariance of the process is given by

$$\mathrm{cov}(M_{H,t}, M_{H,s}) = s^{2H} + t^{2H} + \frac{E(M_1^2)}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}]. \tag{2.2}$$

2) M_H is not a martingale. For a large class of Levy processes, M_H is neither a semimartingale. 3)M_H is Hölder continuous of any order β less than $H - \frac{1}{2}$. 4) M_H has nonstationary increments. 5) M_H is symmetric. 6) M is self-similar, but M_H is not self-similar. 7) M_H has infinite total variation on compacts.

Thus SFLP is a generalization and a natural counterpart of SFBM. Sub-fractional stable motion is a special case of SFLP. First we discuss estimation in partially observed models and then we discuss estimation in directly observed model in finite dimensional set up. In finance, the log-volatility process can be modeled as a sub-fractionally integrated moving average (SFIMA) process which is defined as

$$Y_H(t) = \int_{-\infty}^{t} g_H(t-u)dM_u, \ t \in \mathbb{R} \ \text{ where } \ g_H(t) = \frac{1}{\Gamma(H-\frac{1}{2})}\int_0^t g(t-s)s^{H-\frac{3}{2}}ds, \ t \in \mathbb{R} \tag{2.3}$$

which is the Riemann-Liouville sub-fractional integral of order H and the kernel g is the kernel of a short memory moving average process. The log-volatility process will have slow (hyperbolic rate) decay of the auto-correlation function (acf).

The process $Y_H(t)$ can be written as

$$Y_H(t) = \int_{-\infty}^{t} g(t-u)dM_{H,u}, \ t \in \mathbb{R}. \tag{2.4}$$

We assume the following conditions on the kernel $g : \mathbb{R} \to \mathbb{R}$, namely: $g(t) = 0$ for all $t < 0$ (causality) and $|g(t)| \leq Ce^{-ct}$ for some constants $C > 0$ and $c > 0$ (short memory).

The SFIMA process is stationary and is infinite divisible. It has long memory and jumps which agree empirically with stochastic volatility models. The asset return can be modeled as a SCOG-ARCH process

$$dX(t) = \sqrt{e^{Y_H(t)}}dL_t \tag{2.5}$$

where $(L_t, t \in \mathbb{R})$ is another Levy process and the initial value $Y_H(0)$ is independent of the process L.

Consider the kernel

$$g(t-s) = \sigma e^{-\theta(t-s)}I_{(0,\infty)}(t-s), \ \theta > 0 \tag{2.6}$$

then

$$g_H(t) = \frac{\sigma}{\Gamma(H-\frac{1}{2})}\int_0^\infty e^{\theta(t-s)} I_{(0,\infty)}(t-s) s^{H-\frac{3}{2}} ds, \ \ t \in \mathbb{R}. \tag{2.7}$$

Note that

$$U_t^{H,\theta,\sigma} = \int_{\mathbb{R}} g_H(t-u)dM_u, \ \ t \in \mathbb{R} \tag{2.8}$$

is the sub-fractional Levy Ornstein-Uhlenbeck (SFLOU) process satisfying the sub-fractional Langevin equation

$$dU_t = -\theta U_t dt + \sigma dM_{H,t}, \ \ t \in \mathbb{R}. \tag{2.9}$$

The process has long memory. Consider the asset return driven by sub-fractional Levy process

$$dS_{H,t} = \sigma_{t-} dL_{H,t}, \ \ t > 0, \ \ S_0 = 0, \tag{2.10}$$

with log-volatility

$$\log \sigma_t^2 = \mu + X_t, \ \ t \geq 0 \tag{2.11}$$

where the Levy driven OU process X satisfies

$$dX_t = -\theta X_t dt + dM_t, \ \ t > 0 \tag{2.12}$$

with $\theta \in \mathbb{R}^+$ and the driving compound Poisson process M is a Levy process with Levy symbol

$$\psi_M(u) = -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1)\lambda \Phi_{0,1/\lambda}(dx), \tag{2.13}$$

where $\Phi_{0,1/\lambda}$ being a normal distribution with mean 0 and variance $1/\lambda$. This means that M is the sum of a standard Brownian motion W and a compound Poisson process $J_t = \sum_{k=1}^{N_t} Z_k, \ \ J_{-t} = \sum_{k=1}^{-N_{-t}} Z_{-k}, \ \ t \geq 0$ where $(N_t, t \in \mathbb{R})$ is an independent Poisson process with intensity $\lambda > 0$ and jump times $(t_k)_{k\in\mathbb{Z}}$, i.e., $M_t = W_t + J_t$. The Poisson process N is also independent from the i.i.d. sequence of jump sizes $(Z_k)_{k\in\mathbb{Z}}$ with $Z_1 \sim \mathcal{N}(0, 1/\lambda)$. The Levy process M in this case is given by

$$M_t = \sum_{k=1}^{N_t} (\alpha Z_k + \gamma |Z_k|) - \varpi t, \ \ t > 0 \ \text{ and } \ \ \varpi := \gamma \int_{\mathbb{R}} |x| \lambda \Phi_{0,1/\lambda}(dx) = \sqrt{\frac{2\lambda}{\pi}}\gamma. \tag{2.14}$$

The process $\{M_{-t}, t \geq 0\}$ is defined analogously. The stationary log-volatility is given by

$$\log \sigma_t^2 = \mu + \int_{-\infty}^t e^{-\theta(t-s)} dM_s = \mu + \sum_{k=-\infty, k\neq 0}^{N_t} e^{-\theta(t-t_k)}[\alpha Z_k + \gamma |Z_k|] - \frac{\varpi}{\theta}, \ \ t \geq 0. \tag{2.15}$$

We observe S at n consecutive jump times $0 = t_0 < t_1 < \ldots < t_n < T < t_{n+1}, n \in \mathbb{Z}$ over the time interval $[0, T]$. The state process X has then the following autoregressive representation

$$\begin{aligned} X_{t_i} &= e^{-\theta \Delta t_i} X_{t_{i-1}} + \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} e^{-\theta(t_i - t_k)}[\alpha Z_k + \gamma |Z_k|] - \int_{t_{i-1}}^{t_i} e^{-\theta(t_i - s)} \varpi ds \\ &= e^{-\theta \Delta t_i} X_{t_{i-1}} + \alpha Z_i + \left(|Z_i| - \frac{\varpi}{\theta}(1 - e^{-\theta \Delta t_i})\right) \end{aligned} \tag{2.15}$$

where $\Delta t_i := t_i - t_{i-1}, \ \ i = 1, 2, \ldots, n$ and $N_{t_{i-1}} + 1 = N_{t_i} = i$.

We do the parameter estimation in two steps. The rate λ of the Poisson process N can be estimated given the jump times t_i, therefore it is done at a first step. Since we observe total number of jumps n of the Poisson process N over the T intervals of length one, the MLE of λ is given by $\widehat{\lambda}_n := \frac{n}{T}$.

To estimate the remaining parameters (α, θ, μ), we use the quasi maximum likelihood estimation procedure in conditionally heteroscedastic time series models developed by Straumann [29] and Straumann and Mikosch [30]. Gaussian quasi-maximum likelihood estimation is a method of estimation under the hypothesis of Gaussian innovation.

Assuming that $S_{H,t_i}^{\Delta t_i}$ given $S_{H,t_{i-1}}^{\Delta t_{i-1}}, \ldots, S_{H,t_1}^{\Delta t_1}, X_0$ is conditionally normally distributed with mean zero and variance $\sigma_{t_i-}^2/\lambda$, the conditional log-likelihood given the initial value X_0 has the representation

$$\mathcal{L}(\vartheta | S_H^{\Delta}, \lambda) := -\frac{n}{2}\log(2\pi) - \frac{1}{2}\left(\sum_{i=1}^{n} \log(\sigma_{t_i-}^2/\lambda) - \sum_{i=1}^{n} \frac{(S_{H,t_i}^{\Delta t_i})^2}{\sigma_{t_i-}^2/\lambda}\right). \tag{2.17}$$

where $S_{H,t_i}^{\Delta t_i} := S_{H,t_i} - S_{H,t_{i-1}}$ is the return at time t_i. Since the volatility is unobservable, this log-likelihood can not be evaluated numerically. The quasi log-likelihood function for $\vartheta = (\theta, \alpha, \gamma, \mu)$ given the data $S_H^{\Delta} := (S_{H,t_1}^{\Delta t_1}, S_{H,t_2}^{\Delta t_2}, \ldots, S_{H,t_n}^{\Delta t_n})$ and the MLE $\widehat{\lambda}_n$ is defined as

$$\mathcal{L}(\vartheta | S_H^{\Delta}, \widehat{\lambda}_n) := -\frac{1}{2}\sum_{i=1}^{n} \log(\widehat{\sigma}_{H,t_i}^2(\vartheta, \widehat{\lambda}_n)) - \frac{1}{2}\sum_{i=1}^{n} \frac{(S_{H,t_i}^{\Delta t_i})^2}{\widehat{\sigma}_{H,t_i}^2(\vartheta, \widehat{\lambda}_n)/\widehat{\lambda}_n} \tag{2.18}$$

where the estimates of the volatility σ_{H,t_i}^2, $i = 1, 2, \ldots, n$ are given by

$$\widehat{\sigma}_{H,t_i}^2(\vartheta, \lambda_n) := \exp(\mu + e^{-\alpha \Delta t_i}\widehat{X}_{H,t_{i-1}}(\vartheta, \lambda) - \widehat{\varpi}\Delta t_i), \ \ i = 1, 2, \ldots, n \tag{2.19}$$

and given the parameters ϑ and λ, the estimates of the state process X are given by the recursion

$$\widehat{X}_{H,t_i} = e^{-\theta \Delta t_i}\widehat{X}_{H,t_{i-1}} + \alpha \frac{S_{H,t_i}}{\widehat{\sigma}_{t_i}(\vartheta, \lambda)} + \left(\frac{|S_{H,t_i}|}{\widehat{\sigma}_{t_i}(\vartheta, \lambda)} - \widehat{\varpi}\Delta t_i\right), \ \ i = 1, 2, \ldots, n \tag{2.20}$$

Note that $E(|W|) = \sqrt{\frac{2}{\pi\lambda}}$, $W \sim \mathcal{N}(0, 1/\lambda)$. Hence $\widehat{\varpi} = \gamma\lambda\sqrt{\frac{2}{\pi\lambda}}$.

Here the approximation $(1 - e^{-z}) \approx z$ for small z is used which is similar to approximate exact scheme by Euler scheme in linear SDE simulation and $S_{H,t_i}/\widehat{\sigma}_{t_i}(\vartheta, \lambda)$ approximates the innovation process Z_i producing $\widehat{X}_{H,t_i} = e^{-\theta\Delta t_i}\widehat{X}_{H,t_{i-1}} + \alpha Z_i + (|Z_i| - \widehat{\varpi}\Delta t_i)$. The recursion needs a starting value $\widehat{X}_{H,0}$ which will be set equal to the mean value of the stationary distribution of X which is zero, the mean value zero of the stationary distribution of X.

The quasi-maximum likelihood estimator (QMLE) of ϑ is defined as

$$\widehat{\vartheta}_n := \arg\max_{\vartheta \in \Theta} \mathcal{L}(\vartheta | S_H^{\Delta}, \widehat{\lambda}_n). \tag{2.21}$$

As a byproduct, we get a parametric estimator of the volatility. If we determine the QMLE $\widehat{\vartheta}_n$, we can plug-in into volatility estimate and get estimates

$$\widehat{\sigma}_{H,t_i}^2(\widehat{\vartheta}_n, \widehat{\lambda}_n) := \exp(\widehat{\mu}_n + e^{-\widehat{\alpha}_n \Delta t_i}\widehat{X}_{H,t_{i-1}}(\widehat{\vartheta}_n, \widehat{\lambda}_n) - \widehat{\varpi}\Delta t_i) \tag{2.22}$$

of the volatility at jump times $t_1, t_2, \ldots, t_n$ based on $\widehat{\vartheta}_n = (\widehat{\theta}_n, \widehat{\alpha}_n, \widehat{\gamma}_n, \widehat{\mu}_n)$ and $\widehat{\lambda}_n$.

3. Generalized Method of Moments

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be the stochastic basis on which is defined the Ornstein-Uhlenbeck process X_t satisfying the Itô stochastic differential equation

$$dX_t = -\theta X_t dt + dM_{H,t}, \quad t \geq 0, \tag{3.1}$$

where $\{M_{H,t}\}_{t\geq 0}$ is a sub-fractional Levy motion with $H > 1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta \in \mathbb{R}^+$ is the unknown parameter to be estimated on the basis of completely directly observed continuous observation of the process $\{X_t\}_{t\geq 0}$ on the time interval $[0, T]$. Observe that

$$X_t = \int_{-\infty}^{t} e^{-\theta(t-s)} dM_{H,s}, \ \ t \geq 0. \tag{3.2}$$

This process is stationary and is a process with long memory. It can be shown that X_{t_i} is a stationary discrete time AR(1) process with autoregression coefficient $\phi \in (0, 1)$ with the following representation

$$X_{t_i} = \phi X_{t_{i-1}} + \epsilon_{t_{i-1}} \ \text{ where } \ \phi = e^{-\theta\Delta} \quad \text{and} \ \ \epsilon_{t_{i-1}} = \int_{t_{i-1}}^{t_i} e^{-\theta(t_i-u)} dM_{H,u}, \ \ i = 1, 2, \ldots, n. \tag{3.3}$$

Then the problem is an AR(1) estimation with non-Gaussian non-martingale error. For equidistant sampling, one can study the least squares estimator which boils down to the study of error distribution for non-semimartingales. One can specialize to the case when M is a either a gamma process or an inverse Gaussian process in order to have infinite number of jumps in a finite time interval unlike the compound Poissoan case which have finite number of jumps in a finite time interval. These sub-fractional Gamma and sub-fractional inverse Gaussian Ornstein-Uhlenbeck (SFLOU) processes are LOU processes which include long memory.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis on which is defined the stochastic volatility model

$$dY_t = \sqrt{X_t} dW_t, \quad dX_t = \theta X_t \ dt + \ dZ_t, \ \ t \geq 0 \tag{3.4}$$

where $\{W_t\}$ is a standard Brownian motion, $\{Z_t\}$ is a homogeneous Levy process, $\theta < 0$. Let the integrated volatility be defined as

$$V_T := \int_0^T X_t dt. \tag{3.5}$$

In IGOU model, calculation of conditional cumulants of the integrated volatility conditioned on the initial value is enough to be able to compute European style options very rapidly.

The kumulant functions of IG-OU process are given by

$$k(\theta) = \log E[e^{-\theta Z(1)}] = -\theta\delta\gamma^{-1}(1 + 2\theta\gamma^{-2})^{-1/2}, \tag{3.6}$$

$$k'(\theta) = \log E(e^{-\theta V_t}) = \delta\gamma - \delta\gamma(1 + 2\theta\gamma^{-2})^{1/2}. \tag{3.7}$$

The *Conditional Mean* is given by

$$E(V_t|V_0) = \epsilon(t, \theta)(V_0 - \kappa_1) + \kappa_1 t \tag{3.8}$$

where

$$\epsilon(t,\theta)=\int_0^t e^{-\theta(t-u)}dZ_u.$$

The *Conditional Second Moment* is given by

$$\text{Var}(V_t|V_0)=\theta^{-2}(\theta t-2+2e^{-\theta t}+\frac{1}{2}-\frac{1}{2}e^{-2\theta t}). \tag{3.9}$$

Now we introduce the *Generalized Method of Moments* (GMM).

Let

$$f_t(\xi)=\left\{\begin{array}{c} E[V_{t+1,t+2}|\mathcal{G}_t]-V_{t+1,t+2} \\ E[V^2_{t+1,t+2}|\mathcal{G}_t]-V^2_{t+1,t+2} \\ E[V_{t+1,t+2}V_{t-1,t}|\mathcal{G}_t]-V_{t+1,t+2}V_{t-1,t} \\ E[V^2_{t+1,t+2}V_{t-1,t}|\mathcal{G}_t]-V^2_{t+1,t+2}V_{t-1,t} \\ E[V_{t+1,t+2}V^2_{t-1,t}|\mathcal{G}_t]-V_{t+1,t+2}V^2_{t-1,t} \\ E[V^2_{t+1,t+2}V^2_{t-1,t}|\mathcal{G}_t]-V^2_{t+1,t+2}V^2_{t-1,t} \end{array}\right\} \tag{3.10}$$

where $\mathcal{G}_t$ is the filtration generated by $\{X_s, 0\le s\le t\}$. By construction $E[f_t(\xi_0)|\mathcal{G}_t]=0$. The GMM estimator is defined as

$$\widehat{\xi}_T=\arg\min g_T(\xi)'Gg_T(\xi) \tag{3.11}$$

where $g_T(\xi)=\frac{1}{T}\sum_{t=1}^{T-2}f_t(\xi)$, and G denotes the asymptotic covariance matrix of $g_T(\xi_0)$.

The GMM estimator can be seen as a minimum Mahalanobis D^2 estimator. The minimized value of the objective function multiplied by the sample size distributed as a chi-square distribution with three degrees of freedom, which allows for an omnibus test of the over identifying restrictions.

The invariant (marginal) distribution of X_t is Inverse Gaussian with parameters (δ,γ) with density function given by

$$f(x)=\frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}x^{-3/2}\exp\left(-\frac{1}{2}(\delta^2x^{-1}+\gamma^2x)\right),\ \gamma\ge 0,\ \delta>0,\ x>0 \tag{3.12}$$

and, mean and variance respectively

$$E(X)=\frac{\delta}{\gamma},\quad \text{Var}(X)=\frac{\delta}{\gamma^3}. \tag{3.13}$$

The normal inverse Gaussian (NIG) process with parameters α,β,μ,δ can be described as the distribution of

$$Y=\mu+\beta X_1+\sqrt{X_1}Z,\ Z\sim\mathcal{N}(0,1) \tag{3.14}$$

with Z is independent of X_1, $\alpha=\sqrt{\beta^2+\gamma^2}$.

$$E(Y)=\mu+\frac{\delta\beta}{\alpha\sqrt{1-(\beta/\alpha)^2}},\quad \text{Var}(Y)=\frac{\delta}{\alpha(1-(\beta/\alpha)^2)^{3/2}}, \tag{3.15}$$

$$\text{Skewness}(Y)=\frac{3\beta}{\alpha^2(\delta(1-(\beta/\alpha)^2))^{1/2}},\quad \text{Kurtosis}(Y)=\frac{3(1+4(\beta/\alpha)^2)}{\alpha\delta(1-(\beta/\alpha)^2)}. \tag{3.16}$$

Let $y_i:=Y_{i\Delta}-Y_{i-1\Delta}, i=1,2,\dots,n$. The cummulants satisfy

$$k^{(1)}_{y_1}=\theta\rho\Delta k^{(1)}_{IG}, \tag{3.17}$$

$$k_{y_1}^{(2)} = \Delta k_{IG}^{(1)} + 2\theta\rho^2 \Delta k_{IG}^{(2)}. \tag{3.18}$$

We know that

$$k_{IG}^{(1)} = \frac{\alpha}{\beta}, \tag{3.19}$$

$$k_{IG}^{(2)} = \frac{\alpha(\alpha+1)}{\beta^2}, \tag{3.20}$$

$$\mathcal{L}(X_0) = \mathcal{L}(X_t) = \mathcal{GIG}(\lambda, \delta, \sqrt{\alpha^2 - \beta^2}). \tag{3.21}$$

The distribution of Y_t is Normal Inverse Gaussian (NIG). Inverting (3.17) and (3.18) and replacing the cummulants by their sample cummulants, we obtain the explicit method of moments estimators.

The moment estimators of ρ and λ are given by

$$\widehat{\rho}_n := \frac{\gamma(\gamma s_y^2 - \Delta\delta)}{2\bar{y}}, \quad \widehat{\theta}_n := \frac{\gamma\bar{y}}{\Delta\delta\widehat{\rho}_n} \tag{3.22}$$

where

$$s_y^2 := \frac{1}{n}\sum_{j=1}^{n}(y_j - \bar{y})^2 = \frac{1}{n}\sum_{j=1}^{n} y_j^2 - (\bar{y})^2, \;\; \bar{y} := \frac{1}{n}\sum_{j=1}^{n} y_j, \;\; y_j := Y_{j\Delta} - Y_{(j-1)\Delta}. \tag{3.23}$$

Proposition 3.1

a) $\widehat{\theta}_n \to \theta_0$ a.s. as $n \to \infty$.

b) $\sqrt{n}(\widehat{\theta}_n - \theta_0) \to^{\mathcal{D}} \mathcal{N}(0,\ J^{-1}(\theta_0))$ as $n \to \infty$.

c) $\widehat{\rho}_n \to \rho_0$ a.s. as $n \to \infty$.

d) $\sqrt{n}(\widehat{\rho}_n - \rho_0) \to^{\mathcal{D}} \mathcal{N}(0,\ I^{-1}(\rho_0))$ as $n \to \infty$

where $J(\theta_0)$ and $I(\rho_0)$ are corresponding Fisher-information.

Next we observe the process $\{Y_t, t \geq 0\}$ at random times $\{t_0, t_1, t_2, ...\}$. We assume that the sampling instants $\{t_i, i = 0, 1, 2...\}$ are generated by a Poisson process on $[0, \infty)$, i.e., $t_0 = 0$, $t_i = t_{i-1} + \alpha_i$, $i = 1, 2, \ldots$ where α_i are i.i.d. positive random variables with a common exponential distribution $F(x) = 1 - \exp(-\lambda x)$. Note that intensity parameter $\lambda > 0$ is the average sampling rate which is assumed to be known. It is also assumed that the sampling process $t_i, i = 0, 1, 2, \ldots$ is independent of the observation process $\{X_t, t \geq 0\}$. We note that the probability density function of $t_{k+i} - t_k$ is independent of k and is given by the gamma density

$$f_i(t) = \lambda(\lambda t)^{i-1}\exp(-\lambda t) I_t/(i-1)!, \;\; i = 0, 1, 2, ... \tag{3.24}$$

where $I_t = 1$ if $t \geq 0$ and $I_t = 0$ if $t < 0$.

We do the parameter estimation in two steps: The rate λ of the Poisson process can be estimated given the arrival times t_i, therefore it is done at a first step. Since we observe total number of arrivals n of the Poisson process over the T intervals of length one, the MLE of λ is given by

$$\widehat{\lambda}_n := \frac{n}{T}. \tag{3.25}$$

Theorem 3.1 We have

$$\text{(a)} \;\; \widehat{\lambda}_n \to \lambda \text{ a.s. as } \;\; n \to \infty.$$

$$\text{(b)} \quad \sqrt{n}(\widehat{\lambda}_n - \lambda) \to^{\mathcal{D}} \mathcal{N}(0,\ e^{\lambda}(1 - e^{-\lambda})) \text{ as } \ n \to \infty.$$

Proof. Let V_i be the number of arrivals in the interval $(i-1, i]$. Then $V_i,\ i = 1, 2, \ldots, n$ are i.i.d. Poisson distributed with parameter λ. Since Φ is continuous, we have $I_{\{0\}}(V_i) = I_{\{0\}}(Y(t_i))$ a.s. $i = 1, 2, \ldots, n$. Note that

$$\frac{1}{n}\sum_{i=1}^{n} I_{\{0\}}(Y_{t_i}) \to^{a.s.} E(I_{\{0\}}V_1) = P(V_1 = 0) = e^{-\lambda} \text{ as } \ n \to \infty. \tag{3.26}$$

LLN and CLT and delta method applied to the sequence $I_{\{0\}}(Y_{t_i}),\ i = 1, 2, \ldots, n$ give the results. □

The CLT result above allows us to construct confidence interval for the jump rate λ.

Corollary 3.1 A $100(1-\alpha)\%$ confidence interval for λ is given by

$$\left[\frac{n}{T} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}},\quad \frac{n}{T} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}}\right]$$

where $Z_{1-\frac{\alpha}{2}}$ is the $(1 - \frac{\alpha}{2})$-quantile of the standard normal distribution.

For random sampling,

$$\widehat{\rho}_n := \frac{\gamma(\gamma s_y^2 - \widehat{\lambda}_n\delta)}{2\bar{y}}, \quad \widehat{\theta}_n := \frac{\gamma\bar{y}}{\widehat{\lambda}_n\delta\widehat{\rho}_n} \tag{3.27}$$

where

$$s_y^2 := \frac{1}{n}\sum_{j=1}^{n}(y_j - \bar{y})^2 = \frac{1}{n}\sum_{j=1}^{n} y_j^2 - (\bar{y})^2,\ \ \bar{y} := \frac{1}{n}\sum_{j=1}^{n} y_j,\ \ y_j := Y_{j\widehat{\lambda}_n} - Y_{(j-1)\widehat{\lambda}_n}. \tag{3.28}$$

Theorem 3.2

a) $\widehat{\theta}_n \to \theta_0$ a.s. as $n \to \infty$,

b) $\sqrt{n}(\widehat{\theta}_n - \theta_0) \to^{\mathcal{D}} \mathcal{N}(0,\ J^{-1}(\theta_0))$ as $n \to \infty$,

c) $\widehat{\rho}_n \to \rho_0$ a.s. as $n \to \infty$,

d) $\sqrt{n}(\widehat{\rho}_n - \rho_0) \to^{\mathcal{D}} \mathcal{N}(0,\ I^{-1}(\rho_0))$ as $n \to \infty$

where $J(\theta_0)$ and $I(\rho_0)$ are corresponding Fisher-information.

The robust estimators of ρ and λ are given by

$$\widetilde{\rho}_n := \frac{\gamma(\gamma\widetilde{a}_y^2 - \Delta\delta)}{2\widetilde{y}}, \quad \widetilde{\theta}_n := \frac{\gamma\widetilde{y}}{\Delta\delta\widetilde{\rho}_n} \tag{3.29}$$

where

$$a_y := \frac{1}{n}\sum_{j=1}^{n}|y_j - \widetilde{y}| \tag{3.30}$$

is the sample mean absolute deviation (MAD) from median,

$$\widetilde{y} := \text{median of } \{y_j,\ 1 \le j \le n\}$$

which is defined as

$$\widetilde{y} = \begin{cases} \frac{y_k + y_{k+1}}{2} & : \quad n = 2k \\ y_{k+1} & : \quad n = 2k+1 \end{cases}$$

and median absolute deviation (MDAD) from median is defined as

$$\widetilde{a}_y := \text{median of } \{y_j - \widetilde{y},\ 1 \le j \le n\}$$

Let $\vartheta := (\rho, \lambda)$ and $\widetilde{\vartheta}_n := (\widetilde{\rho}_n, \widetilde{\lambda}_n)$. By using the standard theory of order statistics, see Theorem 5.9 and 5.21 in Van der Vaart [31] and mixing property of the process, along with Glivenko-Cantelli argument and Delta method, we obtain, for deterministic sampling the following properties of the estimators:

Theorem 3.3 For fixed $\Delta > 0$ as $n \to \infty$,

$$(a) \quad \widetilde{\vartheta}_n \to \vartheta_0 \ \text{ a.s.} \quad \text{as } n \to \infty.$$

$$(b) \quad \sqrt{n}(\widetilde{\vartheta}_n - \vartheta_0) \to^{\mathcal{D}} \mathcal{N}_2(0, \frac{\pi}{2}(2\theta\rho^2\Delta^2\delta^2\gamma^{-4})^{-2}D(\vartheta_0)) \text{ as } n \to \infty$$

where $D(\vartheta_0)$ is the limiting covariance matrix.

For Poisson random sampling, we have

$$\widetilde{\widetilde{\rho}}_n := \frac{\gamma(\gamma\widetilde{a}_y - \widehat{\lambda}_n\delta)}{2\widetilde{y}}, \quad \widetilde{\widetilde{\lambda}}_n := \frac{\gamma\widetilde{y}}{\widehat{\lambda}_n\delta\widetilde{\widetilde{\rho}}_n} \tag{3.31}$$

Theorem 3.4 Let $\widetilde{\widetilde{\vartheta}}_n := (\widetilde{\widetilde{\rho}}_n, \widetilde{\widetilde{\lambda}}_n)$. As $n \to \infty$,

$$(a) \quad \widetilde{\widetilde{\vartheta}}_n \to \vartheta_0 \ \text{ a.s.} \quad \text{as } n \to \infty.$$

$$(b) \quad \sqrt{n}(\widetilde{\widetilde{\vartheta}}_n - \vartheta_0) \to^{\mathcal{D}} \mathcal{N}_2(0, \frac{\pi}{2}(2\theta\rho^2\Delta^2\delta^2\gamma^{-4})^{-2}D(\vartheta_0)) \text{ as } n \to \infty$$

where $D(\vartheta_0)$ is the limiting covariance matrix.

For sub-fractional Gamma process with deterministic sampling, we have

$$\bar{\theta}_n = \frac{2\alpha^3(\alpha+1)}{\beta^4\Delta} \frac{(\frac{1}{n}\sum_{i=1}^n (Y_{i\Delta} - Y_{(i-1)\Delta}))^2}{\frac{1}{n}\sum_{i=1}^n (Y_{i\Delta} - Y_{(i-1)\Delta})^2 - \Delta\frac{1}{n}\sum_{i=1}^n (Y_{i\Delta} - Y_{(i-1)\Delta})^2}. \tag{3.32}$$

Direct application of Birkoff Ergodic Theorem and Mixing CLT (see Durrett [21]) give the following results:

Theorem 3.5

$$\text{(a)} \quad \bar{\theta}_n \to \theta_0 \ \text{ a.s.} \quad \text{as} \quad n \to \infty.$$

$$\text{(b)} \quad \sqrt{n}(\bar{\theta}_n - \theta_0) \to^{\mathcal{D}} \mathcal{N}(0, I^{-1}(\theta_0)) \quad \text{as} \quad n \to \infty.$$

For Poisson random sampling, we have

$$\bar{\bar{\theta}}_n = \frac{2\alpha^3(\alpha+1)}{\beta^4\widehat{\lambda}_n} \frac{(\frac{1}{n}\sum_{i=1}^n (Y_{i\widehat{\lambda}_n} - Y_{i-1\widehat{\lambda}_n}))^2}{\frac{1}{n}\sum_{i=1}^n (Y_{i\widehat{\lambda}_n} - Y_{(i-1)\widehat{\lambda}_n})^2 - \widehat{\lambda}_n \frac{1}{n}\sum_{i=1}^n (Y_{i\widehat{\lambda}_n} - Y_{(i-1)\widehat{\lambda}_n})^2}. \tag{3.33}$$

Theorem 3.5 along with Theorem 3.1 gives the following results:

Theorem 3.6

$$\text{(a)} \quad \bar{\bar{\theta}}_n \to \theta_0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

$$\text{(b)} \quad \sqrt{n}(\bar{\bar{\theta}}_n - \theta_0) \to^{\mathcal{D}} \mathcal{N}(0, I^{-1}(\theta_0)) \quad \text{as} \quad n \to \infty.$$

Conclusion

Another possible generalization of the paper is the following: Hawkes processes (see Hawkes [26]) are an efficient generalization of the Poisson processes to model a sequence of arrivals over time of some types of events, that present self-exciting feature, in the sense that each arrival increases the rate of future arrivals for some period of time. This class of counting processes allows one to capture self-exciting phenomena in a more accurate way compared to inhomogeneous Poisson processes or Cox processes. This is the case with aftershocks of earthquakes; an earthquake increases the geophysical tension in the region and can cause a second earthquake. In finance, they are accurate to model for example credit risk contagion, order book or microstructure noises's feature of financial markets.

A Hawkes process is a counting process A_t with stochastic intensity λ_t given by $\lambda_t = \mu + \int_0^t \Phi(t-s)dA_s$ where $\mu > 0$ and $\Phi : \mathbb{R} \to \mathbb{R}_+$ are two parameters. The parameter $\mu > 0$ is called the *background intensity* and the function Φ is called the *excitation function*. When $\Phi = 0$, this a homogeneous Poisson process.

A sub-fractional Hawkes process $\{A_H(t), t > 0\}$ with Hurst parameter $H \in (1/2, 1)$ is defined as

$$A_H(t) = \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^t u^{\frac{1}{2}-H} \left(\int_u^t \tau^{H-\frac{1}{2}} (\tau - u)^{H-\frac{3}{2}} d\tau \right) dR(u)$$

where $R(u) := A(u)/\sqrt{\lambda_t} - \sqrt{\lambda_t}u$ and $A(u)$ is a Hawkes process with stochastic intensity λ_t.

It would be interesting to investigate QML and GMM estimation in stochastic volatility model driven by sub-fractional Hawkes process which would incorporate self-excitation, jumps and long memory of financial models.

Competing interests: The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] O.E. Barndorff-Nielsen, N. Shephard, Non-Gaussian Ornstein–Uhlenbeck-Based Models and Some of Their Uses in Financial Economics, J. R. Stat. Soc. Ser. B: Stat. Methodol. 63 (2001), 167–241. https://doi.org/10.1111/1467-9868.00282.

[2] O.E. Barndorff-Nielsen, N. Shephard, Econometric Analysis of Realized Volatility and Its Use in Estimating Stochastic Volatility Models, J. R. Stat. Soc. Ser. B: Stat. Methodol. 64 (2002), 253–280. https://doi.org/10.1111/1467-9868.00336.

[3] O.E. Barndorff-Neilsen, N. Shephard, How Accurate Is the Asymtotic Approximation to the Distribution of Realised Variance? in: D.W.F. Andrews, J.L. Powell, P.A. Ruud, J.H. Stock, Eds. Identification and Inference for Econometric Models, Cambridge University Press, (2002).

[4] O.E. Barndorff-Nielsen, N. Shephard, Estimating Quadratic Variation Using Realized Variance, J. Appl. Econ. 17 (2002), 457–477. https://doi.org/10.1002/jae.691.

[5] O.E. Barndorff-Nielsen, N. Shephard, Realized Power Variation and Stochastic Volatility Models, Bernoulli 9 (2003), 243-265. https://doi.org/10.3150/bj/1068128977.

[6] B.M. Bibby, M. Sørensen, M. Sorensen, Martingale Estimation Functions for Discretely Observed Diffusion Processes, Bernoulli 1 (1995), 17-39. https://doi.org/10.2307/3318679.

[7] J.P.N. Bishwal, Parameter Estimation in Stochastic Differential Equations, Springer, Berlin, Heidelberg, 2008. https://doi.org/10.1007/978-3-540-74448-1.

[8] J.P.N. Bishwal, Berry–Esseen Inequalities for Discretely Observed Diffusions, Monte Carlo Methods Appl. 15 (2009), 229-239. https://doi.org/10.1515/MCMA.2009.013.

[9] J.P.N. Bishwal, M-Estimation for Discretely Sampled Diffusions, Theory Stoch. Processes 15 (2009), 62-83.

[10] J.P.N. Bishwal, Conditional Least Squares Estimation in Diffusion Processes Based on Poisson Sampling, J. Appl. Prob. Stat. 5 (2010), 169-180.

[11] J.P.N. Bishwal, Milstein Approximation of Posterior Density of Diffusions, Int. J. Pure Appl. Math. 68 (2011), 403-414.

[12] J.P.N. Bishwal, Some New Estimators of Integrated Volatility, Open J. Stat. 01 (2011), 74–80. https://doi.org/10.4236/ojs.2011.12008.

[13] J.P.N. Bishwal, Stochastic Moment Problem and Hedging of Generalized Black–Scholes Options, Appl. Numer. Math. 61 (2011), 1271–1280. https://doi.org/10.1016/j.apnum.2011.08.005.

[14] J.P.N. Bishwal, Sufficiency and Rao-Blackwellization of Vasicek Model, Theory Stoch. Processes 17 (2011), 12-15.

[15] J.P.N. Bishwal, Maximum Quasi-Likelihood Estimation in Fractional Levy Stochastic Volatility Model, J. Math. Finance 1 (2011e), 58-62.

[16] J.P.N. Bishwal, Berry–Esseen Inequalities for the Fractional Black–Karasinski Model of Term Structure of Interest Rates, Monte Carlo Methods Appl. 28 (2022), 111–124. https://doi.org/10.1515/mcma-2022-2111.

[17] J.P.N. Bishwal, Parameter Estimation in Stochastic Volatility Models, Springer, Cham, 2022. https://doi.org/10.1007/978-3-031-03861-7.

[18] J.P.N. Bishwal, Quasi-Likelihood Estimation in Fractional Levy SPDEs from Poisson Sampling, Eur. J. Math. Anal. 2 (2022), 15. https://doi.org/10.28924/ada/ma.2.15.

[19] J.P.N. Bishwal, Parameter Estimation for SPDEs Driven by Cylindrical Stable Processes, Eur. J. Math. Anal. 3 (2022), 4. https://doi.org/10.28924/ada/ma.3.4.

[20] T. Bojdecki, L.G. Gorostiza, A. Talarczyk, Sub-Fractional Brownian Motion and Its Relation to Occupation Times, Stat. Prob. Lett. 69 (2004), 405–419. https://doi.org/10.1016/j.spl.2004.06.035.

[21] R. Durrett, Probability: Theory and Examples, Cambridge University Press, (2019).

[22] M. Fukasawa, Central Limit Theorem for the Realized Volatility Based on Tick Time Sampling, Finance Stoch. 14 (2010), 209–233. https://doi.org/10.1007/s00780-008-0087-3.

[23] M. Fukasawa, Asymptotic Analysis for Stochastic Volatility: Martingale Expansion, Finance Stoch. 15 (2011), 635–654. https://doi.org/10.1007/s00780-010-0136-6.

[24] M. Fukasawa, M. Rosenbaum, Central Limit Theorems for Realized Volatility under Hitting Times of an Irregular Grid, Stoch. Processes Appl. 122 (2012), 3901–3920. https://doi.org/10.1016/j.spa.2012.08.005.

[25] M. Hallin, C. Koell, B.J.M. Werker, Optimal Inference for Discretely Observed Semiparametric Ornstein-Uhlenbeck Processes, J. Stat. Plan. Inference 91 (2000), 323–340. https://doi.org/10.1016/S0378-3758(00)00185-3.

[26] A.G. Hawkes, Spectra of Some Self-Exciting and Mutually Exciting Point Processes, Biometrika 58 (1971), 83–90. https://doi.org/10.1093/biomet/58.1.83.

[27] Y.S. Mishura, Stochastic Calculus for Fractional Brownian Motion and Related Processes, Springer, Berlin, Heidelberg, 2008. https://doi.org/10.1007/978-3-540-75873-0.

[28] Y.S. Mishura, Stochastic Analysis of Mixed Fractional Gaussian Processes, Elsevier, San Diego, 2018.

[29] D. Straumann, Estimation in Conditionally Heteroscedastic Time Series Models, Springer, Berlin, Heidelberg, 2005. https://doi.org/10.1007/b138400.

[30] D. Straumann, T. Mikosch, Quasi-Maximum-Likelihood Estimation in Conditionally Heteroscedastic Time Series: A Stochastic Recurrence Equations Approach, The Annals of Statistics 34 (2006), 2449-2495. https://doi.org/10.1214/009053606000000803.

[31] A.W. van der Vaart, Asymptotic Statistics, Cambridge University Press, 2000.