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# A New Extension of Inverted Exponential Distribution with Applications

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Abstract. This article introduces the MTI inverted exponential distribution (MTIIE), a two-parameter generalization of the inverted exponential distribution, developed using the MTI technique. The MTI technique is named after (Murtiza, Tariq, Ishfaq) who pioneered this approach to enhance the flexibility and applicability of statistical models. The paper explores key properties of the distribution, including moments, the quantile function, the hazard rate, the reliability function, and the moment-generating function. The distribution parameters are estimated using the method of maximum likelihood estimation (MLE). It is applied to two real data sets to demonstrate the practical utility of the new distribution, showcasing its effectiveness in modeling real-world data.

## 1. Introduction

In applied statistics and life sciences, probability distributions play a critical role in reliability and survival analysis. However, many existing models for survival data fail to provide an optimal fit in certain scenarios. To overcome these limitations, researchers often introduce additional shape parameters, enhancing the models' flexibility and accuracy. One such model is the inverted exponential (IE) distribution, introduced by [6] and later explored by [8]. The IE distribution is a continuous probability model characterized by its inverted bathtub-shaped hazard function, making it an improvement over the classical exponential distribution. Specifically, the IE distribution addresses the exponential distribution's limitations, such as its constant failure rate and memoryless property. The IE distribution has demonstrated its utility in modeling Poisson processes between events, where the exponential distribution proves inadequate The probability density function (PDF) and cumulative density function (CDF) of IE distribution are given in Eqs.(1) and (2) respectively.

$$f(x) = \frac{\lambda}{x^2} e^{\frac{-\lambda}{x}}; \quad x > 0, \lambda > 0 \tag{1}$$

$$F(x) = e^{\frac{-\lambda}{x}}; \quad x > 0, \lambda > 0 \tag{2}$$

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In recent years, the inverted exponential distribution has been used in medicine, engineering, biology, business, electronic systems and insurance. In the research, some statisticians have made efforts to enhance the modeling capabilities of the IE distribution. The generalized inverted exponential distribution was introduced by [1].The Bayes estimators of the parameter and reliability function of inverted exponential distribution were obtained by [13].Exponentiated generalized inverted exponential distribution was proposed by [11]. [4] proposed the exponentiated inverted exponential distribution.The alpha power inverted exponential distribution was introduced by [16]. Topp Leone exponentiated inverse exponential distribution was obtained by [14]. [2] introduced the logistic inverse exponential distribution with properties and applications. The alpha power exponentiated inverse exponential distribution and its application on Italy's Covid-19 mortality rate data was derived by [5]. Modified inverse generalized exponential distribution was introduced by [15]. In this manuscript, our aim is to generalize the inverted exponential distribution by using the MTI transformation technique to develop a new probability distribution known as MTI inverted exponential distribution (MTIIE). Here are the key reasons for using the MTI transformation method in practice.

- The proposed model is highly efficient and adaptable for incorporating a new parameter to generalize the existing distributions.
- To provide a better fit as compared to other competing models.
- The additional parameter can give various desirable properties and is more flexible in the form of hazard and density functions.

The rest of the manuscript is organized as follows: Section 2 introduces the new family of probability distributions known as MTI. In Section 3, the MTI-inverted exponential (MTIIE) distribution is presented. Section 4 discusses the reliability measures of the model, while Section 5 derives its mathematical properties. Section 6 explores entropy measures, and Section 7 investigates order statistics. In Section 8, we determine parameter estimation using maximum likelihood estimation. A simulation study is conducted in Section 9, followed by real dataset applications in Section 10. Finally, Section 11 provides concluding remarks.

## 2. MTI transformation technique

The MTI transformation was recently proposed by [10] whose CDF and PDF are given by the following equations respectively.

$$G_{MTI}(x) = \frac{\alpha F(x)}{\alpha - \log \alpha \bar{F}(x)}; \quad \alpha \in \mathbb{R}^+ \tag{3}$$

$$g_{MTI}(x) = \frac{\alpha(\alpha - \log \alpha) f(x)}{(\alpha - \log \alpha \bar{F}(x))^2}; \quad \alpha \in \mathbb{R}^+ \tag{4}$$

Where the PDF and CDF of the base line distribution are denoted by $f(x)$ and $F(x)$ in the Eqs. (1) and (2) respectively, where $\bar{F}(x) = 1 - F(x)$

## 3. MTI inverted exponential distribution (MTIIE)

A random variable X is said to follow a two-parameter MTIIE distribution, denoted by $(\alpha, \lambda)$ if its CDF and PDF for $x > 0$, are, respectively, given as follows:

$$G_{MTIIE}(x) = \frac{\alpha e^{\frac{-\lambda}{x}}}{\alpha - \log \alpha (1 - e^{\frac{-\lambda}{x}})}; \quad x > 0, \lambda > 0, \alpha \in \mathbb{R}^{+} \tag{5}$$

and

$$g_{MTIIE}(x) = \frac{\alpha(\alpha - \log \alpha)\frac{\lambda}{x^2} e^{\frac{-\lambda}{x}}}{\left(\alpha - \log \alpha (1 - e^{\frac{-\lambda}{x}})\right)^2}; \quad x > 0, \lambda > 0, \alpha \in \mathbb{R}^{+} \tag{6}$$

**Remark**: When $\alpha = 1$, the MTIIE distribution reduces to Inverted exponential distribution.
The graphical illustration of the probability density function (PDF) of the MTI-IE distribution is shown in Fig 1. From the figure, it is evident that the distribution can exhibit increasing, decreasing, or unimodal behaviour, and it can be positively skewed, depending on the values of the parameters.

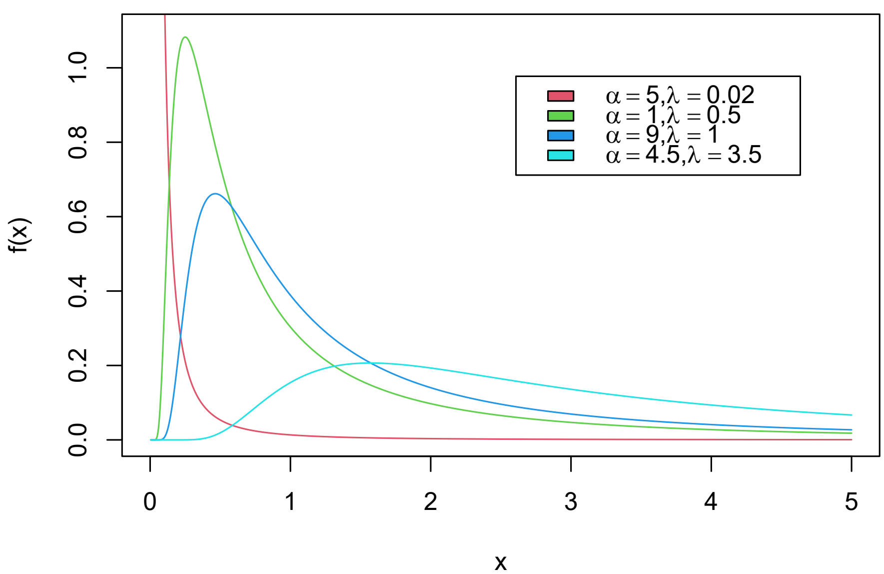


Figure 1. PDF plot for MTIIE distribution

## 4. Reliability Analysis Of MTIIE

In this section, we derive the reliability function, hazard rate, reverse hazard function, and mills ratio expressions for MTI Inverted exponential distribution

4.1. **Reliability function.** The reliability function represents the probability that an item does not fail before a specified time, x. For the MTIIE distribution, it is given as

$$R(x;\alpha,\lambda) = 1 - G_{MTIIE}(x;\alpha,\lambda) = \frac{(\alpha - \log\alpha)(1 - e^{\frac{-\lambda}{x}})}{\alpha - \log\alpha(1 - e^{\frac{-\lambda}{x}})} \tag{7}$$

4.2. **Hazard rate.**

$$h(x;\alpha,\lambda) = \frac{g_{MTIIE}(x;\alpha,\lambda)}{R(x;\alpha,\lambda)} = \frac{\alpha\frac{\lambda}{x^2}e^{\frac{-\lambda}{x}}}{(1 - e^{\frac{-\lambda}{x}})\left(\alpha - \log\alpha(1 - e^{\frac{-\lambda}{x}})\right)} \tag{8}$$

Fig. 2 depicts the graphs of the hazard rate of the MTIIE distribution for various parameter combinations. It demonstrates that the hazard function can exhibit unimodal, increasing, decreasing, and constant shapes, depending on the specific values of the parameters.

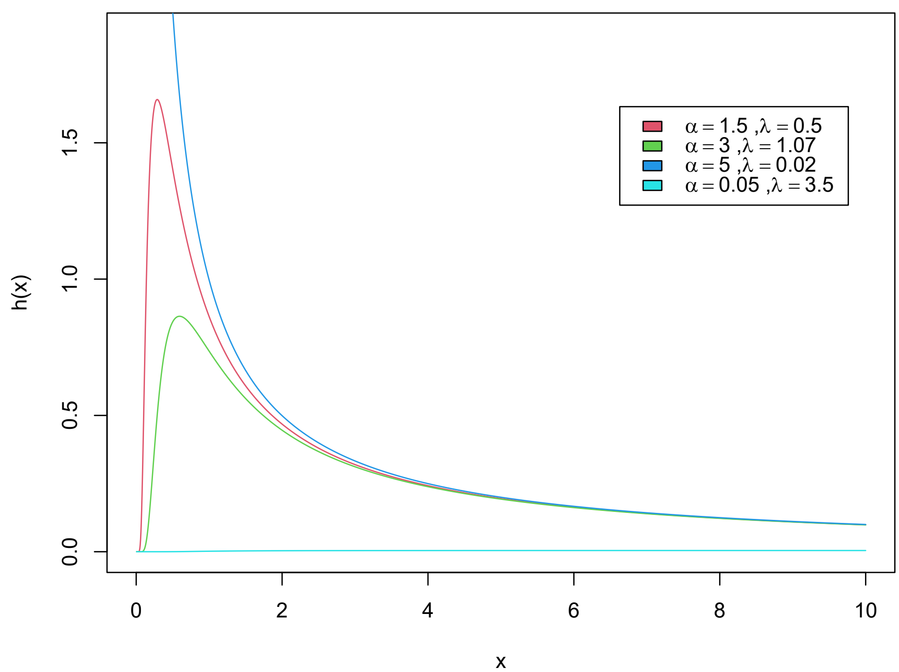


Figure 2. Hazard plot for MTIIE distribution

4.3. **Reverse hazard rate.**

$$h_r(x;\alpha,\lambda) = \frac{g_{MTIIE}(x;\alpha,\lambda)}{G_{MTIIE}(x;\alpha,\lambda)} = \frac{\lambda(\alpha - \log\alpha)}{x^2\left(\alpha - \log\alpha(1 - e^{\frac{-\lambda}{x}})\right)} \tag{9}$$

4.4. **Mills ratio.** The mills ratio is given as

$$M.R = \frac{G_{MTIIE}\,(x;\alpha,\lambda)}{R_{MTIIE}\,(x;\alpha,\lambda)} = \frac{\alpha e^{\frac{-\lambda}{x}}}{(\alpha-\log\alpha)(1-e^{\frac{-\lambda}{x}})} \tag{10}$$

4.5. **Quantile function.**

**Theorem 4.1.** *If $X \sim MTIIE(\alpha,\lambda)$ distribution, then the quantile function of $X$ is given as*

$$x = \frac{-\lambda}{\log\left\{\frac{u(\alpha-\log\alpha)}{\alpha-u\log\alpha}\right\}} \tag{11}$$

*Proof.* Let $G_{MTIIE}(x) = u$. Then the quantile function of MTIIE distribution is

$$\frac{\alpha e^{\frac{-\lambda}{x}}}{\alpha-\log\alpha(1-e^{\frac{-\lambda}{x}})} = u$$

$$\Rightarrow e^{\frac{-\lambda}{x}}\,(\alpha-u\log\alpha) = u\,(\alpha-\log\alpha)$$

$$\Rightarrow x = \frac{-\lambda}{\log\left\{\frac{u(\alpha-\log\alpha)}{\alpha-u\log\alpha}\right\}} \tag{12}$$

Where u is considered as the uniform random variable on the unit interval (0,1). □

4.6. **Median.** Median of MTIIE is obtained by substituting u= 0.5 in Equation (12),we get

$$\text{Median} = \frac{-\lambda}{\log\left\{\frac{0.5(\alpha-\log\alpha)}{\alpha-0.5\log\alpha}\right\}}$$

## 5. Statistical properties of MTIIE

In this section, the structural properties of the proposed MTI Inverted exponential distribution are obtained. This includes the moments, Harmonic mean, mode, moment-generating and characteristic functions.

5.1. **Moments.** The $r^{th}$ moment for MTIIE distribution can be obtained as;

$$\mu_r' = E(x^r) = \int_0^\infty x^r g_{MTIIE}(x;\alpha,\lambda)\,dx$$

$$= \frac{\lambda\,(\alpha-\log\alpha)}{\alpha}\int_0^\infty x^{r-2}e^{\frac{-\lambda}{x}}\left(1-\frac{\log\alpha}{\alpha}\left(1-e^{\frac{-\lambda}{x}}\right)\right)^{-2}dx \tag{13}$$

using series representation $(1-x)^{-2} = \sum_{j=0}^{\infty}(j+1)\,x^j, \quad |x|<1$,in equation (13), we get

$$\mu_r' = \frac{\lambda\,(\alpha-\log\alpha)}{\alpha}\sum_{j=0}^{\infty}(j+1)\left(\frac{\log\alpha}{\alpha}\right)^j\int_0^\infty x^{r-2}e^{\frac{-\lambda}{x}}\left(1-e^{\frac{-\lambda}{x}}\right)^j dx$$

$$= \frac{\lambda^r\,(\alpha-\log\alpha)}{\alpha}\sum_{j=0}^{\infty}\sum_{k=0}^{j}(j+1)\left(\frac{\log\alpha}{\alpha}\right)^j(-1)^k\binom{j}{k}\Gamma\,(1-r)\,(k+1)^{r-1} \tag{14}$$

5.2. **Harmonic-Mean.** The harmonic mean (H) of the MTIIE distribution is given as

$$\frac{1}{H} = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} g_{MTIIE}\left(x; \alpha, \lambda\right)\, dx \tag{15}$$

$$= \frac{\lambda\left(\alpha - \log\alpha\right)}{\alpha} \sum_{j=0}^{\infty}\sum_{k=0}^{j} (j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \int_0^\infty \frac{1}{x^{2+1}} (e^{-\frac{\lambda}{x}})^{k+1}\, dx$$

$$\frac{1}{H} = \frac{\left(\alpha - \log\alpha\right)}{\lambda\alpha} \sum_{j=0}^{\infty}\sum_{k=0}^{j} (j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \frac{1}{(k+1)^2}$$

5.3. **Mode.** The Mode is denoted by $x_m$ and is defined as the maximal value of the MTIIE distribution and is given as:

$$\log f(x) = \log\alpha + \log(\alpha - \log\alpha) + \log\lambda - 2\log x - \frac{\lambda}{x} - 2\log\left(\alpha - \log\alpha\left(1 - e^{\frac{-\lambda}{x}}\right)\right) \tag{16}$$

differentiating equation(16) with respect to x and equate to zero we get,

$$\frac{\partial \log f\left(x\right)}{\partial x} = 0$$

$$= \frac{\lambda}{x^2} - \frac{2}{x} - \frac{2\log\alpha \cdot \frac{\lambda}{x^2} e^{\frac{-\lambda}{x}}}{\alpha - \log\alpha\left(1 - e^{\frac{-\lambda}{x}}\right)} = 0$$

This equation is implicit and solving it analytically might be complex due to the presence of x in both the linear and exponential terms. We will employ the Newton-Raphson method to find an approximate solution for x.

5.4. **Moment generating function of MTIIE.**

**Theorem 5.1.** *LetX ∼ MTIIE(α, λ), then the moment-generating function , $M_X(t)$ of MTIIE is given by:*

$$M_x(t) = \frac{\lambda^r\left(\alpha - \log\alpha\right)}{\alpha} \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{t^r}{r!} (j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \Gamma(1-r)(k+1)^{r-1}$$

*Proof.* The moment-generating function of MTIIE distribution is defined as

$$M_x(t) = \int_0^\infty e^{tx} g_{MTIIE}(x; \alpha, \lambda)\, dx$$

$$M_x(t) = \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) g_{MTIIE}(x; \alpha, \lambda)\, dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^\infty x^r g_{MTIIE}(x; \alpha, \lambda)\, dx$$

$$M_x(t) = \frac{\lambda^r\left(\alpha - \log\alpha\right)}{\alpha} \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{t^r}{r!} (j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \Gamma(1-r)(k+1)^{r-1}$$

□

### 5.5. Characteristic function of MTIIE distribution.

**Theorem 5.2.** *Let* $X \sim MTIIE(\alpha, \lambda)$, *then the characteristic function,* $\phi_X(t)$ *of MTIIE is given by:*

$$\phi_x(t) = \frac{\lambda^r(\alpha - \log\alpha)}{\alpha} \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{(it)^r}{r!}(j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \Gamma(1-r)(k+1)^{r-1}$$

*Proof.* The characteristic function of MTIIE distribution is defined as

$$\phi_x(t) = \int_0^\infty e^{itx} g_{MTIIE}(x;\alpha,\lambda)\, dx$$

$$\phi_x(t) = \int_0^\infty \left(1 + itx + \frac{(itx)^2}{2!} + \dots\right) g_{MTIIE}(x;\alpha,\lambda)\, dx$$

$$\phi_x(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^\infty x^r g_{MTIIE}(x;\alpha,\lambda)\, dx$$

$$\phi_x(t) = \frac{\lambda^r(\alpha - \log\alpha)}{\alpha} \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{(it)^r}{r!}(j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \Gamma(1-r)(k+1)^{r-1}$$

□

**Lemma 1.** Let us suppose that a random variable X follows MTI-IE $(\alpha, \lambda)$ with PDF given in equation (6) and let $I_r(t) = \int_0^t x^r g_{MTIIE}(x;\alpha,\lambda)dx$ denote the r$^{\text{th}}$ incomplete moment, then we have

$$I_r(t) = \frac{\lambda^r(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty}\sum_{k=0}^{j} (j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \Gamma\left(1-r, \frac{\lambda}{t}\right) \tag{17}$$

where $\gamma(a,b) = \int_b^\infty z^{a-1}e^{-z}dz$ denotes the upper incomplete gamma function.

*Proof.*

$$I_r(t) = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \int_0^t x^r \left(e^{\frac{-\lambda}{x}}\right)^{k+1} \frac{\lambda}{x^2}\, dx$$

Put $\frac{\lambda}{x} = z$ in above equation, we get

$$I_r(t) = \frac{\lambda^r(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty}\sum_{k=0}^{j} (j+1)\left(\frac{\log\alpha}{\alpha}\right)^j (-1)^k \binom{j}{k} \Gamma\left(1-r, \frac{\lambda}{t}\right) \tag{18}$$

□

## 6. Renyi entropy

The entropy of a random variable X is a measure of variation of the uncertainty. Among the various entropy measures explored in literature, the Renyi entropy stands out as one of the most widely recognized. The Renyi entropy of X with PDF in equation (6) is given as

$$I_v = \frac{1}{1-v}\log\left(\int_0^\infty g^v(x)\, dx\right)$$

$$I_v = \log\left(\frac{(\alpha - \log\alpha)}{\alpha}\right)^v \lambda^v \int_0^\infty \left(e^{-\frac{\lambda}{x}}\right)^v \frac{1}{x^{2v}} \left[\left(1 - \frac{\log\alpha}{\alpha}\left(1 - e^{-\frac{\lambda}{x}}\right)\right)^{-2}\right]^v dx$$

Substitute $\frac{\lambda}{x} = u$ and simplify the integral

$$I_v = \log\left(\frac{(\alpha - \log\alpha)}{\alpha}\right)^v \lambda^{1-v} \int_0^\infty e^{-uv} \left(1 - \frac{\log\alpha}{\alpha}\left(1 - e^{-u}\right)\right)^{-2v} u^{2v-2}\, du$$

Using the binomial expansion $(1-x)^b$ and simplifying further

$$I_v = \log\left(\frac{(\alpha - \log\alpha)}{\alpha}\right)^v \lambda^{1-v} \sum_{k=0}^{\infty} \binom{-2v}{k} (-1)^k \left(\frac{\log\alpha}{\alpha}\right)^k \int_0^\infty e^{-uv} u^{2v-2} \left(1 - e^{-u}\right)^k du$$

$$I_v = \log\left(\frac{(\alpha - \log\alpha)}{\alpha}\right)^v \lambda^{1-v}\Gamma(2v-1) \sum_{k=0}^{\infty}\sum_{m=0}^{k} \binom{-2v}{k} (-1)^k \left(\frac{\log\alpha}{\alpha}\right)^k \binom{k}{m} (-1)^m (v+m)^{-(2v-1)}$$

## 7. Order Stataistics

Let $X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(n)}$ be the random sample of size n and let $X_{i:n}$ denote the $i^{th}$ order statistics, then the PDF of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1} (1 - F(x))^{n-i} f(x) \tag{19}$$

Using Eq. (5) and Eq. (6) in Eq. (19), we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \left\{\frac{\alpha e^{-\frac{\lambda}{x}}}{\alpha - \log\alpha\left(1 - e^{-\frac{\lambda}{x}}\right)}\right\}^{i-1} \left\{\frac{(\alpha - \log\alpha)(1 - e^{-\frac{\lambda}{x}})}{\alpha - \log\alpha\left(1 - e^{-\frac{\lambda}{x}}\right)}\right\}^{n-i} \left[\frac{\alpha(\alpha - \log\alpha)\frac{\lambda}{x^2} e^{-\frac{\lambda}{x}}}{\left(\alpha - \log\alpha\left(1 - e^{-\frac{\lambda}{x}}\right)\right)^2}\right]$$

$$f_{i:n}(x) = \frac{\lambda\left(\alpha e^{-\frac{\lambda}{x}}\right)^i (\alpha - \log\alpha)^{n-i+1} \left(1 - e^{-\frac{\lambda}{x}}\right)^{n-i}}{B(i, n-i+1)\, x^2 \left(\alpha - \log\alpha\left(1 - e^{-\frac{\lambda}{x}}\right)\right)^{n+1}} \tag{20}$$

From Eq. 20, for i=1 and i= n, we obtain the pdf of the smallest (minimum) order statistics and the largest(maximum) order statistics respectively.

$$f_{1:n}(x) = \frac{\lambda\alpha e^{\frac{-\lambda}{x}} (\alpha - log\alpha)^n \left(1 - e^{\frac{-\lambda}{x}}\right)^{n-1}}{B(1, n)\, x^2 \left(\alpha - log\alpha(1 - e^{\frac{-\lambda}{x}})\right)^{n+1}}$$

$$f_{n:n}(x) = \frac{\lambda\left(\alpha e^{-\frac{\lambda}{x}}\right)^n (\alpha - \log\alpha)}{B(n, 1)\, x^2 \left(\alpha - \log\alpha\left(1 - e^{-\frac{\lambda}{x}}\right)\right)^{n+1}}$$

## 8. Estimation of Parameters

In this section, we consider the method of maximum likelihood estimation to estimate the unknown parameters of the MTI Inverted exponential distribution.

8.1. **Maximum likelihood estimation.** Let $x_1, x_2, x_3, ..., x_n$ be a random sample of size n from MTI Inverted exponential distribution then the likelihood function is given as follows

$$L = \prod_{i=1}^{n}\left[\frac{\alpha(\alpha-\log\alpha)\frac{\lambda}{x^2}e^{-\frac{\lambda}{x}}}{\left(\alpha-\log\alpha\left(1-e^{-\frac{\lambda}{x}}\right)\right)^2}\right]$$

and the log-likelihood function is given as follows

$$\begin{aligned}\log L(x;\alpha,\lambda) &= n\log\alpha + n\log(\alpha-\log\alpha) + n\log\lambda - 2\sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}\frac{1}{x_i}\\ &\quad -2\sum_{i=1}^{n}\log\left(\alpha-\log\alpha\left(1-e^{-\frac{\lambda}{x_i}}\right)\right)\end{aligned}$$

The MLEs of $(\alpha, \lambda)$ are obtained by partially differentiating above equation with respect to model parameters and equating to zero, we have

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \frac{n(\alpha-1)}{\alpha(\alpha-\log\alpha)} - 2\sum_{i=1}^{n}\frac{1-\frac{1}{\alpha}\left(1-e^{-\frac{\lambda}{x_i}}\right)}{\alpha-\log\alpha\left(1-e^{-\frac{\lambda}{x_i}}\right)} = 0$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{1}{x_i} + 2\sum_{i=1}^{n}\frac{\log\alpha\, e^{-\frac{\lambda}{x_i}}}{x_i\left(\alpha-\log\alpha\left(1-e^{-\frac{\lambda}{x_i}}\right)\right)} = 0$$

Since the above equations are not in a closed form, we will employ the Newton-Raphson method and hence using R software to solve these equations and estimate the parameters.

## 9. Simulation study

In this section, the performance of the maximum likelihood estimators for the unknown parameters is scrutinized through simulation studies using 1000 samples generated. The accuracy of the estimates is evaluated based on several criteria, including maximum likelihood estimation (MLE) values, bias and mean square error (MSE). The simulation study is conducted with different sample sizes, namely n=25, 75, and 500. The true parameter values selected for the study are $(\alpha, \lambda)$. The simulation outcomes for various parameter combinations are presented in Table 1. The results indicate that as the sample size increases, the MSE decreases and the average parameter estimates become closer to the true parameter values.

Table 1. MLE, Bias, and MSE for the parameters $\alpha$ and $\lambda$

| Sample size | Parameters | | MLE | | Bias | | MSE | |
|---|---|---|---|---|---|---|---|---|
| $n$ | $\alpha$ | $\lambda$ | $\hat{\alpha}$ | $\hat{\lambda}$ | $\hat{\alpha}$ | $\hat{\lambda}$ | $\hat{\alpha}$ | $\hat{\lambda}$ |
| 25 | 0.50 | 0.20 | 0.77884 | 0.25728 | 0.38331 | 0.10137 | 0.43907 | 0.01916 |
| 75 | | | 0.56220 | 0.22093 | 0.14738 | 0.05503 | 0.04987 | 0.00511 |
| 500 | | | 0.50738 | 0.20248 | 0.04995 | 0.01972 | 0.00405 | 0.00061 |
| 25 | 0.60 | 0.25 | 0.97979 | 0.31046 | 0.50527 | 0.11063 | 0.67459 | 0.02111 |
| 75 | | | 0.71383 | 0.27566 | 0.20737 | 0.06192 | 0.12568 | 0.0067 |
| 500 | | | 0.60841 | 0.25213 | 0.00695 | 0.02208 | 0.00598 | 0.00078 |
| 25 | 0.30 | 1.20 | 0.47391 | 1.85061 | 0.23912 | 0.95308 | 0.20944 | 2.12317 |
| 75 | | | 0.34113 | 1.4096 | 0.10157 | 0.46928 | 0.01931 | 0.39886 |
| 500 | | | 0.30438 | 1.22295 | 0.0342 | 0.1614 | 0.0018 | 0.04149 |
| 25 | 0.30 | 2.0 | 0.47609 | 3.0774 | 0.2483 | 1.60685 | 0.22534 | 6.20121 |
| 75 | | | 0.33736 | 2.32017 | 0.09641 | 0.76494 | 0.01695 | 1.08675 |
| 500 | | | 0.30474 | 2.03539 | 0.03492 | 0.27869 | 0.00195 | 0.11899 |
| 25 | 0.20 | 2.5 | 0.36016 | 4.98979 | 0.20039 | 3.11375 | 0.14176 | 4.42557 |
| 75 | | | 0.24234 | 3.27594 | 0.0816 | 1.35538 | 0.01361 | 3.92376 |
| 500 | | | 0.20576 | 2.60227 | 0.02765 | 0.42825 | 0.00123 | 0.30548 |
| 25 | 1.0 | 2.5 | 1.51063 | 2.69559 | 0.75443 | 0.67057 | 1.00593 | 0.72679 |
| 75 | | | 1.29219 | 2.57583 | 0.48353 | 0.44491 | 0.53096 | 0.30488 |
| 500 | | | 0.80445 | 1.20036 | 0.08000 | 0.03817 | 0.01101 | 0.00236 |
| 25 | 1.0 | 3.0 | 1.56967 | 3.22749 | 0.80343 | 0.78032 | 1.10447 | 0.97728 |
| 75 | | | 1.34224 | 3.12664 | 0.52327 | 0.50715 | 0.60696 | 0.40897 |
| 500 | | | 1.04081 | 3.02382 | 0.13942 | 0.21759 | 0.03882 | 0.07632 |
| 25 | 1.15 | 3.0 | 1.6710 | 3.08491 | 0.83885 | 0.71251 | 1.07216 | 0.82267 |
| 75 | | | 1.52383 | 3.07568 | 0.59673 | 0.4486 | 0.67929 | 0.30272 |
| 500 | | | 1.13253 | 3.02507 | 0.20903 | 0.20547 | 0.09992 | 0.06594 |

## 10. Application

This section evaluates the flexibility of the MTI Inverted Exponential (MTIIE) distribution by comparing it with several existing distributions using two real-world datasets. The comparative analysis includes well-known distributions, such as the Exponentiated Inverted Exponential (EIE) distribution given by [4], the Inverse Exponential Power (IEP) distribution given by [3] and the Inverted Exponential (IE) distribution given by [6]. To assess the compatibility of these distributions with the datasets, we employ several goodness-of-fit criteria, including -2ll, the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the Corrected Akaike Information Criterion (AICC), the Kolmogorov–Smirnov (KS) test and corresponding p-values. In general, a model is considered the best for which these goodness-of-fit statistics have the least and the p value is greater.

**Data set 1 :** The first data set given by [7] represents the remission times, in weeks, for a group of 30 patients with leukemia who received similar treatment. The data is as follows:
1, 1, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31, 42, 45, 50, 57, 60, 71, 85, 91.

**Data set 2 :** The second data set is given by [9] which represents the failure times of the air conditioning system of an airplane. The data are as follows:
23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.

Table 2. MLE and goodness of fit measures for Data-set 1.

| Model | $\hat{\alpha}$ | $\hat{\lambda}$ | $-2ll$ | AIC | BIC | AICC | K-S | P-value |
|---|---|---|---|---|---|---|---|---|
| MTIIE | 0.2904 | 2.4181 | 258.4379 | 262.4379 | 265.2403 | 262.8824 | 0.1240 | 0.7454 |
| EIE | 2.7507 | 2.3136 | 265.4971 | 269.4971 | 272.2995 | 269.9415 | 0.1795 | 0.2882 |
| IEP | 0.5174 | 3.0679 | 268.2681 | 272.2681 | 275.0705 | 265.9415 | 0.19129 | 0.2223 |
| IE | - | 9.6221 | 280.0412 | 282.0412 | 283.4424 | 282.4857 | 0.2684 | 0.0265 |

Table 3. MLE and goodness of fit measures for Data-set 2.

| Model | $\hat{\alpha}$ | $\hat{\lambda}$ | $-2ll$ | AIC | BIC | AICC | K-S | P-value |
|---|---|---|---|---|---|---|---|---|
| MTIIE | 0.1786 | 2.5629 | 305.9145 | 309.9145 | 265.2403 | 312.7169 | 0.13044 | 0.6871 |
| EIE | 3.6204 | 3.0880 | 318.1239 | 322.1239 | 324.9263 | 322.5684 | 0.23296 | 0.0770 |
| IEP | 0.4459 | 5.3024 | 318.1673 | 322.1673 | 324.9697 | 322.6118 | 0.2134 | 0.1298 |
| IE | - | 17.1273 | 339.4545 | 341.4545 | 342.8557 | 341.8989 | 0.33147 | 0.0002 |

The results presented in Tables 2 and 3 indicate that the AIC, BIC, AICC, and -2LL values for the SMPP distribution are lower than those of the other fitted distributions. Therefore, the SMPP distribution outperforms both the competing models and the base model.The promising performance of the proposed distribution is further illustrated in Figures 3 and 4.

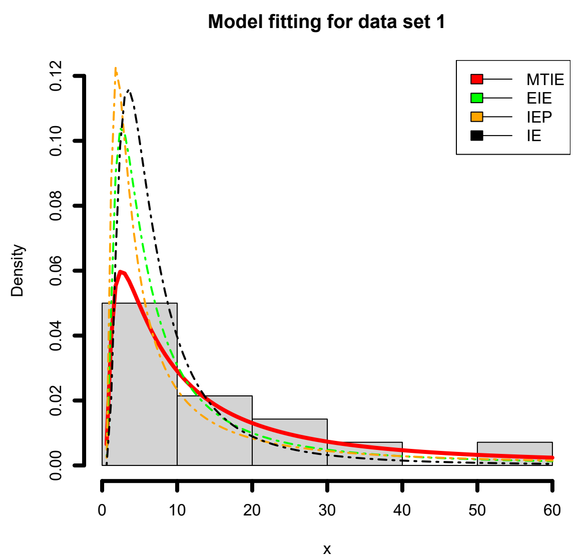


FIGURE 3. Fitted density plot for data set 1

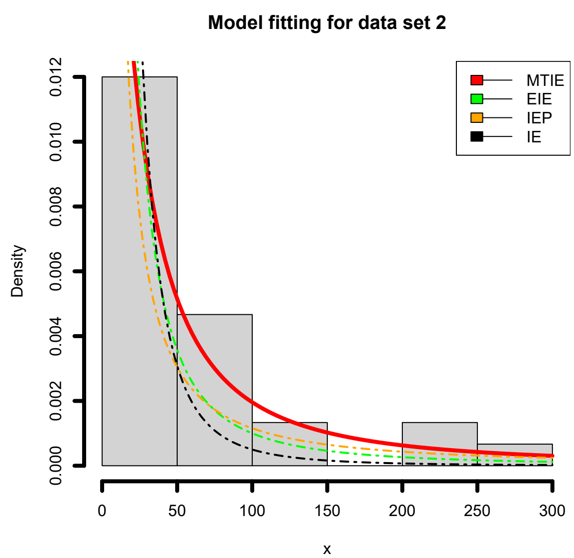


FIGURE 4. Fitted density plot for data set 2

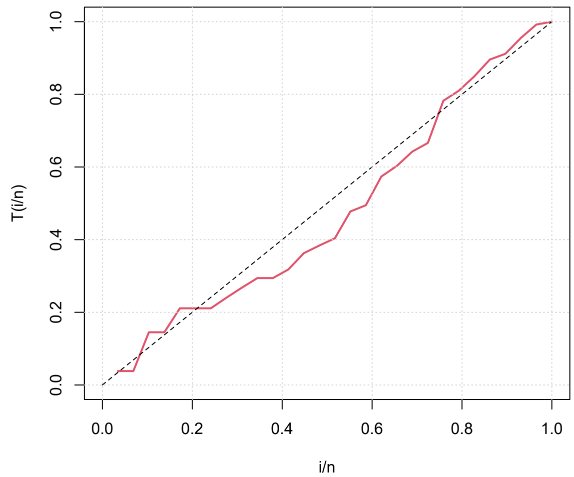


Figure 5. TTT plot for data set 1

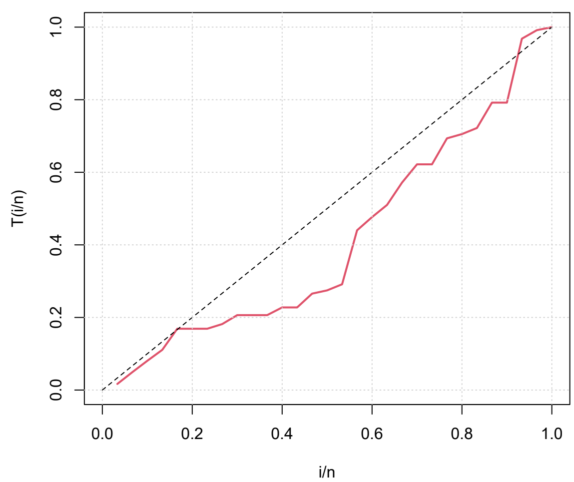


Figure 6. TTT plot for data set 2

## 11. Conclusion

This manuscript introduces and examines a new distribution called the MTI Inverted Exponential Distribution (MTIIE). We derived explicit expressions for several of its statistical properties, including moments, hazard rate, reliability, moment generating function, quantile function, mode, Renyi entropy, and order statistics. Parameter estimation was performed using the maximum likelihood estimation (MLE) method. An extensive simulation study was conducted to assess the accuracy of

MLE in estimating the parameters of the MTIIE distribution. The results showed that the technique effectively recovers the true parameter values, with improved precision and reduced bias as the sample size increases. To evaluate the versatility of the MTIIE distribution, its performance was compared to that of its sub-models using two real-life datasets. The findings from these applications demonstrated that the MTIIE distribution outperformed both its sub-models and other well-known distributions in the study.

**Competing interests:** The authors declare that there is no conflict of interest regarding the publication of this paper.

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