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The Generalized Unit Weibull Distribution: Properties, Estimation, and Applications in Actuarial Science and Insurance

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Abstract. This study introduces the Generalized Unit Weibull (GUW) distribution, an extension of the Unit Weibull distribution achieved through transformation and the inclusion of additional parameters. We explore key theoretical properties of this novel distribution, including stochastic functions, quantile functions and measures, moments, and Rényi entropy. The model's unknown parameters are estimated using the maximum likelihood method. To demonstrate its applicability, we compare the proposed model with existing alternatives using two real-world data sets, particularly in actuarial science and insurance.

1. Introduction

Statistical distributions are extensively used in numerous fields, offering precious tools for decision-making. They are used in life cycle analysis, system trustability, life expectation determination, insurance opinions, engineering, finance, economics, biology, extreme event threat assessment, drug, husbandry, actuarial modeling, demography, administration, sports, and accouterments wisdom.

Among the new statistical distributions proposed recently, those whose domain is bounded by the interval (0, 1) are of particular interest because of their suitability for representing empirical data within this range, such as quotients, ratios, or percentages. This type of quantitative data is frequently encountered in various fields of study, such as hazard assessment, psychology, economics, medical applications, and engineering. Distributions whose probability density functions can adopt specific shapes, such as increasing, decreasing, or bathtub, are particularly valuable for

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modeling complex phenomena. Among the new distributions recently introduced are the follows: unit exponential distribution [1], unit upper truncated Weibull distribution [2], unit Gumbel type-II distribution [3], generalized unit half-logistic geometric distribution [4], unit inverse exponentiated Weibull distribution [5], Unit modified Burr-III distribution [6], kumaraswamy unit-Gompertz distribution [7], unit Muth distribution [8], Marshall–Olkin reduced Kies distribution [9], unit power Weibull distribution [10], unit Xgamma Distribution [11], unit exponentiated Fréchet distribution [12], transmuted Marshall-Olkin extended Topp-Leone distribution [13], unit-exponentiated half-logistic distribution [14], new modified kumaraswamy distribution [15], unit Burr XII [16], extension of J-shaped distribution [17], unit-Chen distribution [18], new regression model for bounded response variable [19], unit generalized log Burr XII distribution [20], unit-Rayleigh distribution [21], unit-Weibull distribution [22], new power Topp-leone distribution [23], power Topp–Leone exponential negative family of distributions [24], Topp-Leone Cauchy family of distributions [25], two-parameter family of distributions [26], Topp-Leone Cauchy Family of distributions, [27].

In addition to these earlier findings, recent developments in statistical distributions have introduced promising new concepts. However, numerous statistical distributions are limited in adapting to various data sets.

Certainly, some datasets show distinct features such as high skewness, kurtosis, heavy tails, inverted J-shapes, multimodality, etc. Distribution generators offer the possibility of efficiently managing, and manipulating these dataset characteristics. We aim to develop a new distribution in this study by generalizing the Weibull distribution and making it unitary. This transformation is motivated by the need for greater flexibility in modeling bounded data on the unit interval, particularly in fields such as reliability analysis, survival modeling, and proportions data. We demonstrate the high degree of adaptability of the distribution to real-world data using two applications: materials engineering and finance. Weibull distribution is widely used because of its advantageous attributes, such as its probabilistic function's mathematical simplicity and flexibility.

The article's remaining sections are organized as follows: Section (2) presents a description of the Generalized Unit Weibull (GUW) distribution. Section (3) addresses some noteworthy characteristics. Sections (4) and (5) provide the methodology for actuarial measures and distribution parameters estimation. Sections (6), (7), and (8) are devoted to the simulations, applications, and new quantile regression model, in that order. Finally, the conclusion is made in the section (9) followed by the perspectives.

2. Generalized Unit Weibull Distribution

We propose a new generalized distribution with support on the unit-interval (0, 1), which arises from a certain transformation on the two-parameter Weibull distribution [28] with probability distribution function (PDF):

$$h(t;\lambda,k)=\frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}e^{-\left(\frac{t}{\lambda}\right)^{k}},t>0,$$

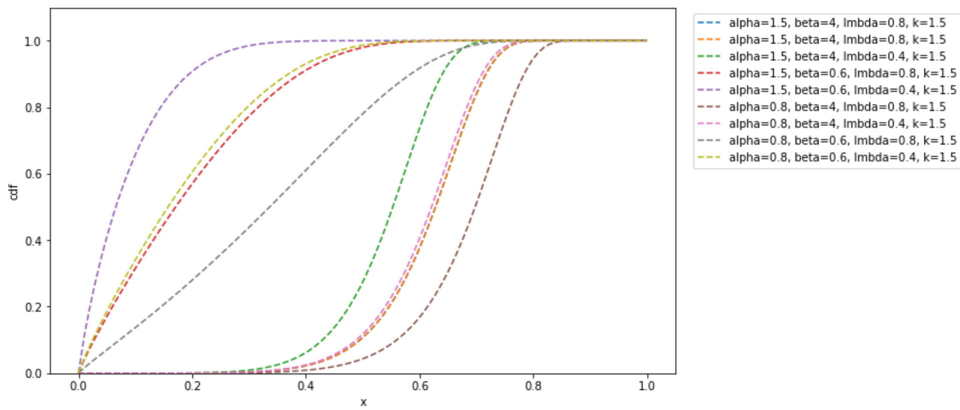


FIGURE 1. CDF of the GUW distribution

and cumulative distribution function (CDF):

$$H(t;\lambda,k) = 1 - e^{-\left(\frac{t}{\lambda}\right)^k}.$$

Where the distribution's shape parameter is $k > 0$, the scale parameter is $\lambda > 0$.

Using the transformation $y = t^{\frac{1}{\alpha}} (t = y^{\alpha})$ and then the transformation $x = \frac{y}{y+\beta} (y = \frac{x\beta}{1-x})$, we have a new generalized distribution on (0, 1), that we call the GUW distribution. Its CDF by is expressed as:

$$F(x;\alpha,\beta,\lambda,k) = 1 - e^{-\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^k}, x \in]0;1[\tag{1}$$

The hazard rate function (hrf) and related PDF are provided by:

$$f(x;\alpha,\beta,\lambda,k) = \frac{\alpha k}{(\lambda)^k} \frac{\beta}{(1-x)^2} \left(\frac{\beta x}{1-x}\right)^{\alpha k - 1} e^{-\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^k}, \tag{2}$$

and

$$hrf(x;\alpha,\beta,\lambda,k) = \frac{\alpha k}{(\lambda)^k} \frac{\beta}{(1-x)^2} \left(\frac{\beta x}{1-x}\right)^{\alpha k - 1},$$

where $\alpha, \beta, \lambda, k > 0$.

Figures (1), (2), and (3) show the CDF, PDF, and hrf of the GUW distribution, respectively. Figure (1) illustrates the flexibility of the cumulative function across different parameter settings. Figure (2) shows that the PDF can take various shapes, including decreasing, reversed J, or asymmetric. Figure (3) highlights the wide range of possible hazard rate behaviors, such as increasing, decreasing, or bell-shaped. This observation is consistent with prior findings in the literature. These curvature characteristics are widely understood and important for developing universal statistical models.

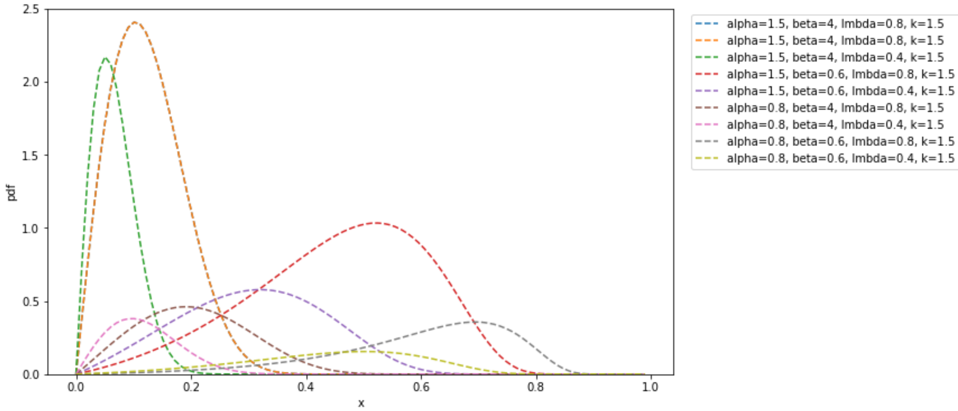


Figure 2. PDF of the GUW distribution

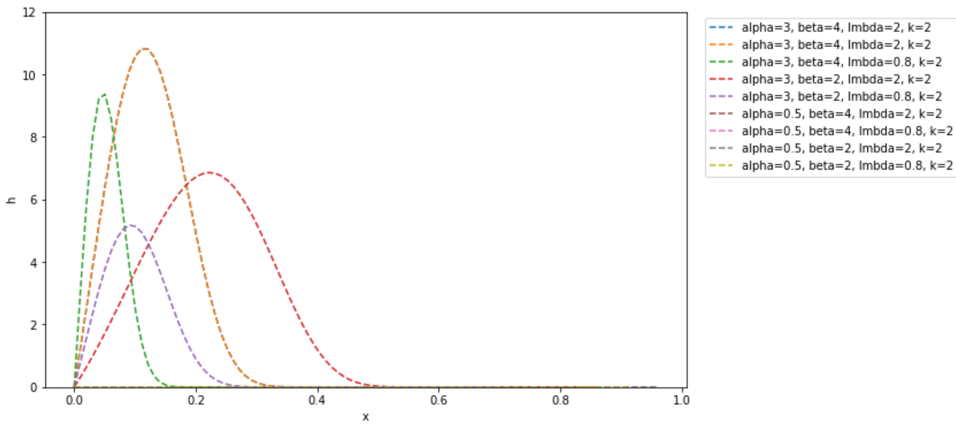


Figure 3. hrf of the GUW distribution

3. Some Mathematical Features of the GUW Distribution

This part focuses on numerous relevant mathematical characteristics of the GUW distribution.

3.1. Series development of the density function f. Proposition 1:

The series development of f is provided by:

$$f(x;\sigma) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n \cdot U(x,\sigma), \tag{3}$$

where

$$T_n = \binom{k+1}{k}\binom{i+j-1}{j}\binom{\alpha mk-1}{i}\left(\frac{1}{\lambda}\right)^{mk} \tag{4}$$

$$\times \frac{[-1]^{m+\alpha mk+i}}{m!}\alpha mk\beta^{\alpha mk}, \tag{5}$$

and

$$U(x;\sigma) = x^{k+j} \tag{6}$$

Proof:

According to (1),

$$F(x;\alpha,\beta,\lambda,k) = 1 - e^{-\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{k}}, x \in]0;1[$$

Considering G(x;σ) $= \left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{k}$

$$F(x;\sigma) = 1 - e^{-[G(x,\sigma)]}$$

knowing that: $e^z = \sum_{m=0}^{+\infty} \frac{z^m}{m!}$

So,

$$\begin{aligned} F(x;\sigma) &= 1 - \sum_{m=0}^{+\infty} \frac{[-G(x,\sigma)]^m}{m!} \\ &= 1 - \sum_{m=0}^{+\infty} \frac{[-1]^m}{m!}[G(x,\sigma)]^m \end{aligned}$$

Let's develop $[G(x;\sigma)]^m$,

$$\begin{aligned} [G(x;\sigma)]^m &= \left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{mk} \\ &= \left(\frac{1}{\lambda}\right)^{mk} \beta^{\alpha mk} \left(\frac{x}{1-x}\right)^{\alpha mk}, \end{aligned}$$

$$F(x;\sigma) = 1 - \sum_{m=0}^{+\infty} \frac{[-1]^m}{m!}\left(\frac{1}{\lambda}\right)^{mk} \tag{7}$$

$$\times \beta^{\alpha mk} \left(\frac{x}{1-x}\right)^{\alpha mk} \tag{8}$$

By differentiating expression (7) with respect to x, we obtain the series expansion of $f(x)$.

$$\begin{aligned} f(x;\sigma) = -\sum_{m=0}^{+\infty} \frac{[-1]^m}{m!}\left(\frac{1}{\lambda}\right)^{mk} \beta^{\alpha mk} \frac{\alpha mk}{(1-x)^2} \\ \times \left(\frac{x}{1-x}\right)^{\alpha mk-1} \end{aligned}$$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \binom{k+1}{k} x^k,$$

More explicitly,

$$\left(\frac{x}{1-x}\right)^{\alpha mk-1} = (-1)^{\alpha mk-1} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} \binom{i+j-1}{j} \times \binom{\alpha mk-1}{i} (-1)^i x^j$$

Finally the development in series of $f(x)$ is given by:

$$f(x;\sigma) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} T_n \cdot U(x,\sigma),$$

where

$$T_n = \binom{k+1}{k} \binom{i+j-1}{j} \binom{\alpha mk-1}{i} \left(\frac{1}{\lambda}\right)^{mk} \times \frac{[-1]^{m+\alpha mk+i}}{m!} \cdot \alpha mk \beta^{\alpha mk},$$

and

$$U(x;\sigma) = x^{k+j},$$

3.2. **Rényi Entropy. Proposition 2:**

The Rényi entropy for the distribution is defined as:

$$ER(X) = \frac{1}{1-\gamma} \log \left\{ \sum_{n=0}^{+\infty} T_n^{'} \cdot I_\gamma(x,\sigma) \right\},$$

where

$$T_n^{'} = \frac{[-\gamma]^n}{n!} \left(\frac{1}{\lambda}\right)^{nk+k\gamma} \beta^{\alpha nk+\gamma\alpha k} (\alpha k)^\gamma,$$

and

$$I_\gamma(x;\sigma) = \int_{\mathbb{R}} \left[\frac{1}{(1-x)^2}\right]^\gamma \left(\frac{x}{1-x}\right)^{\gamma\alpha k-\gamma+\alpha nk} dx$$

Proof:

The Rényi entropy of X in the case of a continuous random variable is defined by:

$$ER(X) = \frac{1}{1-\gamma} \log \left\{ \int_{\mathbb{R}} f(x,\sigma)^\gamma dx \right\}, \gamma \neq 1, \gamma \sup 1 \tag{9}$$

Considering (2)

$$(f(x;\sigma))^{\gamma} = (\alpha k)^{\gamma}\left(\frac{1}{\lambda}\right)^{k\gamma}\beta^{\gamma}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{\beta x}{1-x}\right)^{\gamma\alpha k-\gamma}$$
$$\times\, e^{-\gamma\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{k}}.$$

The terms: $(\alpha k)^{\gamma}$, $\left(\frac{1}{\lambda}\right)^{k\lambda}$, β^{λ} being constant then their series developments remain unchanged.

$$e^{-\gamma\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{k}} = \sum_{n=0}^{+\infty}\frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk}\beta^{\alpha nk}\left(\frac{x}{1-x}\right)^{\alpha nk}$$

So,

$$\begin{aligned}(f(x;\sigma))^{\gamma} &= \sum_{n=0}^{+\infty}\frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk}\beta^{\alpha nk}(\alpha k)^{\gamma}\left(\frac{1}{\lambda}\right)^{k\gamma}\\ &\quad\times\beta^{\gamma}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{\beta x}{1-x}\right)^{\gamma\alpha k-\gamma}\left(\frac{x}{1-x}\right)^{\alpha nk}\\ &= \sum_{n=0}^{+\infty}\frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk+k\gamma}\beta^{\alpha nk+\gamma+\gamma\alpha k-\gamma}\\ &\quad\times(\alpha k)^{\gamma}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{x}{1-x}\right)^{\gamma\alpha k-\gamma+\alpha nk}\end{aligned}$$

Thus, the series expansion of $(f(x,\sigma))^{\gamma}$ is:

$$(f(x;\sigma))^{\gamma} = \sum_{n=0}^{+\infty}\frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk+k\gamma}\beta^{\alpha nk+\gamma\alpha k}(\alpha k)^{\gamma}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{x}{1-x}\right)^{\gamma\alpha k-\gamma+\alpha nk}. \quad (10)$$

Replacing (10) in (9) leads to:

$$ER(X) = \frac{1}{1-\gamma}\log\left\{\int_{\mathbb{R}}\sum_{n=0}^{+\infty}\frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk+k\gamma}\beta^{\alpha nk+\gamma\alpha k}(\alpha k)^{\gamma}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{x}{1-x}\right)^{\gamma\alpha k-\gamma+\alpha nk}dx\right\}.$$

Renyi's Entropy finally is provided by:

$$ER(X) = \frac{1}{1-\gamma}\log\left\{\sum_{n=0}^{+\infty}\frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk+k\gamma}\beta^{\alpha nk+\gamma\alpha k}(\alpha k)^{\gamma}\int_{\mathbb{R}}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{x}{1-x}\right)^{\gamma\alpha k-\gamma+\alpha nk}dx\right\}.$$

Let:

$$T'_n = \frac{[-\gamma]^n}{n!}\left(\frac{1}{\lambda}\right)^{nk+k\gamma}\beta^{\alpha nk+\gamma\alpha k}(\alpha k)^{\gamma},$$

$$I_{\gamma}(x;\sigma) = \int_{\mathbb{R}}\left[\frac{1}{(1-x)^2}\right]^{\gamma}\left(\frac{x}{1-x}\right)^{\gamma\alpha k-\gamma+\alpha nk}dx$$

We have:

$$ER(X) = \frac{1}{1-\gamma}\log\left\{\sum_{n=0}^{+\infty} T_n^{'} \cdot I_\gamma(x,\sigma)\right\}$$

3.3. **Moments and associated measures.** At this stage, we'll look more closely at the moments of the new distribution. Momentum is a crucial statistical concept that aids in understanding a distribution's characteristics and movement as well as its form.

Proposition 3:

The GUW distribution's moment of order s can be calculated as:

$$M_s = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty}\frac{T_n}{s+j+k+1}, \tag{11}$$

where T_n is defined in (4).

Proof:

A variable's moment of order s is determined as follows:

$$M_s = \mathbb{E}(X^s).$$

So,

$$M_s = \int_0^1 x^s \times f(x)dx$$

Using the series development (3), we obtain:

$$\begin{aligned} M_s &= \int_0^1 x^s \times \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n U(x,\sigma)dx \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n \int_0^1 x^{k+j+s}dx \\ M_s &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty}\frac{T_n}{s+j+k+1} \end{aligned}$$

By setting $s = 1$, 2, 3, and 4, we successively obtain the first four moments of the GUW distribution, namely the mean, variance, skewness, and kurtosis.

Figure 4 shows the mean and variance, while Figure 5 presents the skewness and kurtosis of the GUW model, for different combinations of the parameters α, β, λ, and k.

FIGURE 4. Mean and variance of the GUW model with $\lambda = 1$ and $k = 0.09$

FIGURE 5. Skewness and kurtosis of the GUW model with $\lambda = 2$ and $k = 20$

3.4. **Moment Generating Function (MGF).** The MGF is used to fully describe the distribution of a random variable in terms of its moments.

Proposition 4:
Let X be a random variable following the GUW distribution. Then its moment generating function $M_X(t)$ can be expressed as:

$$M_X(t) = \sum_{s=0}^{+\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} \frac{t^s}{s!} \frac{T_n}{s+j+k+1}$$

Proof:

The moment-generating function is defined by:

$$M_X(t) = \mathbb{E}\left(e^{Xt}\right).$$

Knowing that the development in a series of exponential is given by:

$$e^{tx} = \sum_{s=0}^{+\infty} \frac{(tx)^s}{s!}.$$

We can write:

$$\begin{aligned} M_X(t) &= \mathbb{E}\left(\sum_{s=0}^{+\infty} \frac{(tX)^s}{s!}\right) \\ &= \sum_{s=0}^{+\infty} \mathbb{E}\left(\frac{(tX)^s}{s!}\right) \end{aligned}$$

$$M_X(t) = \sum_{s=0}^{+\infty} \frac{t^s}{s!} \mathbb{E}\left(X^s\right) \tag{12}$$

The moment of order s of the distribution is represented by $\mathbb{E}\left(X^s\right)$:

$$M_s = \mathbb{E}\left(X^s\right)$$

By replacing (11) in (12) we have:

$$M_X(t) = \sum_{s=0}^{+\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} \frac{t^s}{s!} \frac{T_n}{s+j+k+1},$$

3.5. **Quantile function. Proposition 5:**

The quantile function associated with the GUW distribution is defined as follows:

$$\mathbb{Q}(p;\sigma) = \frac{\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)}{\left[\beta + \left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)\right]}. \tag{13}$$

Proof:

Let $\pi_t = \mathbb{Q}(t;\sigma)\ \forall\ y \in [0,1]$.

The quantile function is defined as π_t, which is the solution to the following nonlinear equation:

$$t = F(x;\sigma).$$

So,

$$1 - t = e^{-\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^k}.$$

Applying a log transformation to each member of the equation, we obtain:

$$-\ln(1-t)=\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{k} \tag{14}$$

Let's raise each number in the equation (14) to the power $(1/k)$

$$\lambda\left[-\ln(1-t)\right]^{\frac{1}{k}}=\left(\frac{\beta x}{(1-x)}\right)^{\alpha}. \tag{15}$$

Let's raise each number in the equation (15) to the power $(1/\alpha)$

$$\beta x=\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-t)\right]^{\frac{1}{\alpha k}}\right)-x\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-t)\right]^{\frac{1}{\alpha k}}\right).$$

Let's arrange terms containing x in a single member

$$\beta x+x\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-t)\right]^{\frac{1}{\alpha k}}\right)=\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-t)\right]^{\frac{1}{\alpha k}}\right),$$

$$\left[\beta+\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-t)\right]^{\frac{1}{\alpha k}}\right)\right]x=\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-t)\right]^{\frac{1}{\alpha k}}\right).$$

Knowing that α , λ, β are strictly greater than 0, then we have:

$$\pi=\frac{\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)}{\left[\beta+\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)\right]}.$$

The UWG distribution's 25%, 50%, and 75% quartiles may be found by adjusting p =0.25, p =0.5, and p =0.75, respectively, in equation (13).

Assume that p is evenly distributed $(0,1)$, in this case, the following random data sets of size n can be generated by the QF using the GUW distribution:

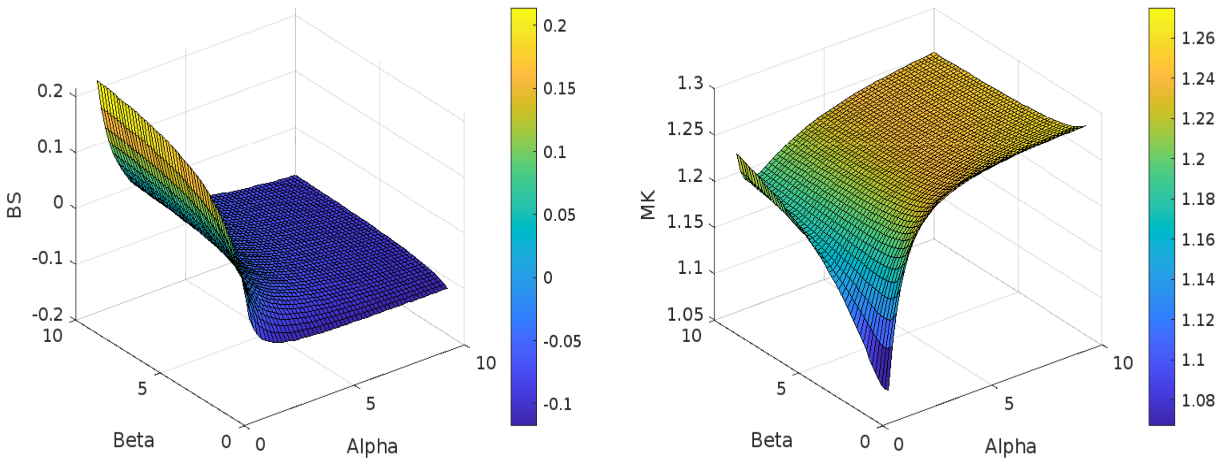
$$\pi_i=\frac{\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-y_i)\right]^{\frac{1}{\alpha k}}\right)}{\left[\beta+\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-y_i)\right]^{\frac{1}{\alpha k}}\right)\right]},\ i=1,2,....,n$$

Graphs of Bowley and Moor skewness and kurtosis are shown in (6).

3.6. **Survival function (suf).** The GUW distribution is characterized by its survival function, which is expressed as:

$$suf(x)=1-F(x),$$

$$suf(x)=e^{-\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^{k}}.$$



(A) Plot of the Bowley's coefficient of skewness

(B) Plot of the Moor's coefficient of kurtosis

FIGURE 6. Skewness (left) and Kurtosis (right) plots

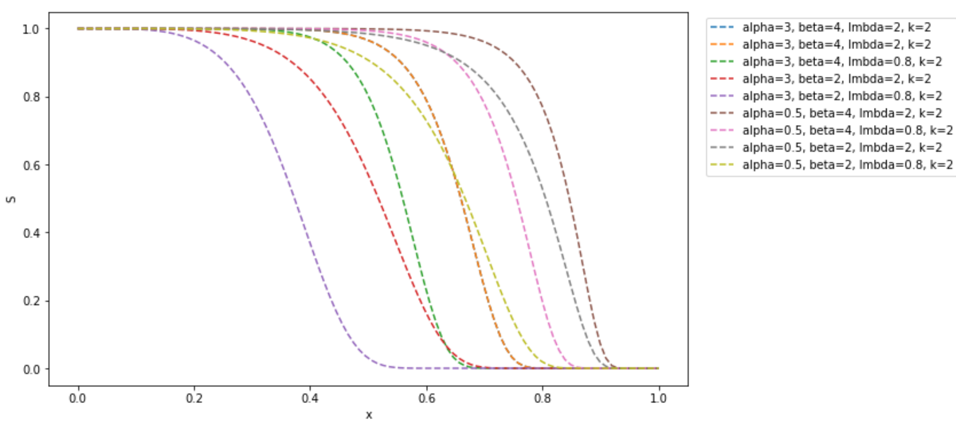


FIGURE 7. suf of the GUW distribution

3.7. **Hazard function (haf).** The GUW distribution's hazard function may be described as follows:

$$haf(x) = \frac{f(x)}{suf(x)},$$

$$haf(x) = \frac{\alpha k}{(\lambda)^k} \frac{\beta}{(1-x)^2} \left(\frac{\beta x}{1-x} \right)^{\alpha k - 1}.$$

3.8. **Cumulative hazard function (cf).** The GUW distribution is characterized by its cf, which is defined as follows:

$$cf(x) = -\log(suf(x)),$$

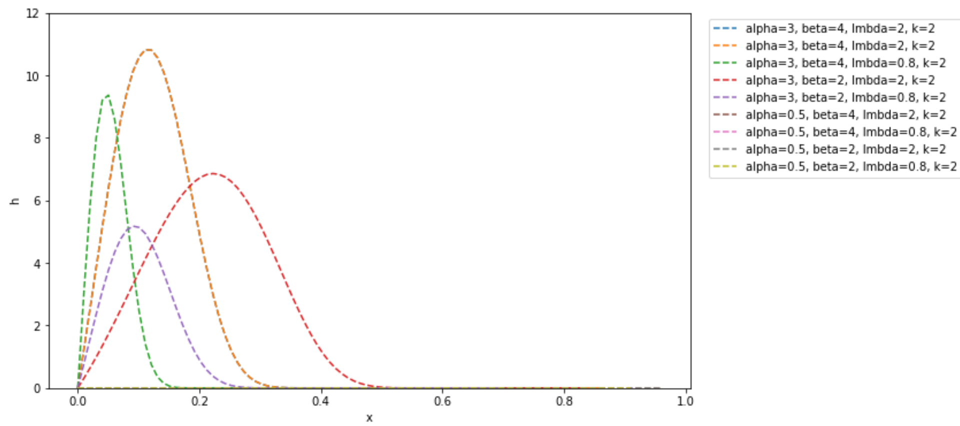


FIGURE 8. haf of the GUW distribution

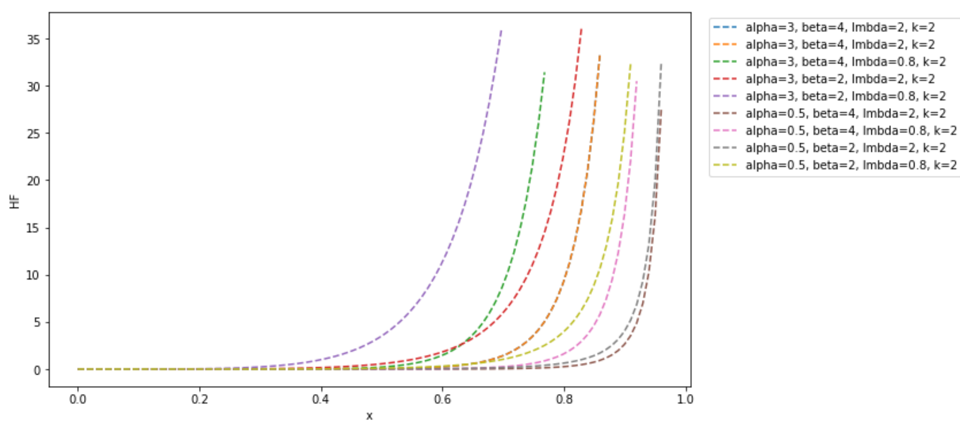


FIGURE 9. Cf of the GUW distribution

So, the cf of the GUW distribution is as follows:

$$cf(x) = \left[\frac{\left(\frac{\beta x}{(1-x)} \right)^{\alpha}}{\lambda} \right]^{k}.$$

3.9. **Reserve hazard function (Rf).** The GUW distribution is characterized by its Rf:

$$Rf(x) = \frac{f(x)}{F(x)},$$

Let: $A(x) = e^{-\left[\frac{\left(\frac{\beta x}{(1-x)} \right)^{\alpha}}{\lambda} \right]^{k}}$.

So,

$$Rf(x) = \frac{\alpha k}{(\lambda)^{k}} \frac{\beta}{(1-x)^{2}} \left(\frac{\beta x}{1-x} \right)^{\alpha k - 1} \frac{A(x)}{1 - A(x)}$$

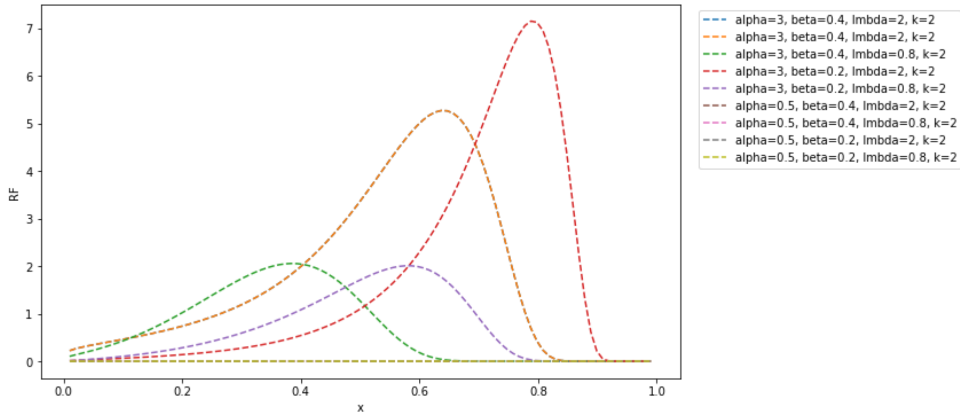


FIGURE 10. Rf of the GUW distribution

3.10. **Average absolute deviation (mad).** The mean absolute deviation indicates how far, on average, each piece of data in a set is from the mean of that set. If we consider a GUW distribution with a mean of μ, the average absolute deviation is calculated as:

$$mad(\mu) = \mathbb{E}(|X - \mu|) \tag{16}$$

By using (16), we have:

$$\begin{aligned}
\text{mad}(\mu) &= \int_0^1 |x - \mu| f(x)\, dx \\
&= \int_0^{\mu} (-x + \mu) f(x)\, dx + \int_{\mu}^1 (x - \mu) f(x)\, dx \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} T_n \left[\int_0^{\mu} (-x^{k+j+1} + \mu x^{k+j})\, dx + \int_{\mu}^1 (x^{k+j+1} - \mu x^{k+j})\, dx \right] \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} T_n \left(\frac{1 - 2\mu^{k+j+2}}{k+j+2} + \frac{2\mu^{k+j+2} - \mu}{k+j+1} \right).
\end{aligned}$$

So, the average absolute deviation is expressed as:

$$\text{mad}(\mu) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} T_n M_{\mu}(\sigma),$$

where

$$M_{\mu}(\sigma) = \frac{1 - 2\mu^{k+j+2}}{k+j+2} + \frac{2\mu^{k+j+2} - \mu}{k+j+1}.$$

3.11. **Median absolute deviation (MD).** If we have a GUW distribution with a median of me, the MD may be stated as follows:

$$MD(me) = \mathbb{E}(|X - me|), \tag{17}$$

By using (17), we have:

$$\begin{aligned}
\mathrm{MD}(me) &= \int_0^1 |x - me| \times f(x)\,dx \\
&= \int_0^{me} (-x + me) \times f(x)\,dx + \int_{me}^1 (x - me) \times f(x)\,dx \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n \left[\int_0^{me} (-x^{k+j+1} + me \cdot x^{k+j})\,dx + \int_{me}^1 (x^{k+j+1} - me \cdot x^{k+j})\,dx\right] \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_{i,j,k,n} \times \left(\frac{1 - 2me^{k+j+2}}{k+j+2} + \frac{2me^{k+j+2} - me}{k+j+1}\right).
\end{aligned}$$

So, the MD is given by:

$$\mathrm{MD}(me) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n M_{me}(x,\sigma),$$

where

$$M_{me}(\sigma) = \frac{1 - 2me^{k+j+2}}{k+j+2} + \frac{2me^{k+j+2} - me}{k+j+1}.$$

4. Actuarial Measures

This section presents both the theoretical foundations and practical aspects of several essential risk measures, including the Value at Risk (VaR), the Tail Value at Risk (TVaR), the tail conditional variance (TV), and the tail variance risk (TVP), as applied to the new distribution.

4.0.1. *VaR measure.* The VaR of the GUW distribution is defined by:

$$\mathrm{VaR}_q = \frac{\left(\lambda^{\alpha}\left[-\ln(1-q)\right]^{\frac{1}{\alpha k}}\right)}{\left[\beta + \left(\lambda^{\alpha}\left[-\ln(1-q)\right]^{\frac{1}{\alpha k}}\right)\right]}.$$

Proof. The VaR of a random variable is the quantile of its distribution function, denoted by VaR_q, and can be expressed as follows:

Then using (13), we have:

$$\mathrm{VaR}_q = Q(q).$$

So we have:

$$\mathrm{VaR}_q = \frac{\left(\lambda^{\alpha}\left[-\ln(1-q)\right]^{\frac{1}{\alpha k}}\right)}{\left[\beta + \left(\lambda^{\alpha}\left[-\ln(1-q)\right]^{\frac{1}{\alpha k}}\right)\right]}.$$

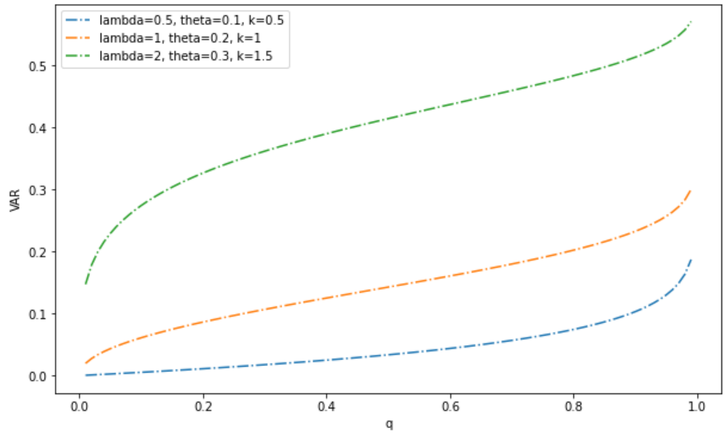


FIGURE 11. The VaR plot of the GUW distribution

4.0.2. *TVaR measure.* **Proposition 6:**

The TVaR of the GUW distribution is described by:

$$\mathrm{TVaR}_q = \frac{1}{1-q}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha nk-1}\sum_{j=0}^{\infty} T_n \mathrm{IV}(\sigma),$$

where

$$\mathrm{IV}(\sigma) = \left(\frac{1-\mathrm{VaR}_q^{k+j+2}}{k+j+2}\right).$$

Proof

TVaR is defined by:

$$\mathrm{TVaR}_q = \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{1} x f(x)\,dx.$$

Knowing that $f(x)$ is given by:

$$\begin{aligned} f(x,\sigma) &= \frac{\alpha k}{(\lambda)^k}\frac{\beta}{(1-x)^2}\left(\frac{\beta x}{1-x}\right)^{\alpha k-1} e^{-\left[\frac{\left(\frac{\beta x}{(1-x)}\right)^{\alpha}}{\lambda}\right]^k} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n U(x,\sigma) \end{aligned}$$

We have:

$$\begin{aligned} \mathrm{TVaR}_q &= \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{1} x f(x)\,dx \\ &= \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{1} x \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n U(x,\sigma) \\ &= \frac{1}{1-q}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n \int_{\mathrm{VaR}_q}^{1} x^{k+j+1} \end{aligned}$$

$$= \frac{1}{1-q}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n\left(\frac{1-\mathrm{VaR}_q^{k+j+2}}{k+j+2}\right).$$

Hence,

$$\mathrm{TVaR}_q = \frac{1}{1-q}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n\left(\frac{1-\mathrm{VaR}_q^{k+j+2}}{k+j+2}\right).$$

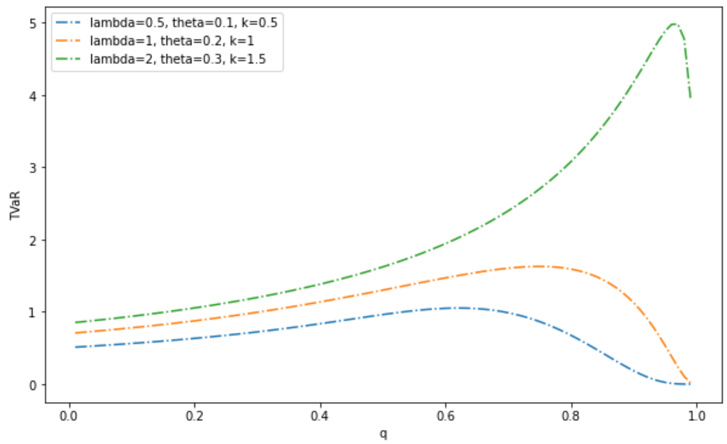


FIGURE 12. The TVaR plot of the GUW distribution

4.0.3. *TV measure.* **Proposition 7:**

The GUW distribution TV is defined by:

$$\mathrm{TV}_q = \frac{1}{1-q}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n \mathrm{IV}'(\sigma) - (\mathrm{TVaR}_q)^2,$$

where

$$\mathrm{IV}'(\sigma) = \left(\frac{1-\mathrm{VaR}_q^{k+j+3}}{k+j+3}\right)$$

Proof.

The TV distribution may be characterized as:

$$\begin{aligned}
\mathrm{TV}_q(X) &= \mathbb{E}(X^2|X > \mathrm{VaR}_q) - (\mathrm{TVaR}_q)^2\\
&= \frac{1}{1-q}\int_{\mathrm{VaR}_q}^{1} x^2 f(x)\,dx - (\mathrm{TVaR}_q)^2\\
&= \frac{1}{1-q}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\alpha mk-1}\sum_{j=0}^{\infty} T_n
\end{aligned}$$

$$\times \left(\frac{1 - \text{VaR}_q^{k+j+3}}{k+j+3} \right) - (\text{TVaR}_q)^2$$

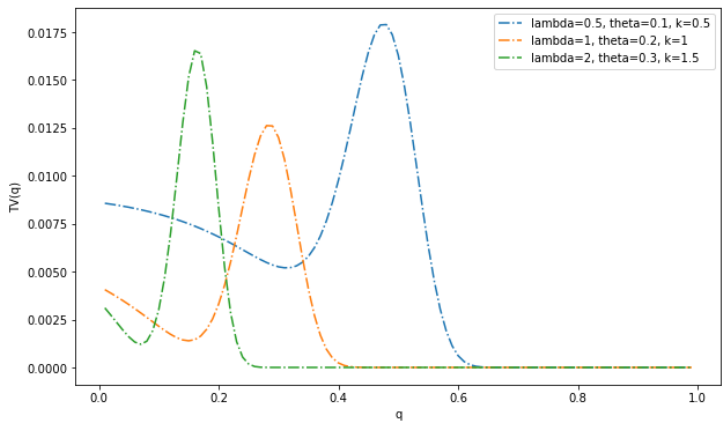


FIGURE 13. The TV plot of the GUW distribution

4.0.4. *TVP measure.* TVP is another key metric used in insurance and is obtained by:

$$\text{TVP}_q = \text{TVaR}_q + \lambda \text{TV}_q,$$

where

$$\text{TV}q = \frac{1}{1-q} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} T_n \text{IV}'(\sigma) - (\text{TVaR}_q)^2,$$

and

$$\text{TVaR}q = \frac{1}{1-q} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\alpha mk-1} \sum_{j=0}^{\infty} T_n \text{IV}(\sigma)$$

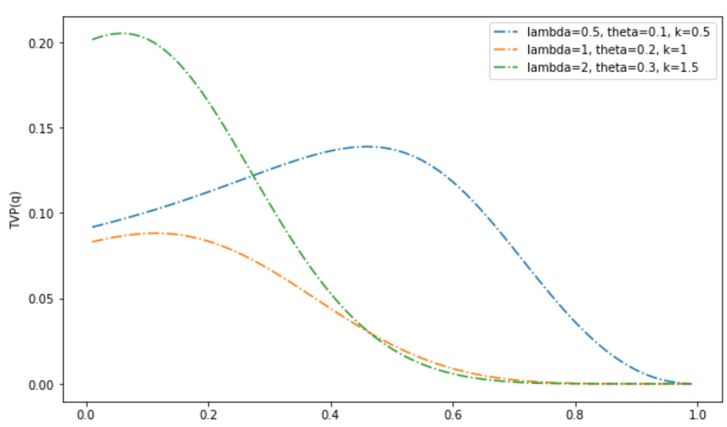


FIGURE 14. The TVp plot of the GUW distribution

5. Estimation

Let $x_1, x_2, \ldots, x_m$ be a random sample of size m from the variable X. Employing the PDF provided in (2), the likelihood function may be expressed as follows:

$$\ell(\alpha, \beta, \lambda, k) = \prod_{j=1}^{m} f(x_j),$$

So, we have:

$$\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k}) = \sum_{j=1}^{m} \ln(\alpha k) - k\sum_{j=1}^{m} \ln(\lambda) - \sum_{j=1}^{m} \ln(\beta) - 2\sum_{j=1}^{m} \ln(1-x_j) + (\alpha k - 1)\sum_{j=1}^{m} \ln\left(\frac{\beta x_j}{1-x_j}\right) - \sum_{j=1}^{m} e^{k\alpha \ln\left(\frac{\beta x_j}{(1-x_j)}\right)} \left(\frac{1}{\lambda}\right)^k$$

The logarithmic likelihood function may be stated in the form of:

$$\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k}) = \ln\left[L(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k}))\right],$$

We obtain:

$$\begin{aligned}
\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k}) &= \sum_{j=1}^{m} \ln\left(\frac{\alpha k}{(\lambda)^k}\frac{\beta}{(1-x_j)^2}\left(\frac{\beta x_j}{1-x_j}\right)^{\alpha k-1} e^{-\left[\frac{\left(\frac{\beta x_j}{(1-x_j)}\right)^{\alpha}}{\lambda}\right]^k}\right) \\
&= \sum_{j=1}^{m} \ln(\alpha k) - k\sum_{j=1}^{m} \ln(\lambda) - \sum_{j=1}^{m} \ln(\beta) - 2\sum_{j=1}^{m} \ln(1-x_j) + (\alpha k - 1)\sum_{j=1}^{m} \ln\left(\frac{\beta x_j}{1-x_j}\right) \\
&\quad - \sum_{j=1}^{m}\left[\frac{\left(\frac{\beta x_j}{(1-x_j)}\right)^{\alpha}}{\lambda}\right]^k
\end{aligned}$$

Introducing the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$, and $\hat{k}$.

We have:

$$\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k}) = max_{(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k})\in[0,+\infty]^4} l(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k})$$

The first partial derivatives of $l(\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{k})$ with regard to zero are provided as follows:

$$\frac{\partial l}{\partial \alpha} = \frac{1}{\alpha} + k\sum_{j=1}^{m} \ln\left(\frac{\beta x_j}{1-x_j}\right) + \sum_{j=1}^{m} k\ln\left(\frac{\beta x_j}{1-x_j}\right) e^{k\alpha \ln\left(\frac{\beta x_j}{1-x_j}\right)} \left(\frac{1}{\lambda}\right)^k,$$

$$\frac{\partial l}{\partial \beta} = -\frac{1}{\beta} + \frac{(\alpha k - 1)}{\beta} - \sum_{j=1}^{m} \alpha k(\beta)^{\alpha k - 1}\left[\frac{\left(\frac{x_j}{(1-x_j)}\right)^{\alpha}}{\lambda}\right]^k,$$

$$\frac{\partial l}{\partial k}=\frac{1}{k}-\ln(\lambda)+(\alpha)\sum_{j=1}^{m}\ln\left(\frac{\beta x_j}{1-x_j}\right)-\sum_{j=1}^{m}\alpha\ln\left(\frac{\beta x_j}{1-x_j}\right)e^{k\alpha\ln\left(\frac{\beta x_j}{1-x_j}\right)}\left(\frac{1}{\lambda}\right)^{k}$$
$$+\sum_{j=1}^{m}e^{k\alpha\ln\left(\frac{\beta x_j}{(1-x_j)}\right)}\ln(\lambda)e^{-k\ln(\lambda)},$$

$$\frac{\partial l}{\partial \lambda}=-\frac{k}{\lambda}+\sum_{j=1}^{m}e^{k\alpha\ln\left(\frac{\beta x_j}{(1-x_j)}\right)}k\lambda^{-k-1}.$$

We first derived explicit formulas for the partial derivatives with respect to each parameter in the log-likelihood of the model. The resulting system is analytically intractable due to the complexity and non-linearity of the equations. In such situations, numerical methods become essential. These methods offer a practical and efficient approach for obtaining approximate solutions iteratively.

6. A New Quantile Regression Model

The concept of parametric quantile regression recently gained popularity due to its robustness in modeling asymmetric data or data with extreme values. This type of regression is also effective in dealing with asymmetric and high-tail response variables, which are defined on the interval (0,1). To implement these regressions, it is necessary to re-parameterize PDFs of the distribution in terms of quantiles, to obtain the quantile PDF (QPDF) [4, 29–32]. To formulate the quantile regression model of the GUW distribution, we begin by substituting the parameter β in terms of the quantile function of the GUW distribution. Then we substitute it in the expressions of the CDF and the PDF. Thus, after simplification, we obtain the cumulative distribution function (QCDF) and quantile probability density (QPDF) of the GUW distribution.

Poses $\mathbb{Q}(p,\sigma)=\mu$

$$\mu=\frac{\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)}{\left[\beta+\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)\right]}$$

$$\mu\beta+\mu\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)=\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)$$

$$\mu\beta=\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)(1-\mu)$$

So,

$$\beta=G(p)\left(\frac{1-\mu}{\mu}\right),$$

where

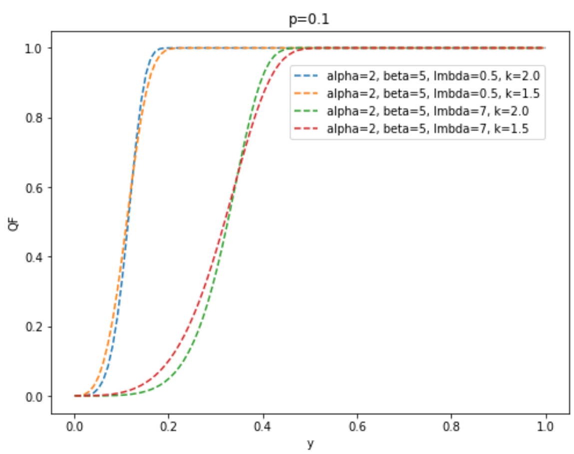
$$G(p)=\left(\lambda^{\frac{1}{\alpha}}\left[-\ln(1-p)\right]^{\frac{1}{\alpha k}}\right)$$

QCDF and QPDF are, respectively, given by

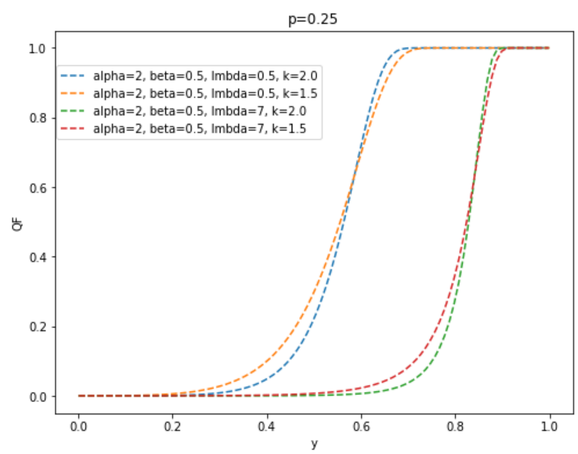
$$QF(y;\alpha,\beta,\lambda,\mu,k,p) = 1 - e^{-\left[\frac{\left(\frac{G(p)\left(\frac{1-\mu}{\mu}\right)y}{(1-y)}\right)^{\alpha}}{\lambda}\right]^{k}}, \; y \in]0;1[$$

$$Qf(y;\alpha,\beta,\lambda,k,\mu,p) = \frac{\alpha k}{(\lambda)^k}\frac{G(p)\left(\frac{1-\mu}{\mu}\right)}{(1-y)^2}\left(\frac{G(p)\left(\frac{1-\mu}{\mu}\right)y}{1-y}\right)^{\alpha k-1} e^{-\left[\frac{\left(\frac{G(p)\left(\frac{1-\mu}{\mu}\right)y}{(1-y)}\right)^{\alpha}}{\lambda}\right]^{k}} \; y \in]0;1[$$

Where $\mu \in (0,1)$ and $p \in (0,1)$. Figures (15), (16), (17), and (18) show, respectively, the plots of QCDFs and QPDFs for different quantiles and parameter values. QPDFs come in many shapes, including left- and right-tilted, decreasing, increasing, symmetrical, J-shaped, and bathtub-shaped. This shows that the regression model developed from this PDF is flexible enough to deal with short-interval data with such properties.

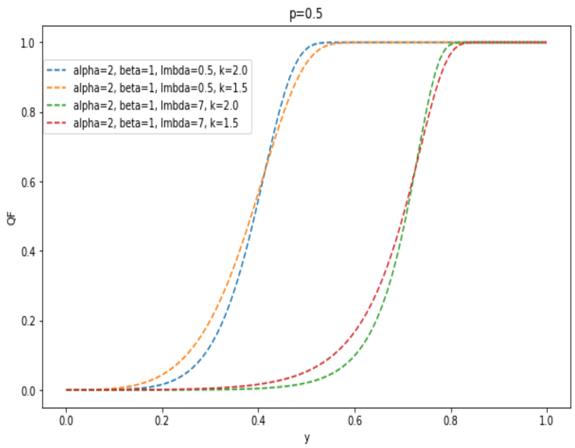


(A) p=0.1

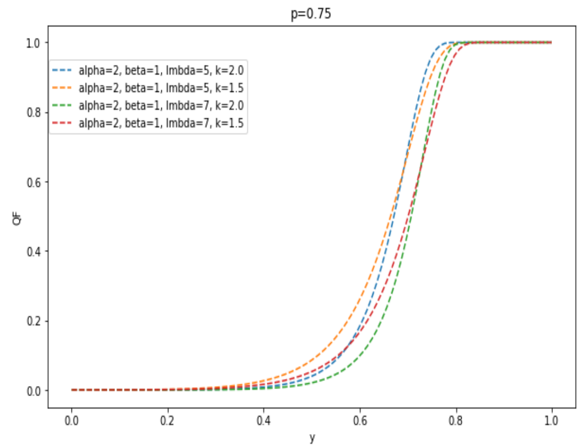


(B) p=0.25

FIGURE 15. QCDFs plots of GUW distribution for $p = 0.10$ and $p = 0.25$

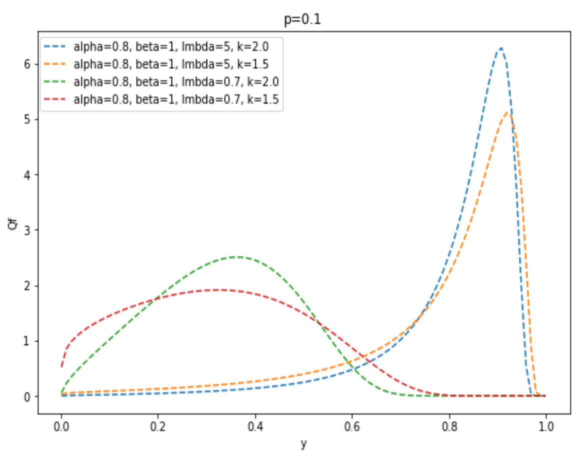


(A) $p = 0.50$

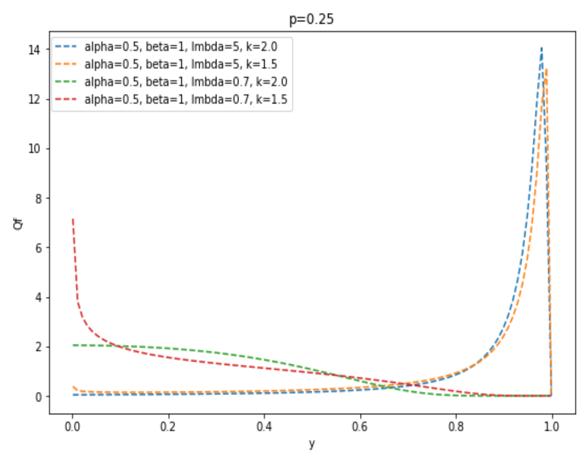


(B) $p = 0.75$

FIGURE 16. QCDFs plots of GUW distribution for $p = 0.50$ and $p = 0.75$



(A) $p = 0.1$



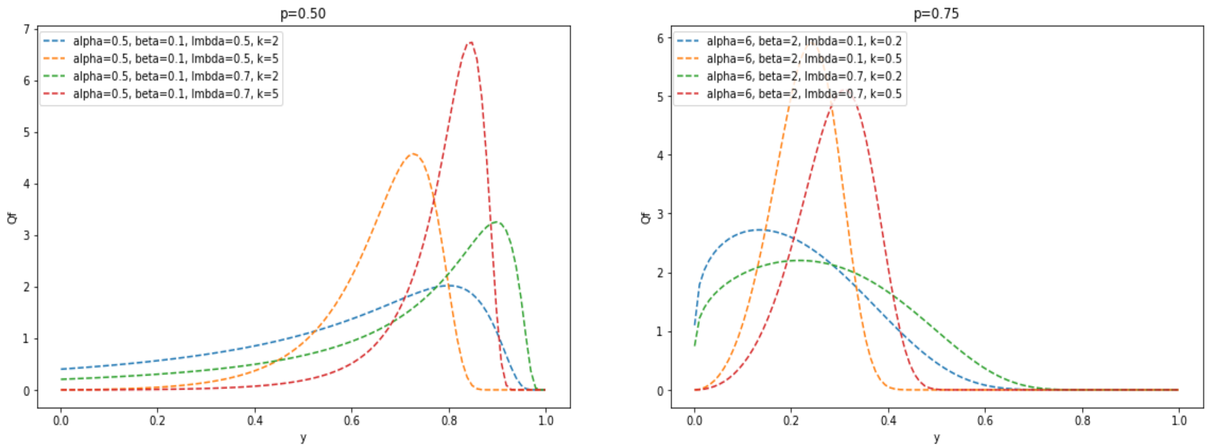
(B) $p = 0.25$

FIGURE 17. QPDFs plots of GUW distribution for $p = 0.10$ and $p = 0.25$

When we have random observations $y_1, y_2, y_3, ..., y_n$ from the GUW distribution, and predictor variables $x_1, x_2, x_3, ..., x_n$, the GUW quantile regression is established by associating the conditional quantile of the dependent variable with the predictor variables through a suitable link function, as follows:

$$g(\mu_i) = x_i^T \gamma$$

In this regression model, γ is the vector of coefficients for the independent variables, and x_i represents the vector of predictor variables for each observation i. The function g is the link function used. For this study, the logit link function is preferred for its ease of interpretation of the coefficients of the exogenous variables. Thus, the regression structure takes the following form:



(A) $p = 0.50$ (B) $p = 0.75$

FIGURE 18. QPDFs plots of GUW distribution for $p = 0.50$ and $p = 0.75$

$$logit(\mu_i) = log\left(\frac{\mu_i}{1-\mu_i}\right) = x_i^T\gamma$$

6.1. **Estimation of regression parameters.** To estimate the unidentified regression parameters using the ML technique, we compute the logarithmic likelihood by replacing:

$$\mu_i = \frac{exp(x_i^T\gamma)}{1+exp(x_i^T\gamma)}$$

in the quantile PDF. The log-likelihood function is defined as:

$$\ell(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{k},\hat{p}) = \sum_{i=1}^{n}\ln(\alpha k) - k\sum_{i=1}^{n}\ln(\lambda) + \sum_{i=1}^{n}\ln(G(p)) + \sum_{i=1}^{n}\ln\left(\frac{1-\mu_i}{\mu_i}\right) \tag{18}$$

$$-2\sum_{i=1}^{n}\ln(1-y_i) + (\alpha k-1)\sum_{i=1}^{n}\ln(G(p)) + (\alpha k-1)\sum_{i=1}^{n}\ln\left(\frac{1-\mu_i}{\mu_i}\right)$$

$$+(\alpha k-1)\sum_{i=1}^{n}\ln(y_i) - (\alpha k-1)\sum_{i=1}^{n}\ln(1-y_i) - \left[\frac{\left(\frac{G(p)\cdot\left(\frac{1-\mu_i}{\mu_i}\right)y_i}{(1-y_i)}\right)^{\alpha}}{\lambda}\right]^{k} \tag{19}$$

To obtain parameter estimates, we set the components of the score vector to zero while concurrently solving the resultant system of equations. To fit the median regression, we set p=0.50 in equation (19) and maximize the log-likelihood function. The parameter standard error estimates are computed using the ML method's large-sample characteristic. As per [33], the Fisher information matrix for parameter standard error estimation is:

$$I(\hat{\eta}) = -\left.\frac{\partial^2 \ell(\eta|y)}{\partial \eta^T \partial \eta}\right|_{\eta=\hat{\eta}}$$

6.2. **Regression modeling for educational data.** In this section, we carry out an analysis of the real data to compare the new regression with the Generalized Unit Half-Logistic Geometric (GUHLG) regression model. The data can be accessed via the link https://stats.oecd.org/index.aspx?DataSetCode=BLI [32]. They include three variables: level of education (expressed as a percentage of the 35 OECD countries, y), homicide rate (as a ratio, x_1), and life satisfaction (measured by the mean score of the Cantril scale, also known as the Self-Anchoring Striving Scale, x_2).

The regression model is formulated as follows:

$\text{logit}(\mu_i) = \gamma_0 + \gamma_1 x_1 + \gamma_2 x_2$

where μ_i represents the median for the GUW and GUHLG distributions. We calculate maximum likelihood estimates (MLE), associated standard errors, and estimated log-likelihood values for all models, as shown in table (1). This reveals that only the coefficients γ_0 and γ_1 of the GUW model are significant at the 0.05 threshold, while for the GUHLG model, none of the coefficients is significant. We also observe a negative relationship between the level of education (represented by percentage) and the country's homicide rate, but a positive relationship between the level of education and satisfaction with life. These results suggest that an increase in life satisfaction is associated with an increase in the percentage of educational achievement, while an increase in the homicide rate corresponds to a decrease in the percentage of educational achievement.

Table 1. The result for fitted regression models.

	GUW			GUHLG		
	Estimate	Standard deviation	p-value	Estimate	Standard deviation	p-value
γ_0	12.8942	6.3630	0.0427*	-1.720182	4.454259	0.699
γ_1	-2.3384	1.0181	0.0216*	0.010572	0.653999	0.987
γ_2	0.2337	0.1547	0.1309	-0.001193	0.076603	0.988

7. Simulation study

In this section, we check whether the estimates obtained by the maximum likelihood method are consistent for the GUW distribution. To do this, we carry out a simulation study using R software. We create a thousand independent samples of different sizes (100, 200, 300, 400, and 500) from the GUW distribution. For each sample, we compute the MLEs of the parameters of concern. Next, we evaluate two important statistical metrics that show how reliable and accurate the estimations are: mean bias (Bias) and root mean square error (RMSE). The outcomes of this investigation are shown in table (2). This simulation technique improves our understanding of MSE performance and consistency across multiple sample sizes for the GUW distribution.

Table 2. Values for mean, mean bias, and RMSE of simulations for GUW distribution.

	Sample size	$\alpha=0.5$ $\beta=2$ $\lambda=5$ $k=0.8$			$\alpha=0.6$ $\beta=1.5$ $\lambda=3$ $k=5$		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	100	0.3865726	0.3403587	0.003719803	0.5160878	0.09829673	-0.0007436222
	200	0.3871267	0.3338191	0.002655435	0.5160524	0.09050732	-0.0060478341
α	300	0.3875297	0.3378054	0.006666082	0.5161166	0.09409248	-0.0024559742
	400	0.3877250	0.3397468	0.007578518	0.5161333	0.09463990	-0.0026778599
	500	0.3875603	0.3411975	0.010500618	0.5161374	0.09624244	-0.0009811381
	100	0.4703165	0.3581328	0.003394391	0.7537197	0.08557860	-0.0006667927
	200	0.4708478	0.3486794	-0.001702143	0.7536183	0.07560987	-0.0061186074
β	300	0.4711307	0.3545124	0.004974935	0.7536426	0.07831139	-0.0029596014
	400	0.4713905	0.3554896	0.006533760	0.7536522	0.07828130	-0.0030667419
	500	0.4712644	0.3582302	0.011011426	0.7536705	0.08027758	-0.0014813712
	100	0.2917571	0.3026817	0.003763988	0.3319138	0.08227289	-0.0003214238
	200	0.2922474	0.3003471	0.004936670	0.3319190	0.07712006	-0.0041476335
λ	300	0.2927800	0.3025652	0.006881418	0.3319927	0.07986454	-0.0013732919
	400	0.2928630	0.3055884	0.006582468	0.3320070	0.08059554	-0.0016638122
	500	0.2926784	0.3062275	0.008259311	0.3320030	0.08181845	-0.0003755340
	100	0.3932002	0.3125611	0.0031226059	0.4643264	0.1986186	-0.0006312249
	200	0.3936681	0.3055806	-0.0001105596	0.4644523	0.1885544	-0.0086263304
	300	0.3940392	0.3101794	0.0048216971	0.4646319	0.1952523	-0.0020652042
k	400	0.3942010	0.3120068	0.0052970049	0.4646903	0.1962480	-0.0024236803
	500	0.3940626	0.3139784	0.0085886883	0.4646540	0.1987249	0.0008041739

8. Data Handling

We will utilize two appropriate data sets and compare them to some competitive models to assess the efficacy of the distribution in practical settings:

(1) Exponentiated Topp-Leone distribution (ETPLD) [34]:

$$f(x;\alpha,\beta) = 2\alpha\beta(1-x)\left(x(2-x)\right)^{\alpha-1} \\ (1-(x^{\alpha}(2-x)^{\alpha}))^{\beta-1}$$

(2) Kumaraswamy distribution (KwD) [35]:

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$$

(3) Rayleigh distribution(RD) [36].:

$$f(x;\alpha) = \frac{x}{\alpha^2}\exp\left(-\frac{x^2}{2\alpha^2}\right)$$

The predicted values of the PDF parameters were calculated and visualized using R, Matlab online and Python under Spyder. Examination of the data allows us to check whether the distribution behaves as expected in real-life circumstances.

8.1. **Dataset I.** Glass fiber strength is a measure of a glass fiber's ability to resist breakage or deformation under stress. This property is essential in many applications such as the manufacture of composite materials used in the aerospace, automotive, and construction industries.

The National Physical Laboratory (NPL) is the UK's national metrology laboratory. They carry out precise measurements in a variety of fields, including materials characterization. In this case, they measure the strength of glass fibers.

Data on the strength of 1.5 cm glass fibers measured at the National Physical Laboratory in England represent important information for understanding and characterizing the mechanical properties of glass fibers, with potential implications in various industrial and research fields. The data are presented below (https://catalog.data.gov/dataset/unemployment-insurance-data-july-2008-to-april-2013) : 0.055, 0.093, 0.125, 0.136, 0.149, 0.152, 0.158, 0.161, 0.164, 0.168, 0.173, 0.181, 0.200, 0.074, 0.104, 0.127, 0.139, 0.149, 0.153, 0.159,0.161, 0.166, 0.168, 0.176, 0.182, 0.201, 0.077, 0.111, 0.128, 0.142,0.150, 0.154, 0.160, 0.162, 0.166, 0.169, 0.176, 0.184, 0.224, 0.081,0.113, 0.129, 0.148, 0.150, 0.155, 0.161, 0.162, 0.166, 0.170, 0.177,0.184, 0.084, 0.124, 0.130, 0.148, 0.151, 0.155, 0.161, 0.163, 0.167, 0.170, 0.178, 0.189. Table (3) shows the predicted values for dataset I, with initial values of $\alpha = 2$, $\beta = 6$, $\lambda = 8$, and $k = 5$. The information criteria obtained by various models for this dataset are summarized in table (4). In addition, the empirical PDFs for dataset I are visible in the figure (19).

Table 3. Predicted values for various models

Model	α	β	λ	k
UWG	2.042932	6.166541	7.907704	4.632182
ETPLD	1.857044	6.317736	-	-
Kumaraswamy	1.656934	5.580659	-	-
Rayleigh	1.597854	-	-	-

Table 4. Measures of selection for different models

Model	**AIC**	**CAIC**	**BIC**	**HQIC**
UWG	-66.57406	-65.8844	-58.00152	-63.20243
ETPLD	82.25297	82.45297	86.53924	83.93878
Kumaraswamy	-41.42698	-41.22698	-37.14071	-39.74117
Rayleigh	122.5984	122.6639	124.7415	123.4413

8.2. **Dataset II.** 58 observations, or monthly unemployment insurance measures, were made between July 2008 and April 2013. The State of Maryland, USA's Department of Labor, Licensing and Regulation, provided these statistics.

Variable number 5, entitled "New.Claims.Filed...UCX", represents the number of new claims filed.

This variable, therefore, measures the monthly frequency with which new claims are filed by veterans for unemployment insurance benefits. It makes it possible to monitor unemployment trends among veterans and the evolution of unemployment insurance claims in this specific demographic group.

Analysis of this variable can be crucial in understanding the financial support needs of veterans, and in assessing the effectiveness of unemployment insurance programs aimed at them. It can also help identify seasonal, economic, or geographic trends that influence veterans' claims. The data are presented below ([37]):0.129, 0.103, 0.129, 0.125, 0.103, 0.111, 0.149, 0.115, 0.131, 0.106, 0.102, 0.138, 0.141, 0.140, 0.155, 0.149, 0.106, 0.132, 0.137, 0.118, 0.136, 0.157, 0.124, 0.177, 0.170,

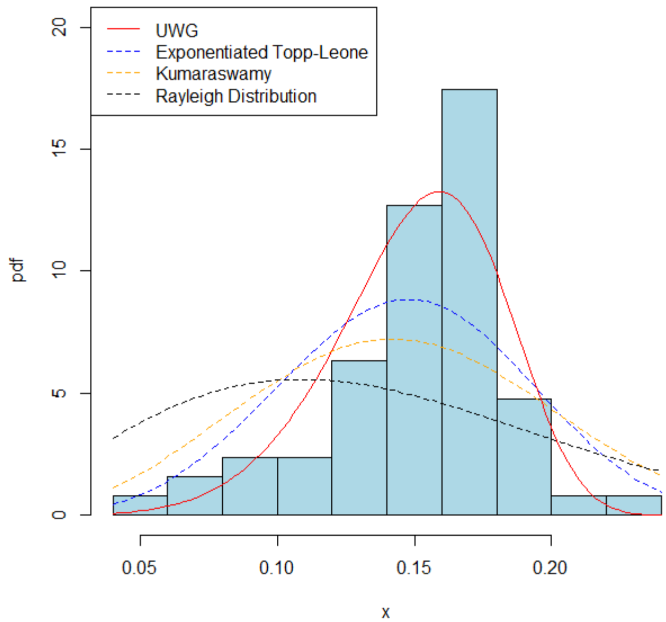


FIGURE 19. Visualization of PDFs for various models

0.203, 0.184, 0.173, 0.153, 0.153, 0.166, 0.145, 0.145, 0.148, 0.144, 0.164, 0.166, 0.178, 0.171, 0.179, 0.166, 0.127, 0.207, 0.168, 0.192, 0.182, 0.193, 0.191, 0.195, 0.194, 0.156, 0.267, 0.180, 0.145, 0.207, 0.159, 0.149, 0.172

Table (5) shows the estimated values for dataset II, with initial values of α =8, β =2, λ =5, and k =3. The information criteria obtained by various models for this dataset are listed in table (6). In addition, the figure (20) presents the empirical probability (PDFs) for dataset II.

TABLE 5. Predicted values for various models

Model	α	β	λ	k
UWG	7.902857	2.305981	4.927164	2.806308
ETPLD	7.651511	2.087676	–	–
Kumaraswamy	7.652575	2.151124	–	–
Rayleigh	7.613136	–	–	–

Table 6. Measures of selection for various models

Model	AIC	CAIC	BIC	HQIC
UWG	-57.56362	-56.8089	-49.32184	-54.35328
ETPLD	599.1901	599.4082	603.3109	600.7952
Kumaraswamy	1121.127	1121.345	1125.248	1122.732
Rayleigh	692.6526	692.724	694.713	693.4551

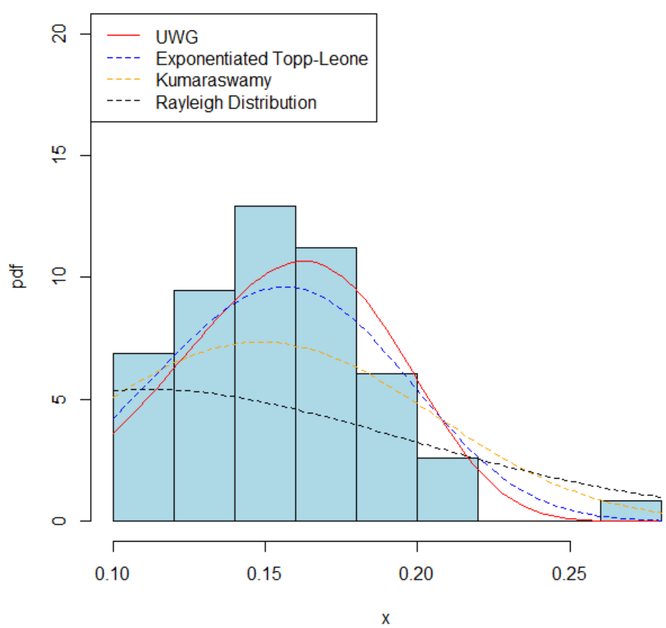


Figure 20. Visualization of PDFs for various models

After examining tables (3), (4), (5), (6), as well as figures (19) and (20), we can conclude that the GUW model proves to be more compatible with datasets I and II than competing models. Its flexibility enables it to adapt to domains as varied as materials engineering and unemployment insurance data.

9. Conclusion

In this study, we introduced and investigated the GUW distribution, a novel statistical model derived as a generalization of the Weibull distribution. We derived several statistical properties of the GUW distribution, including order statistics, quantile , Rényi entropy, and moments. Parameter estimation was carried out using the maximum likelihood estimation method. In addition, we developed key actuarial risk measures such as value at Risk, tail Value at Risk, tail conditional variance, and tail variance risk for the new distribution.

The practical relevance of the GUW distribution was demonstrated through applications to real-world datasets from the banking and materials engineering sectors. Its performance was compared against existing models, such as the ETPLD, KwD, and RD distributions. Furthermore, we proposed a quantile regression model based on the GUW quantile function. The results suggest that the GUW distribution can be a valuable tool for professionals in materials science and financial risk analysis, particularly those involved in modeling and statistical decision-making.

10. Future work

We envisage several promising avenues of research. Firstly, we could extend this study by exploring other methods of validating customized distributions, such as specific statistical tests or cross-validation techniques. This would enable us to deepen our understanding of the performance of these distributions in different contexts.

In addition, we could seek to improve the flexibility and accuracy of the GUW distribution by introducing additional parameters or exploring alternative distribution functions. This could enable us better to model complex phenomena or data with particular characteristics.

Finally, we could explore applying the data simulation approach in specific fields such as health, the environment, or the social sciences. By adapting the proposed methodology to the needs and particularities of these fields, we could contribute to concrete problem-solving and informed decision-making.

By combining these approaches, we could extend the potential applications of this work and provide valuable tools for research and practice in various fields.

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