Eur. J. Stat. 1 (2021) 108-131
doi: 10.28924/ada/stat.1.108

The Alpha Power Transformed Dagum Distribution: Properties and Applications

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Abstract. In this study, we propose a new extension of the Dagum distribution called the alpha power transformed Dagum distribution. Basic statistical properties of the new distribution such as; quantile function, raw and incomplete moments, moment generating function, order statistics, Rényi entropy, stochastic ordering and stress strength model are investigated. The characterizations of the new model is investigated. The method of maximum likelihood is used to estimate the model parameters of the new distribution and the observed information matrix is also obtained. A Monte Carlo simulation is presented to examine the behavior of the parameter estimates. The applicability of the new model is demonstrated by means of three applications.

1. Introduction

The Dagum distribution was introduced by Dagum (1977) for analysing income data as an alternative model to the log-normal and Pareto distributions. This distribution has been used in different fields such as, reliability and survival analysis,

Received: 24 Sep 2021.
Key words and phrases. alpha-power transformed family; characterizations; Dagum distribution; probability weighted moments; Rényi entropy; stochastic ordering; stress strength model.

wealth and income data and meteorological data. The Dagum distribution has cumulative distribution function (cdf) given by

$$F(x)=\left(1+\lambda x^{-\beta}\right)^{-\theta},\ x\geq 0, \tag{1}$$

and its corresponding probability density function (pdf) is

$$f(x)=\lambda\beta\theta x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1},\ x>0, \tag{2}$$

where $\lambda>0$ is the scale parameter and $\beta,\theta>0$ are the shape parameters.

There are many extensions of the Dagum distribution such as; Beta-Dagum by Domma and Condino (2013), weighted Dagum by Oluyede and Ye (2014), Weibull Dagum by Tahir et al. (2016), power log Logistic Dagum by Bakouch et al. (2017) and odd log Logistic Dagum by Domma et al. (2018).

Moreover, if $g(x;\phi)$ and $G(x;\phi)$ denote the pdf and cdf of a baseline model with parameter vector ϕ. Mahdavi and Kundu (2017) introduced a new method of generating continuous distributions called the alpha-power transformation (APT for short) with cdf and pdf given by

$$F(x;\phi)=\frac{\alpha^{G(x;\phi)}-1}{\alpha-1},\ \alpha>0,\alpha\neq 1,x\in R, \tag{3}$$

$$f(x;\phi)=\frac{\log(\alpha)}{\alpha-1}g(x;\phi)\alpha^{G(x;\phi)},\ \alpha>0,\alpha\neq 1,x\in R. \tag{4}$$

The goal of this article is to derive a new distribution from the Dagum distribution by Alpha-power transformation as suggested by Mahdavi and Kundu (2017), called alpha power transformed Lindley (APTL) distribution . This concept of generalization is well established in the statistical literature (see Dey et al. (2017a, 2017b)). The proposed distribution encompasses the behavior of and provides better fits than some well known lifetime distributions in the literature; please see in application section 6. We are motivated to introduce the alpha power transformed Dagum (APTD for short) distribution because *(i)* it is capable of modeling increasing, decreasing, constant, bathtub and upside-down bathtub shaped hazard rates; *(ii)* it can be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other known distributions and can also be used in a variety of problems in different areas; see also in application section 6; and *(iii)* three real data applications show that it compares well with other competing lifetime distributions. The cdf and pdf of the APTD distribution are given, respectively, by

$$F(x;\phi)=\frac{\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}}-1}{\alpha-1},\ \alpha>0,\alpha\neq 1,\lambda,\beta,\theta>0,x\geq 0, \tag{5}$$

$$f(x;\phi)=\frac{\lambda\beta\theta\log(\alpha)}{\alpha-1}x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}},\ x>0. \tag{6}$$

Henceforth, a random variable with density (6) will be denoted by $X\sim \mathrm{APTD}(\alpha,\beta,\lambda,\theta)$. The hazard function $\tau(x)$ for the APTD distribution is given by

$$\tau(x)=\frac{\lambda\beta\theta\log(\alpha)x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}}}{\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}}-1},\ x>0. \tag{7}$$

The remainder of this paper is orgainzed as follows. In Section 2, main mathematical properties of the APTD model are studied. In Section 3, certain characterizations of the new disatribution are presented. In Section 4, the maximum likelihood estimates are obtained for the model parameters. A simulation study is conducted in Section 5. In Section 6, we provide three applications. Section 7 offers some concluding remarks.

2. Mathematical Properties

In this section, we will study some main properties of the APTD distribution.

2.1. Quantile Function

The quantile function of the APTD distribution $Q(u)=F^{-1}(u)$ for $u\in(0,1)$, $\alpha>0$, $\beta>0$, $\lambda>0$ and $\theta>0$ is the solution of the non-linear equation

$$Q(u)=G^{-1}\left\{\left(\frac{1}{\lambda}\left[\left\{\frac{\log\left(1+(\alpha-1)u\right)}{\log(\alpha)}\right\}^{-1/\theta}-1\right]\right)^{-1/\beta}\right\}. \tag{8}$$

2.2. The Shape of the APTD Distribution

The shape of the density and hazard functions of the APTD distribution can be described mathematically. The critical points of the density function are the roots of the following equation:

$$-\left(\frac{\beta+1}{x}\right)+\left(\frac{\lambda\beta(\theta+1)x^{-\beta-1}}{1+\lambda x^{-\beta}}\right)+\left\{\frac{\lambda\beta\theta\log(\alpha)x^{-\beta-1}}{\left(1+\lambda x^{-\beta}\right)^{\theta+1}}\right\}=0.$$

Further, the critical points of the hazard function are the roots of the following equation:

$$-\left(\frac{\beta+1}{x}\right)+\left(\frac{\lambda\beta(\theta+1)x^{-\beta-1}}{1+\lambda x^{-\beta}}\right)+\left\{\frac{\lambda\beta\theta\log(\alpha)x^{-\beta-1}}{\left(1+\lambda x^{-\beta}\right)^{\theta+1}}\right\}$$

$$-\left\{\frac{\lambda\beta\theta\log(\alpha)x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}}}{\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}}-1}\right\}=0.$$

Some plots of the density and hazard functions are displayed in Figures 1 and 2.

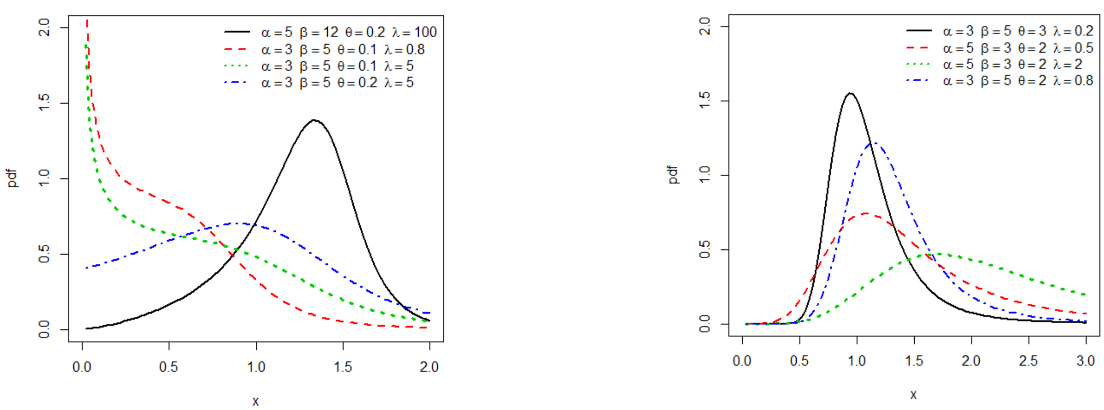


Figure 1: Plots of the APTD pdf for selected parameter values.

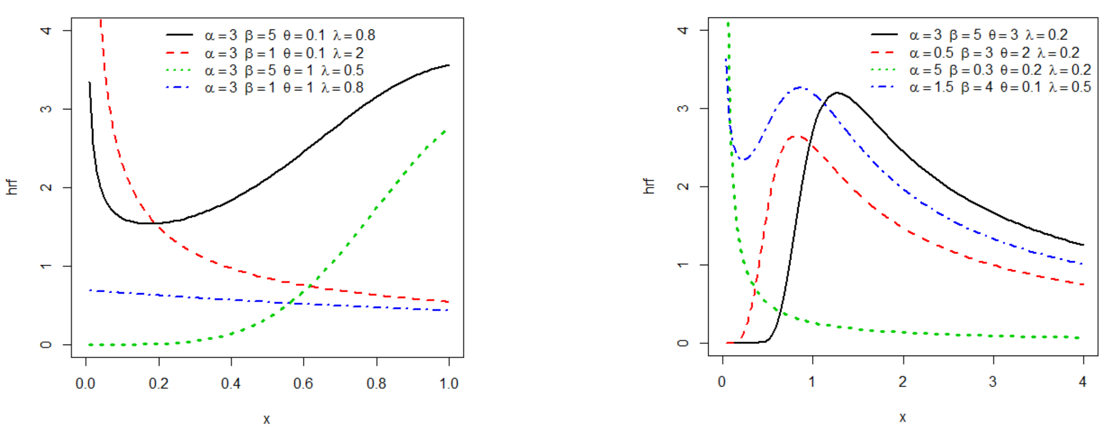


Figure 2: Plots of the APTD hrf for selected parameter values.

2.3. Moments and Moment Generating Function

Let X be a random variable with the APTD distribution, then the ordinary moment, say μ_r', is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \sum_{w=0}^{\infty} \frac{\lambda\,\beta\,\theta\left(\log\alpha\right)^{w+1}}{(\alpha-1)\,w!}\int_0^{\infty} x^{r-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta(w+1)-1} dx$$

$$= \sum_{w=0}^{\infty} \frac{\lambda^{-\theta(w+1)}\,\beta\,\theta\left(\log\alpha\right)^{w+1}}{(\alpha-1)\,w!}\int_0^{\infty} x^{r+\beta\theta(w+1)-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\theta(w+1)-1} dx.$$

Letting $y=\left(1+x^{\beta}/\lambda\right)^{-1}$, we obtain

$$\mu_r' = \sum_{w=0}^{\infty} \frac{\lambda^{r/\beta}\,\theta\left(\log\alpha\right)^{w+1}}{(\alpha-1)\,w!}\int_0^{1} y^{-r/\beta}\left(1-y\right)^{r/\beta+\theta(w+1)-1} dy$$

$$= \sum_{w=0}^{\infty} \frac{\lambda^{r/\beta}\,\theta\left(\log\alpha\right)^{w+1}}{(\alpha-1)\,w!} B\left(1-r/\beta, r/\beta+\theta(w+1)\right), \tag{9}$$

where, $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is the beta function. Substituting $r=1,2,3,4$ in (9), we obtain the $\text{mean}=\mu_1'$, $\text{variance}=\mu_2'-\mu_1'^2$, $\text{skewness}=\mu_3'/\mu_1'^{3/2}$ and $\text{kurtosis}=\mu_4'/\mu_1'^2$.

The plots of the skewness and kurtosis of the APTD distribution for parameters $\alpha=0.5, \beta=0.8$ are displayed in Figure (3). Based on these plots, we conclude: if the parameters α and β increase, the skewness decreases and kurtosis increases. Moreover, parameter β has more significant effect on skewness and kurtosis measures than parameter α.

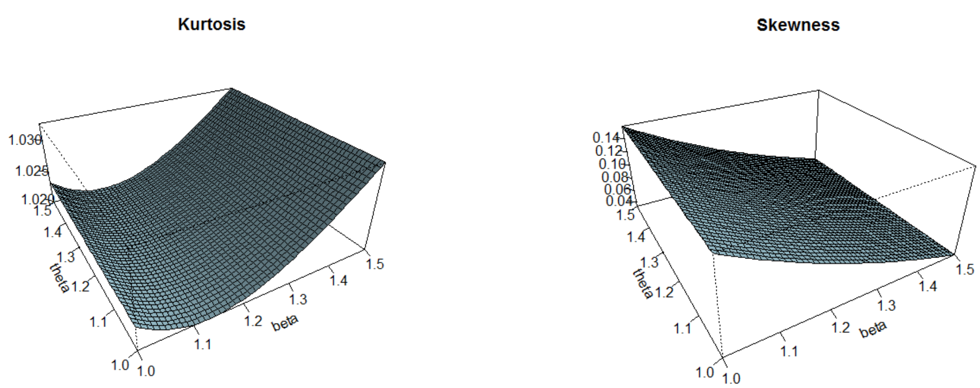


Figure 3: The skewness and kurtosis of the APTD distribution.

The nth central moment of the APTD distribution, say μ_n, can be obtained from

$$\mu_n=\sum_{r=0}^{n}\binom{n}{r}(-\mu_1')^{n-r}E(x^r)$$
$$=\sum_{r=0}^{n}\sum_{w=0}^{\infty}\binom{n}{r}\frac{(-\mu_1')^{n-r}\lambda^{r/\beta}\theta(\log\alpha)^{w+1}}{(\alpha-1)w!}B\left(1-r/\beta,r/\beta+\theta(w+1)\right). \tag{10}$$

The rth incomplete moment of the APTD distribution, denoted by $\varphi_s(t)$, is

$$\varphi_s(t)=\int_{-\infty}^{t}x^s f(x)\,dx$$
$$=\sum_{w=0}^{\infty}\frac{\lambda\beta\theta(\log\alpha)^{w+1}}{(\alpha-1)w!}\int_0^t x^{s-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta(w+1)-1}dx$$
$$=\sum_{w,j=0}^{\infty}\frac{\lambda^{j+1}\beta\theta(\log\alpha)^{w+1}}{(\alpha-1)w!}\binom{-\theta(w+1)-1}{j}\int_0^t x^{s-\beta(j+1)-1}dx$$
$$=\sum_{w,j=0}^{\infty}\binom{-\theta(w+1)}{j}\frac{\lambda^{j+1}\beta\theta(\log\alpha)^{w+1}t^{s-\beta(j+1)}}{(\alpha-1)\left(s-\beta(j+1)\right)w!}. \tag{11}$$

The moment generating function of the APTD distribution, denoted by $M_X(t)$, can be obtained by

$$M_X(t)=E(e^{tX})=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r)$$
$$=\sum_{r,w=0}^{\infty}\frac{t^r\lambda^{r/\beta}\theta(\log\alpha)^{w+1}}{(\alpha-1)w!r!}B\left(1-\frac{r}{\beta},\frac{r}{\beta}+\theta(w+1)\right). \tag{12}$$

2.4. Probability Weighted Moments

The $(r+s)$th PWM of a random variable X with the APTD distribution, say $M_{r,s}$, is given by

$$M_{r,s}=E\left(X^rF(x)^s\right)=\int_{-\infty}^{\infty}X^rF(x)^s f(x)dx. \tag{13}$$

From (5) and (6), we have

$$f(x)F(x)^s = \frac{\lambda\beta\theta\log(\alpha)}{\alpha-1} x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1} \alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}} \left\{\frac{\alpha^{\left(1+\lambda x^{-\beta}\right)^{-\theta}}-1}{\alpha-1}\right\}^s$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \lambda\beta\theta\log(\alpha)}{(\alpha-1)^{s+1}} \binom{s}{j} x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1} \alpha^{(s-j+1)\left(1+\lambda x^{-\beta}\right)^{-\theta}}$$

$$= \sum_{j,w=0}^{\infty} \frac{(-1)^j \lambda\beta\theta\left(\log(\alpha)\right)^{w+1}(s-j+1)^w}{(\alpha-1)^{s+1} w!} \binom{s}{j}$$
$$\times x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta(w+1)-1}. \quad (14)$$

Subsituting from (14) in (13), we obtain

$$M_{r,s} = \sum_{j,w=0}^{\infty} \frac{(-1)^j \lambda^{r/\beta}\theta\left(\log(\alpha)\right)^{w+1}(s-j+1)^w}{(\alpha-1)^{s+1} w!} \binom{s}{j}$$
$$\times B\left(1-r/\beta, r/\beta+\theta(w+1)\right). \quad (15)$$

2.5. Order Statistics

Let $X_{1:n} \le X_{2:n}, \ldots \le X_{n:n}$ be order statistics corresponding to a sample of size n from the APTD distribution. The pdf of $X_{k:n}$, the kth order statistic, is given by

$$f_{X_{k:n}}(x) = \frac{1}{B(k,n-k+1)} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} f(x)F(x)^{k+i-1}. \quad (16)$$

Based on (5) and (6), we have

$$f(x)F(x)^{k+i-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda\beta\theta\log(\alpha)}{(\alpha-1)^{k+i}} \binom{k+i-1}{j}$$
$$\times x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1} \alpha^{(k+i-j)\left(1+\lambda x^{-\beta}\right)^{-\theta}}. \quad (17)$$

Using (17) in (16), we have

$$f_{X_{k:n}}(x) = \sum_{i=0}^{n-k}\sum_{j=0}^{\infty} \frac{(-1)^{j+j} \lambda\beta\theta\log(\alpha)}{(\alpha-1)^{k+i} B(k,n-k+1)} \binom{n-k}{i}\binom{k+i-1}{j}$$
$$\times x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1} \alpha^{(k+i-j)\left(1+\lambda x^{-\beta}\right)^{-\theta}}. \quad (18)$$

Furthermore, the rth moment of kth order statistic for the APTD distribution is given by

$$E\left(x_{k:n}^{r}\right)=\sum_{i=0}^{n-k}\sum_{j=0}^{\infty}\frac{(-1)^{j+j}\lambda\beta\theta\log(\alpha)}{(\alpha-1)^{k+i}B(k,n-k+1)}\binom{n-k}{i}\binom{k+i-1}{j}$$
$$\times\int_0^\infty x^{r-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}\alpha^{(k+i-j)\left(1+\lambda x^{-\beta}\right)^{-\theta}}dx$$

$$=\sum_{i=0}^{n-k}\sum_{j,w=0}^{\infty}\frac{(-1)^{j+j}\lambda\beta\theta\left(\log(\alpha)\right)^{w+1}(k+i-j)^{w}}{(\alpha-1)^{k+i}w!B(k,n-k+1)}\binom{n-k}{i}\binom{k+i-1}{j}$$
$$\times\int_0^\infty x^{r-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta(w+1)-1}dx$$

$$E\left(x_{k:n}^{r}\right)=\sum_{i=0}^{n-k}\sum_{j,w=0}^{\infty}\frac{(-1)^{j+j}\lambda^{r/\beta}\theta\left(\log(\alpha)\right)^{w+1}(k+i-j)^{w}}{(\alpha-1)^{k+i}w!}\binom{n-k}{i}\binom{k+i-1}{j}$$
$$\times\frac{B\left(1-r/\beta,r/\beta+\theta(w+1)\right)}{B(k,n-k+1}. \tag{19}$$

2.6. Rényi Entropy

The entropy is a useful approach used in different areas such as queuing theory and statistics.The Rényi entropy is defined as

$$I_R(X)=(1-\mu)^{-1}\log\int_{-\infty}^{\infty}f(x)^{\mu}\,dx,\ \mu>0,\ \mu\neq 0.$$

Using (6) and after some manipulations, we have

$$I_R(X)=(1-\mu)^{-1}\log\left\{\begin{array}{l}\sum_{w=0}^{\infty}\frac{\beta^{\mu-1}\theta^{\mu}\left(\log(\alpha)\right)^{\mu+w}\mu^{w}}{w!\,\lambda^{(\mu-1)/\beta}}\\ \times B\left(\mu+(\mu-1)/\beta,\theta(\mu+w)-(\mu-1)/\beta\right)\end{array}\right\}. \tag{20}$$

2.7. Stochastic Ordering

Stochastic ordering is a vital criterion that is used in different fields to examine the comparative behavior. According to Shaked and Shanthikumar (2007), a random variable X_1 is said to be smaller than another random variable X_2 in the likelihood ratio order $\left(X_1\leq_{lr}X_2\right)$ if $f_1(x)/f_2(x)$ decreases in x. The following theorem shows that the APTD distribution is ordered in likelihood ratio ordering if the appropriate assumptions exist.

Theorem 1: Let $X_1 \sim \text{APTD}(\alpha_1,\beta_1,\lambda_1,\theta_1)$ and $X_2 \sim \text{APTD}(\alpha_2,\beta_2,\lambda_2,\theta_2)$. If $\alpha_1=\alpha_2$, $\lambda_1=\lambda_2,\theta_1=\theta_2$ and $\beta_1\geq\beta_2$, then $X_1 \leq_{lr} X_2$.

Proof: We have

$$\frac{f_1(x)}{f_2(x)}=\frac{\frac{\lambda_1\beta_1\theta_1\log(\alpha_1)}{\alpha_1-1}x^{-\beta_1-1}\left(1+\lambda_1x^{-\beta_1}\right)^{-\theta_1-1}\alpha_1^{\left(1+\lambda_1x^{-\beta_1}\right)^{-\theta_1-1}}}{\frac{\lambda_2\beta_2\theta_2\log(\alpha_2)}{\alpha_2-1}x^{-\beta_2-1}\left(1+\lambda_2x^{-\beta_2}\right)^{-\theta_2-1}\alpha_2^{\left(1+\lambda_2x^{-\beta_2}\right)^{-\theta_2-1}}}.$$

Then

$$\log\frac{f_1(x)}{f_2(x)}=\log\left\{\frac{\lambda_1\beta_1\theta_1\log(\alpha_1)}{\alpha_1-1}\right\}-(\beta_1+1)\log(x)-(\theta_1+1)\log\left(1+\lambda_1x^{-\beta_1}\right)$$

$$+\left(1+\lambda_1x^{-\beta_1}\right)^{-\theta_1}\log(\alpha_1)-\log\left\{\frac{\lambda_2\beta_2\theta_2\log(\alpha_2)}{\alpha_2-1}\right\}+(\beta_2+1)\log(x)$$

$$+(\theta_2+1)\log\left(1+\lambda_2x^{-\beta_2}\right)-\left(1+\lambda_2x^{-\beta_2}\right)^{-\theta_2}\log(\alpha_2).$$

If $\alpha_1=\alpha_2,\lambda_1=\lambda_2,\theta_1=\theta_2$ and $\beta_1\geq\beta_2$, then we have

$$\frac{d}{dx}\log\frac{f_1(x)}{f_2(x)}=-\left(\frac{\beta_1+1}{x}\right)+\lambda_1\beta_1(\theta_1+1)\left(\frac{x^{-\beta_1-1}}{1+\lambda_1x^{-\beta_1}}\right)$$

$$+\lambda_1\beta_1\theta_1\log(\alpha_1)x^{-\beta_1-1}\left(1+\lambda_1x^{-\beta_1}\right)^{-\theta_1-1}+\left(\frac{\beta_2+1}{x}\right)$$

$$-\lambda_2\beta_2(\theta_2+1)\left(\frac{x^{-\beta_2-1}}{1+\lambda_2x^{-\beta_2}}\right)$$

$$-\lambda_2\beta_2\theta_2\log(\alpha_2)x^{-\beta_2-1}\left(1+\lambda_2x^{-\beta_2}\right)^{-\theta_2-1}<0.$$

Consequentely, $f_1(x)/f_2(x)$ decreases in x and hence $X_1 \leq_{lr} X_2$.

2.8. Stress Strength Model

Let X_1 and X_2 be two independent random variables with $X_1 \sim \text{APTD}(\alpha_1,\beta_1,\lambda_1,\theta_1)$ and $X_2 \sim \text{APTD}(\alpha_2,\beta_2,\lambda_2,\theta_2)$ distributions. Then, the stress strength model is given by

$$R=\Pr(X_2<X_1)=\int_0^\infty f_1(\alpha_1,\beta_1,\lambda_1,\theta_1)F_2(\alpha_2,\beta_2,\lambda_2,\theta_2)\,dx$$

$$= \underbrace{\Omega \int_0^\infty x^{-\beta_1 - 1} \left(1 + \lambda_1 x^{-\beta_1}\right)^{-\theta_1 - 1} e^{\left(1+\lambda_1 x^{-\beta_1}\right)^{-\theta_1} \log(\alpha_1) + \left(1+\lambda_2 x^{-\beta_2}\right)^{-\theta_2} \log(\alpha_2)} dx}_{\mathrm{H}}$$

$$-\underbrace{\Omega \int_0^\infty x^{-\beta_1 - 1} \left(1 + \lambda_1 x^{-\beta_1}\right)^{-\theta_1 - 1} e^{\left(1+\lambda_1 x^{-\beta_1}\right)^{-\theta_1} \log(\alpha_1)} dx}_{\mathrm{E}}$$

where,

$$\Omega = \frac{\lambda_1 \beta_1 \theta_1 \log(\alpha_1)}{(\alpha_1 - 1)(\alpha_2 - 1)},$$

$$\mathrm{H} = \sum_{w,h,j=0}^{\infty} \frac{\lambda_1^{-\beta_2 j/\beta_1 - 1} \lambda_2^{j} \left(\log(\alpha_1)\right)^{w-h} \left(\log(\alpha_2)\right)^{h}}{\beta_1 \, w!}$$
$$\times \binom{w}{h} \binom{-\theta_2 h}{j} B\left(1 + \beta_2 j/\beta_1, \theta_1 (w - h + 1) - \beta_2 j/\beta_1\right)$$

and

$$\mathrm{E} = \sum_{w,\ell=0}^{\infty} \frac{(-1)^{\ell} \left(\log(\alpha_1)\right)^{w}}{w! \beta_1 \lambda_1 \left(\theta_1 (w+1) + \ell + 2\right)} \binom{-2}{\ell}.$$

Therefore, the stress strength model for the APTD distribution is

$$R = \sum_{w=0}^{\infty} \eta_w \left(\sum_{h,j=0}^{\infty} \delta_{h,j} - \sum_{\ell=0}^{\infty} \tau_\ell \right), \tag{21}$$

where

$$\eta_w = \frac{\theta_1 \left(\log(\alpha_1)\right)^{w+1}}{(\alpha_1 - 1)(\alpha_2 - 1) w!},$$

$$\delta_{h,j} = \lambda_2^{j} \left(\log \frac{\alpha_2}{\alpha_1} \right)^{h} B\left(1 + \beta_2 j/\beta_1, \theta_1 (w - h + 1) - \beta_2 j/\beta_1\right),$$

and

$$\tau_\ell = \frac{(-1)^{\ell}}{\theta_1 (w+1) + \ell + 2} \binom{-2}{\ell}.$$

3. Characterization Results

This section is devoted to the characterizations of the APTD distribution in two directions: *(i)* based on the ratio of two truncated moments and *(ii)* in terms of the hazard function. Note that *(i)* can be employed also when the cdf does not have a closed form. We present our characterizations *(i)-(ii)* in two subsections.

3.1. Characterizations based on two truncated moments

This subsection deals with the characterizations of APTD distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glanzel (1987), see Theorem 1 of Appendix A. The result, however, holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 3.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let, $q_1=\alpha^{-\left(1+\lambda x^{-\beta}\right)^{-\theta}}$ and $q_2(x)=q_1(x)\left(1+\lambda x^{-\beta}\right)^{-\theta}$ for $x>0$. The random variable X has pdf (6) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x)=\frac{1}{2}\left\{1+\left(1+\lambda x^{-\beta}\right)^{-\theta}\right\},\ x>0.$$

Proof. Suppose the random variable X has pdf (6), then

$$\left(1-F(x)\right)E\left[q_1(X)\middle|X\geq x\right]=\frac{\log(\alpha)}{\alpha-1}\left\{1-\left(1+\lambda x^{-\beta}\right)^{-\theta}\right\},\ x>0,$$

and

$$\left(1-F(x)\right)E\left[q_2(X)\middle|X\geq x\right]=\frac{\log(\alpha)}{2(\alpha-1)}\left\{1-\left(1+\lambda x^{-\beta}\right)^{-2\theta}\right\},\ x>0.$$

Further,

$$\xi(x)q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-\left(1+\lambda x^{-\beta}\right)^{-\theta}\right\}>0,\ \text{for }x>0.$$

Conversely, if ξ is of the above form, then

$$s'(x)=\frac{\xi'(x)q_1(x)}{\xi(x)\,q_1(x)-q_2(x)}=\frac{\lambda\beta\theta\left(1+\lambda x^{-\beta}\right)^{-\theta-1}}{1-\left(1+\lambda x^{-\beta}\right)^{-\theta}},\ x>0,$$

and consequently

$$s(x)=-\log\left\{1-\left(1+\lambda x^{-\beta}\right)^{-\theta}\right\},\ x>0.$$

Now, according to Theorem 1, X has density (6).

Corollary 3.1. Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3.1. The random variable X has pdf (6) if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the following differential equation

$$\frac{\xi'(x)q_1(x)}{\xi(x)\,q_1(x)-q_2(x)}=\frac{\lambda\beta\theta x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}}{1-\left(1+\lambda x^{-\beta}\right)^{-\theta}},\ x>0.$$

Corollary 3.2. The general solution of the differential equation in Corollary 3.1 is

$$\xi(x)=\left\{1-\left(1+\lambda x^{-\beta}\right)^{-\theta}\right\}^{-1}\left[-\int\lambda\beta\theta x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant. We like to point out that one set of functions satisfying the above differential equation is given in Proposition 3.1 with $D=1/2$. Clearly, there are other triplets (q_1,q_2,ξ) which satisfy conditions of Theorem 1.

3.2 Characterization in terms of hazard function

The hazard function, τ_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{\tau_F'(x)}{\tau_F(x)}-\tau_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of APTD distribution in terms of the hazard function.

Proposition 3.2. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The random variable X has pdf (6) if and only if its hazard function $\tau_F(x)$ satisfies the following differential equation

$$\tau_F'(x)-\frac{\lambda\beta\theta\log(\alpha)\,x^{-\beta-1}}{\left(1+\lambda x^{-\beta}\right)^{\theta+1}}\tau_F(x)=\lambda\beta\theta\log(\alpha)\,\alpha^{-\left(1+\lambda x^{-\beta}\right)^{-\theta}}\frac{d}{dx}\left\{\frac{x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}}{\alpha^{-\left(1+\lambda x^{-\beta}\right)^{-\theta}}-1}\right\},\ x>0.$$

Proof. If X has pdf (6), then clearly the above differential equation holds. If the differential equation holds, then

$$\frac{d}{dx}\left\{\alpha^{-\left(1+\lambda x^{-\beta}\right)^{-\theta}}\tau_F(x)\right\}=\lambda\beta\theta\log(\alpha)\frac{d}{dx}\left\{\frac{x^{-\beta-1}\left(1+\lambda x^{-\beta}\right)^{-\theta-1}}{\alpha^{-\left(1+\lambda x^{-\beta}\right)^{-\theta}}-1}\right\},$$

from which we arrive at the hazard function corresponding to pdf (6).

4. Maximum Likelihood Estimation

The maximum likelihood estimates (MLEs) for the model parameters of the APTD distribution will be discussed in this section. Let $x_1, x_2, \ldots, x_n$ be observed values of a random sample from the APTD distribution, then the corresponding log-likelihood function is given by

$$\begin{aligned}\ell = n\{\log(\lambda)+\log(\beta)+\log(\theta)+\log(\log(\alpha))-\log(\alpha-1)\}\\ -(\beta-1)\sum_{i=1}^{n}\log(x_i)-(\theta-1)\sum_{i=1}^{n}\log\left(1+\lambda x_i^{-\beta}\right)\\ +\log(\alpha)\sum_{i=1}^{n}\left(1+\lambda x_i^{-\beta}\right)^{-\theta}.\end{aligned} \tag{22}$$

The components of the score vector $\nabla\ell=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\theta}\right)$ are the following:

$$\frac{\partial\ell}{\partial\alpha}=\frac{1}{\alpha}\left\{\frac{n\left[\alpha\left(1-\log(\alpha)\right)-1\right]}{(\alpha-1)\log(\alpha)}\right\}+\sum_{i=1}^{n}\left(1+x_i^{-\beta}\right)^{-\theta}, \tag{23}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\log(x_i)+\lambda(\theta+1)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)}{1+\lambda x_i^{-\beta}}\right)\\ +\lambda(\theta+1)\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+1}}\right),\end{aligned} \tag{24}$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}-(\theta+1)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}}{1+\lambda x_i^{-\beta}}\right)-\theta\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+1}}\right), \tag{25}$$

and

$$\frac{\partial\ell}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\log\left(1+\lambda x_i^{-\beta}\right)+\log(\alpha)\sum_{i=1}^{n}\left(\frac{\log\left(1+\lambda x_i^{-\beta}\right)}{\left(1+\lambda x_i^{-\beta}\right)^{\theta}}\right). \tag{26}$$

The MLEs, say $\hat{\Theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$, of $\Theta=(\alpha,\beta,\lambda,\theta)^T$ can be obtained by equating the system of nonlinear equations (23) to (26) to zero and solving them simultaneously. The components of the observed information matrix of the model parameters are obtained in Appendix B.

5. Simulation Study

In this section, we evaluate the performance of the MLEs of the model parameters of the APTD distribution via a Monte Carlo simulation. We compare MLEs to minimum spacing absolute distance estimator (MSADE) and minimum spacing absolute-log distance estimator (MSALDE) of the APTD distribution (see Torabi and Bagheri (2010) and Torabi and Montazeri (2014)). The simulation is performed for sample size $n=50,100,200,300$ and 500. The parameters values are $\alpha=1.5, \beta=1.0, \theta=0.5$ and $\lambda=0.8$. For each sample size, we compute the MLEs, MSADEs and MSALDEs of the parameters. We repeat this process 3,000 times and obtain the average estimates (AEs), biases and mean square error (MSEs). The results are reported in Table 1. It is observed that the MSEs of MLE is less than MSEs of MSADEs and MSALDEs for large sample size. We can verify that the estimates are stable and quite close to the true parameter values for these sample sizes. As the sample size increases the MSE decreases in all cases. Accordingly, the sample size n plays an important role in determining the efficiency of the parameters since when n increases some additional information is gathered.

Table 1: Estimated AE and MSE of MLE, MSADE and MSALDE of the model parameters of the APTD distribution.

Different method		MLE			MSADE			MSALDE		
n	Parameter	A.E	Bias	MSE	A.E	Bias	MSE	A.E	Bias	MSE
50	α	2.022	0.522	2.948	1.614	0.114	1.004	0.574	0.074	0.067
	β	2.346	1.346	6.872	1.325	0.325	2.526	1.335	0.335	3.003
	θ	0.551	0.081	0.092	0.517	0.117	0.144	0.600	0.100	0.099
	λ	0.975	0.175	0.248	0.863	0.063	0.093	0.835	0.035	0.169
100	α	1.693	0.193	0.689	1.597	0.097	0.641	1.507	0.007	0.727
	β	1.328	0.328	3.630	1.210	0.210	2.079	1.325	0.325	2.878
	θ	0.553	0.053	0.087	0.531	0.091	0.103	0.598	0.098	0.087
	λ	0.915	0.115	0.197	0.824	0.024	0.092	0.798	-0.002	0.163
200	α	1.549	0.049	0.348	1.492	-0.008	0.326	1.401	-0.099	0.392
	β	1.164	0.164	0.909	1.147	0.147	0.505	1.226	0.226	0.938
	θ	0.560	0.051	0.084	0.549	0.081	0.097	0.603	0.083	0.083
	λ	0.873	0.073	0.162	0.800	-0.001	0.086	0.776	-0.024	0.136
300	α	1.543	0.043	0.232	1.482	-0.018	0.252	1.420	-0.080	0.300
	β	1.084	0.084	0.584	1.091	0.091	0.377	1.147	0.147	0.667
	θ	0.530	0.030	0.031	0.540	0.038	0.049	0.579	0.079	0.066
	λ	0.870	0.070	0.123	0.802	0.002	0.074	0.794	-0.006	0.117
500	α	1.520	0.020	0.033	1.451	-0.197	0.049	1.419	-0.081	0.241
	β	1.000	-0.001	0.306	1.077	0.077	0.311	1.081	0.081	0.417
	θ	0.519	0.019	0.027	0.541	0.031	0.035	0.565	0.065	0.055
	λ	0.866	0.066	0.050	0.796	-0.004	0.069	0.800	0.001	0.103

6. Applications

In this section, we introduce three applications to real data sets to illustrate the usefulness of the APTD distribution. The first data set consists of 63 observations of the strengths of 1.5 cm glass fibers which obtained by workers at the UK National Physical Laboratory. The data are: *0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84,* *1.89, 2.00, 2.01, 2.24.* This data have also been used by Smith and Naylor (1987) and Merovci et al. (2016).

The second data set (Cooray and Ananda, 2008) represents the failure times of Kevlar 49/epoxy strands when the pressure is at 90% stress level: *0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09,0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.*

The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli (Bjerkedal, 1960). Guinea pigs are known to have high susceptibility of human tuberculosis, which is one of the reasons for choosing this species. The survival times of the Guinea pigs in days are: *0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.341.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.*

We estimate the unknown parameters of all competitive models by maximum likelihood. We compute the log-likelihood function evaluated at the MLEs using a limited memory quasi-Newton code for bound-constrained optimization (L-BFGS-B). For model comparison, we consider six well-known statistics: the maximized log-likelihood (LL), Akaike information criterion (AIC), BIC (Bayesian information criterion), Anderson-Darling (A^*), Cram´er von Mises, (W^*) and Kolmogorov-Smirnov (K-S) measures, where lower values of these statistics and higher p-values of K-S indicate good fits.

We compare the APTD distribution with those of alpha power Weibull (APW) (Nassar et al., 2016), exponentiated Kumaraswamy Dagum (EKD) (Huang and Oluyede;

2014), Beta Dagum (BD)(Domma and Condino; 2013) and Weibull Dagum (WD) (Tahir et al., 2016).

The densities of the competitive models are, respectively, given by

$$f_{APW}=\frac{\lambda\beta\log(\alpha)}{\alpha-1}x^{\beta-1}e^{-\lambda x^{\beta}}\alpha^{1-e^{-\lambda x^{\beta}}},x,\alpha,\beta,\lambda>0,\alpha\neq 1,$$

$$f_{EKD}=\alpha\lambda\delta\phi\theta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\alpha-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi-1}$$

$$\times\left\{1-\left[1-\left(1+\lambda x^{-\delta}\right)^{-\alpha}\right]^{\phi}\right\}^{\theta-1},x,\alpha,\lambda,\delta,\phi,\theta>0,$$

$$f_{BD}=\frac{\beta\lambda\delta}{B(a,b)}x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-a\beta-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{b-1},x,\lambda,\beta,\delta,a,b>0,$$

and

$$f_{WD}=b\lambda\delta\beta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-b\beta-1}\left[1-\left(1+\lambda x^{-\delta}\right)^{-\beta}\right]^{-b-1}$$

$$\times e^{\left[\left(1+\lambda x^{-\delta}\right)^{\beta}-1\right]^{-b}},x,\lambda,\beta,\delta,b>0.$$

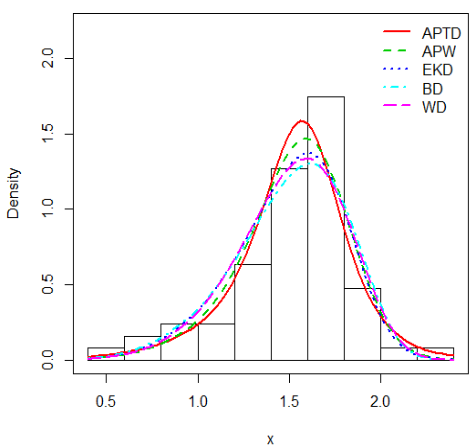
The MLEs and some statistics of the models for the data sets are introduced in Tables (2), (3), (4), (5), (6) and (7) respectively. The estimated pdfs and cdfs plots of all competitive distributions for the three data sets are displayed in Figures 4, 5 and 6 respectively.

Table 2: The MLEs for the first data set.

Distribution	Estimates with standard error in parenthesis							
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{a}$	$\hat{b}$
APTD	16.345 (3.195)	12.479 (1.311)	217.251 (11.724)	0.246 (0.108)	---	---	---	---
APW	10.843 (1.658)	4.483 (0.761)	0.195 (0.108)	8.792 (0.011)	3.196 (0.011)	193.570 (0.011)		
EKD	0.177 (0.022)	---	5.068 (0.033)	---	---	---	---	---
BD	---	72.416 (10.247)	2.198 (2.183)	---	0.455 (0.256)	---	4.826 (1.822)	2.652 (1.383)
WD	---	14.938 (3.267)	52.659 (34.284)	---	0.158 (0.028)	---	---	1.262 (0.360)

Table 3: Some statistics for the models fitted to the first data set.

Distribution	Goodness-of-fit statistics						
	LL	AIC	BIC	A^*	W^*	K-S	P-value
APTD	11.662	31.324	39.190	0.817	0.149	0.117	0.356
APW	13.474	32.948	39.378	0.928	0.169	0.123	0.300
EKD	13.823	37.646	48.362	0.997	0.181	0.151	0.113
BD	14.283	38.566	49.282	1.038	0.187	0.144	0.146
WD	14.736	37.471	46.044	1.164	0.211	0.147	0.134



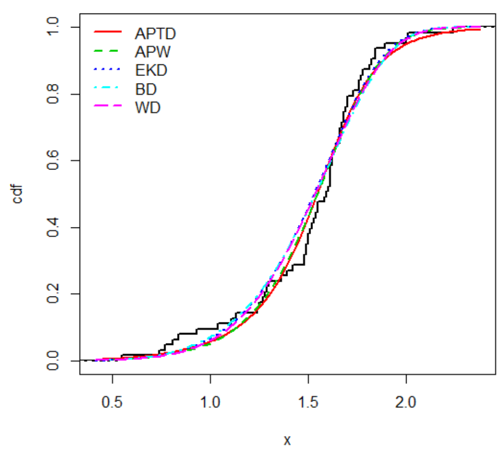


Figure 4: Estimated pdfs and cdfs plots of the APTD distribution for data set 1.

Table 4: The MLEs for the second data set.

Distribution	Estimates with standard error in parenthesis							
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{a}$	$\hat{b}$
APTD	1.452 (1.521)	3.370 (0.687)	7.277 (2.778)	0.202 (0.060)	---	---	---	---
APW	3.559 (3.937)	0.806 (0.128)	1.404 (0.373)	---	---	---	---	---
EKD	0.343 (0.715)	---	14.762 (5.768)	0.975 (0.138)	2.462 (2.308)	3.933 (0.042)		
BD	---	54.532 (12.323)	0.821 (0.014)	---	0.806 (0.423)	---	1.281 (0.945)	58.215 (4.171)
WD	---	5.999 (6.111)	55.406 (5.427)	---	0.415 (0.474)	---	---	0.232 (0.246)

Table 5: Some statistics for the models fitted to the second data set.

Distribution	Goodness-of-fit statistics						
	LL	AIC	BIC	A*	W*	K-S	P-value
APTD	100.044	208.089	218.549	0.464	0.066	0.065	0.782
APW	102.369	210.738	218.583	0.923	0.159	0.076	0.602
EKD	102.253	214.505	227.581	0.839	0.141	0.077	0.594
BD	102.754	215.508	228.584	0.986	0.172	0.084	0.475
WD	102.883	213.765	224.226	0.969	0.167	0.084	0.475

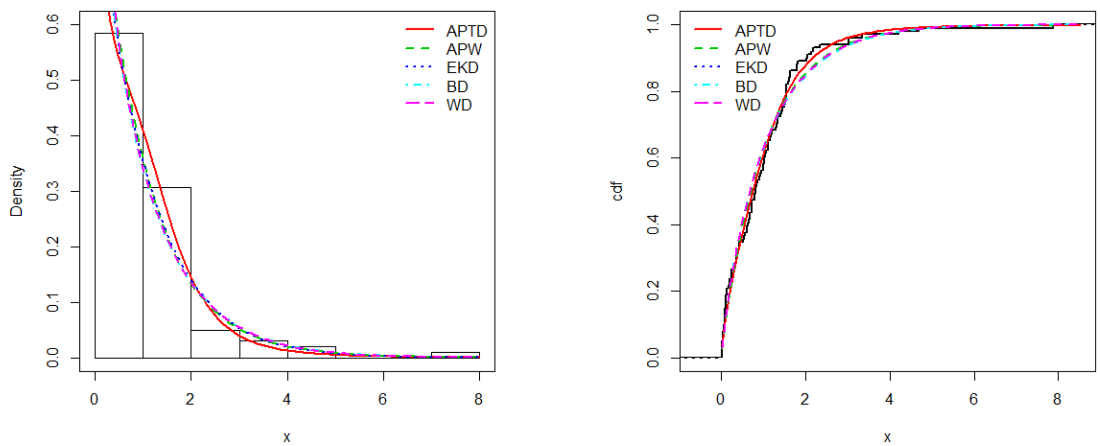


Figure 5: Estimated pdfs and cdfs plots of the APTD distribution for data set 2.

Table 6: The MLEs for the third data set.

Distribution	Estimates with standard error in parenthesis							
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\delta}$	$\hat{\phi}$	$\hat{a}$	$\hat{b}$
APTD	51.948 (11.254)	3.065 (0.476)	2.980 (2.369)	0.331 (0.224)	---	---	---	---
APW	243.092 (94.320)	0.898 (0.135)	1.357 (0.289)	---	---	---	---	---
EKD	4.175 (2.085	---	0.239 (0.331)	9.295 (1.202)	0.009 (0.027)	1.774 (4.241)	---	---
BD	---	1.014 (0.814)	1.187 (1.249)	---	0.484 (0.476)	---	3.335 (1.580)	2.035 (0.489)
WD	---	12.128 (4.430)	20.565 (8.774)	---	0.794 (0.388)	---	---	0.144 (0.191)

Table 7: Some statistics for the models fitted to the third data set.

Distribution	Goodness-of-fit statistics						
	LL	AIC	BIC	A*	W*	K-S	P-value
APTD	99.223	207.847	216.953	0.252	0.040	0.065	0.920
APW	101.156	208.312	215.142	0.489	0.078	0.082	0.719
EKD	99.987	209.174	220.558	0.278	0.048	0.069	0.882
BD	101.231	212.462	223.846	0.407	0.059	0.0870	0.648
WD	104.081	216.161	225.268	0.985	0.162	0.114	0.303

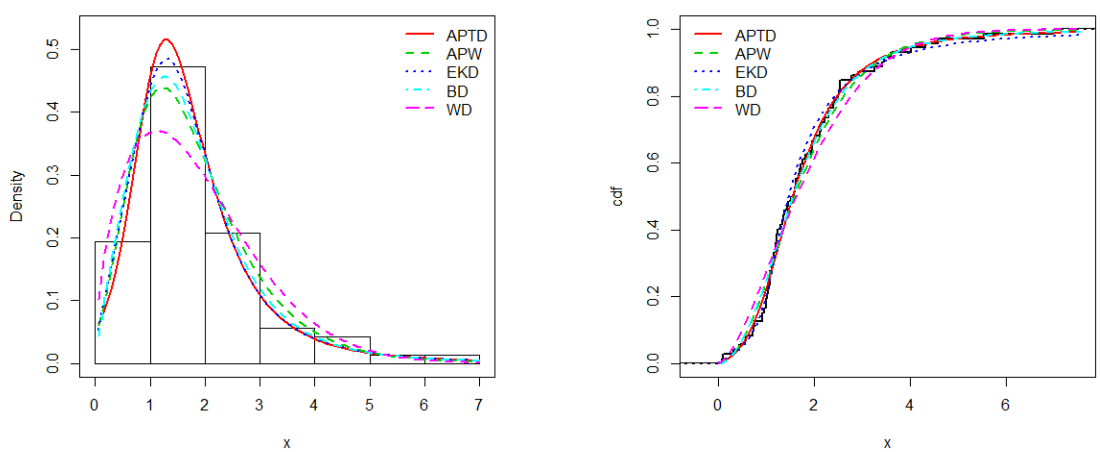


Figure 6: Estimated pdfs and cdfs plots of the APTD distribution for data set 3.

The values in Tables 3, 5 and 7 showed that the APTD distribution has the smallest values for A*, W*, AIC, BIC, KS and largest p-values among all competetive models then, it could be chosen as the best model. It is clear from Figures 4, 5 and 6 that the new APTD distribution provides the best fits for the three data sets.

7. Conclusions

We introduce a new four parameter Dagum distribution called the alpha power transformed Dagum distribution distribution. The main features of the new model such as the quantile function, ordinary and incomplete moments, moment generating function, order statistics, Rényi entropy, stress strength model and stochastic ordering are investigated. The characterizations of the proposed model are studied. The maximum likelihood criterion is used to estimate the model parameters and the importance of these estimates are assesed by means of a simulation study. The usefulness of the new model is illustrated via three real applications. Numerical results show that the new distribution can be considered a good alternative model to the alpha power Weibull, exponentiated Kumaraswamy Dagum, Beta Dagum and Weibull Dagum Distributions.

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Appendix A

Theorem 1. Let (Ω, F, P) be a given probability space and let $H=[a,b]$ be an interval for some $d<b$ $(a=-\infty, b=\infty$ might as well be allowed). Let $X:\Omega \rightarrow H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E\left[q_2(X)\middle| X \geq x\right]=E\left[q_1(X)\middle| X \geq x\right]\xi(x),\ x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^{-1}(H), \xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and ξ, particularly

$$F(x)=\int_a^x C\left|\frac{\xi'(u)}{\xi(u)\, q_1(u)-q_2(u)}\right| \exp(-s(u))\, du,$$

where the function s is a solution of the differential equation $s'=\dfrac{\xi' q_1}{\xi q_1-q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glanzel (1990)), in particular, let us assume that there is a sequence $\left\{X_n\right\}$ of random variables with distribution function $\left\{F_n\right\}$ such that the functions q_{1n}, q_{2n} and $\xi_n(n \in N)$ satisfy the conditions of Theorem 1 and let $q_{1n} \rightarrow q_1, q_{2n} \rightarrow q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\left\{F_n\right\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ, where

$$\xi(x)=\frac{E\left[q_2(X)\middle| X \geq x\right]}{E\left[q_1(X)\middle| X \geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution function is reflected by corresponding convergence of the function q_1, q_2 and ξ_n, respectively. It guarantees, for instance, the convergence of characterization on the Wald distribution to that of the Levy-Smirrnov distribution if $\alpha \rightarrow \infty$.

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

Appendix B

The components of the observed information matrix are the following

$$\frac{\partial^2\ell}{\partial\alpha^2}=n\left(\frac{1}{(\alpha-1)^2}-\frac{(1+\log(\alpha))}{(\alpha\log(\alpha))^2}\right)-\frac{1}{\alpha^2}\sum_{i=1}^{n}\left(1+x_i^{-\beta}\right)^{-\theta},$$

$$\frac{\partial^2\ell}{\partial\beta\,\partial\alpha}=\frac{\lambda\theta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+1}}\right),$$

$$\frac{\partial^2\ell}{\partial\lambda\,\partial\alpha}=\frac{-\theta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+1}}\right),$$

$$\frac{\partial^2\ell}{\partial\theta\,\partial\alpha}=-\left(\frac{1}{\alpha}\right)\sum_{i=1}^{n}\left(\frac{\log\left(1+\lambda x_i^{-\beta}\right)}{\left(1+\lambda x_i^{-\beta}\right)^{\theta}}\right),$$

$$\frac{\partial^2\ell}{\partial\beta^2}=\frac{-n}{\beta^2}-\lambda(\theta+1)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\left(\log(x_i)\right)^2}{\left(1+\lambda x_i^{-\beta}\right)^2}\right)+\lambda\theta\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\left(\log(x_i)\right)^2\left(\theta\lambda x_i^{-\beta}-1\right)}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+2}}\right),$$

$$\frac{\partial^2\ell}{\partial\lambda\,\partial\beta}=(\theta+1)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)}{\left(1+\lambda x_i^{-\beta}\right)^2}\right)+\theta\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)\left(1-\theta\lambda x_i^{-\beta}\right)}{\left(1+\lambda x_i^{-\beta}\right)^{2\theta+1}}\right),$$

$$\frac{\partial^2\ell}{\partial\theta\,\partial\beta}=\lambda\left\{\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)}{1+\lambda x_i^{-\beta}}\right)+\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\log(x_i)\left(1-\log\left(1+\lambda x_i^{-\beta}\right)\right)}{\left(1+\lambda x_i^{-\beta}\right)^{2\theta}}\right)\right\},$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=\frac{-n}{\lambda^2}+(\theta+1)\left\{\sum_{i=1}^{n}\left(\frac{x_i^{-2\beta}}{\left(1+\lambda x_i^{-\beta}\right)^2}\right)+\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-2\beta}}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+2}}\right)\right\},$$

$$\frac{\partial^2\ell}{\partial\theta\,\partial\lambda}=-\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}}{1+\lambda x_i^{-\beta}}\right)-\log(\alpha)\sum_{i=1}^{n}\left(\frac{x_i^{-\beta}\left(1-\theta\log\left(1-\theta\lambda x_i^{-\beta}\right)\right)}{\left(1+\lambda x_i^{-\beta}\right)^{\theta+1}}\right),$$

and

$$\frac{\partial^2\ell}{\partial\theta^2}=\frac{-n}{\theta^2}+\log(\alpha)\sum_{i=1}^{n}\left(\frac{\left(\log\left(1+\lambda x_i^{-\beta}\right)\right)^2}{\left(1+\lambda x_i^{-\beta}\right)^{\theta}}\right).$$