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A Novel Chen Extension: Theory, Characterizations and Different Estimation Methods

Haitham M. Yousof[1], Mustafa C. Korkmaz[2], G.G. Hamedani[3,*], Mohamed Ibrahim[4]

[1]*Department of Statistics, Mathematics and Insurance, Benha University, Egypt*
haitham.yousof@fcom.bu.edu.eg
[2]*Department of Measurement and Evaluation, Artvin Coruh University, Artvin, Turkey*
mcagatay@artvin.edu.tr
[3]*Department of Mathematical and Statistical Sciences, Marquette University, USA*
gholamhoss.hamedani@marquette.edu
[4]*Department of Applied, Mathematical and Actuarial Statistics, Faculty of Commerce, Damietta University, Damietta, Egypt*
mohamed_ibrahim@du.edu.eg
[*]*Correspondence: gholamhoss.hamedani@marquette.edu*

Abstract. In this work, we derive a novel extension of Chen distribution. Some statistical properties of the new model are derived. Numerical analysis for mean, variance, skewness and kurtosis is presented. Some characterizations of the proposed distribution are presented. Different classical estimation methods under uncensored schemes such as the maximum likelihood, Anderson-Darling, weighted least squares and right-tail Anderson–Darling methods are considered. Simulation studies are performed in order to compare and assess the above-mentioned estimation methods. For comparing the applicability of the four classical methods, two application to real data set are analyzed.

1. Introduction

Let X be a non-negative random variable (RV) with an Exponentiated Chen (EC) distribution (see Chaubey and Zhang (2015)), then its cumulative distribution function (CDF) is given by

$$G_{\alpha,\gamma,\beta}\left(x\right)=\left(1-\exp\left\{\gamma\left[1-\exp\left(x^{\beta}\right)\right]\right\}\right)^{\alpha}, \tag{1}$$

where $x>0, \alpha>0, \gamma>0$ and $\beta>0$. Chaubey and Zhang (2015) presented two propositions studying probability density function (PDF) and hazard rate function (HRF). The first proposition shows that the PDF shapes are either "decreasing" or "unimodal". The second proposition concludes that the HRF shapes are either "increasing" or "bathtub". Chaubey and Zhang (2015) also addressed

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the problem of estimation of parameters of the EC distribution, focusing on the maximum likelihood estimation method. The shape of the PDF of the EC distribution may be characterized as follows: for $\alpha < 1$, $1 > \beta$, $g_{\alpha,\gamma,\beta}(x)$ is a decreasing density, for $\alpha > 1$, $\beta > 1$, $g_{\alpha,\gamma,\beta}(x) = dG_{\alpha,\gamma,\beta}(x)/dx$ is a unimodal density and for $1 > \alpha$, $\beta > 1$ and for $\alpha > 1$, $1 > \beta$, $g_{\alpha,\gamma,\beta}(x)$ may be unimodal or decreasing density (see Due to Dey et al. (2017)). Chaubey and Zhang (2015) presented a proof that the failure behavior of the EC distribution are, respectively, bathtub ($1 > \alpha$, $\beta < 1$), increasing ($\alpha > 1$, $\beta > 1$), increasing or bathtub ($1 > \alpha, 1 < \beta$ and $\alpha > 1, \beta < 1$). For $\alpha = 1$, the EC distribution reduces to Chen (C) distribution (Chen (2000)) with $G_{\gamma,\beta}(x) = 1 - \nabla_{\gamma,\beta}(x)$ where

$$\nabla_{\gamma,\beta}(x) = \exp\left\{\gamma\left[1 - \exp\left(x^{\beta}\right)\right]\right\}.$$

Dey et al. (2017) addressed various mathematical properties and estimation methods for the EC model. They described different estimation methods such as the method of maximum likelihood, percentile estimation, ordinary least square and weighted least square, maximum product of spacings estimation, Cramér-von Mises, Anderson–Darling and right-tail Anderson–Darling estimation methods. For more Chen models see Khan et al. (2013 and 2016), Korkmaz et al. (2021) and Almazah et al. (2021). In this work, we shall use the Burr X generator (BX-G) (Yousof et al. (2017)) to derive a new version of the Chen distribution called the Burr type X exponentiated Chen (BXEC) distribution. The CDF of the BX-G is defined as

$$F_{\theta,\underline{\boldsymbol{\xi}}}(x) = \left\{1 - \exp\left[-\mathbf{O}^{2}_{\underline{\boldsymbol{\xi}}}(x)\right]\right\}^{\theta}|_{x\in\mathbb{R}}, \tag{2}$$

where

$$\mathbf{O}_{\underline{\boldsymbol{\xi}}}(x) = \frac{G_{\underline{\boldsymbol{\xi}}}(x)}{\overline{G}_{\underline{\boldsymbol{\xi}}}(x)}.$$

Inserting (1) into (2), the CDF of the BXEC distribution can be expressed as

$$F_{\underline{\Psi}}(x) = \left\{1 - \exp\left[-\mathbf{O}^{-2}_{\alpha,\gamma,\beta}(x)\right]\right\}^{\theta}|_{x>0}, \tag{3}$$

where

$$\mathbf{O}_{\alpha,\gamma,\beta}(x) = \left\{\left[1 - \nabla_{\gamma,\beta}(x)\right]^{-\alpha} - 1\right\}|_{x>0}.$$

The corresponding PDF of the BXEC can be derived as **(for $x \in \mathbb{R}^{+}$)**

$$f_{\underline{\Psi}}(x) = 2\theta\alpha\gamma\beta\frac{x^{\beta-1}\exp\left(x^{\beta}\right)\nabla_{\gamma,\beta}(x)\exp\left[-\mathbf{O}^{-2}_{\alpha,\gamma,\beta}(x)\right]\left[1 - \nabla_{\gamma,\beta}(x)\right]^{2\alpha-1}}{\left\{1 - \left[1 - \nabla_{\gamma,\beta}(x)\right]^{\alpha}\right\}^{3}\left\{1 - \exp\left[-\mathbf{O}^{-2}_{\alpha,\gamma,\beta}(x)\right]\right\}^{1-\theta}}|_{x>0}, \tag{4}$$

where $\underline{\boldsymbol{\Psi}} = (\theta, \alpha, \gamma, \beta)$. The bivariate BXEC (B-BXEC) type extensions can be derived using some copulas such as the Farlie-Gumbel-Morgenstern copula (Morgenstern (1956), Gumbel (1960), Gumbel (1961), Johnson and Kotz (1975) and Johnson and Kotz (1977)), modified Farlie-Gumbel-Morgenstern copula (Rodriguez-Lallena and Ubeda-Flores (2004)), Renyi entropy (Pougaza and Djafari (2011)) and Clayton copula. Different classical estimation methods under uncensored schemes are considered, such as the maximum likelihood (ML), Anderson–Darling (AD), weighted least squares (WLS) and right-tail Anderson–Darling (RTAD) methods (see Ibrahim et al. (2019 and 2020), Mansour et al. (2020d), Ibrahim and Yousof (2020) and Yousof et al. (2021a) for more details). Numerical simulations are performed for comparing the estimation approaches using different sample sizes for three different combinations of parameters (see Section 6). Although all estimation methods perform well, the ML method is the best method among all estimation techniques in modeling the uncensored relief times data and the WLS method is the best method among all estimation techniques in modeling the uncensored minimum flow data (see Section 7).

We are motivated to introduce and study the BXEC model for the following reasons:

- The BXEC model is recommended for modeling the relief times. However, the BXEC model is recommended for modeling the minimum flow data.
- The range of the skewness of the BXEC model is falling in the interval $(-0.1151715{,}27.99215)$. The wide range of the skewness gives the priority to the BXEC model in modeling and future prediction since many real-life datasets are negatively skewed. The kurtosis of the BXEC model is located between 0.674322 and 853.3517.
- The estimation persuaders of the BXEC model can be performed under many estimation methods such as the maximum likelihood (ML), Anderson–Darling (AD), weighted least squares (WLS) and right-tail Anderson–Darling (RTAD) methods. Although all estimation methods perform well in simulations, the ML method is recommended for modeling the relief times data and the ML method is recommended for modeling the minimum flow data.

2. Linear representation

In this section, we provide a very useful linear representation for the BXEC density function. Consider the power series

$$\left(1-\frac{\zeta_1}{\zeta_2}\right)^{\zeta_3}=\sum_{l_1=0}^{+\infty}\frac{(-1)^{l_1}\,\Gamma(1+\zeta_3)}{l_1!\,\Gamma(1+\zeta_3-l_1)}\left(\frac{\zeta_1}{\zeta_2}\right)^{l_1}|_{\left|\frac{\zeta_1}{\zeta_2}\right|<1,\ \zeta_3>0}. \tag{5}$$

Applying (5) to (4) we have

$$\begin{aligned} f_{\underline{\Psi}}(x) &= 2\theta\alpha\gamma\beta x^{\beta-1}\frac{\exp\left(x^{\beta}\right)\nabla_{\gamma,\beta}(x)}{\left(1-\nabla_{\gamma,\beta}(x)\right)^{1-\alpha}}\frac{\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha}}{\left\{1-\left[1-\nabla_{\gamma,\beta}(x)\right]^{\alpha}\right\}^{3}} \\ &\quad\times\sum_{l_1=0}^{+\infty}\frac{(-1)^{l_1}\,\Gamma(\theta)}{l_1!\,\Gamma(\theta-l_1)}\exp\left[-\left(l_1+1\right)\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x)\right]. \end{aligned} \tag{6}$$

Applying the power series to the term $\exp\left\{-\left(l_1+1\right)\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x)\right\}$, equation (6) becomes

$$f_{\underline{\Psi}}\left(x\right) = 2\theta\alpha\gamma\beta x^{\beta-1}\frac{\exp\left(x^{\beta}\right)\nabla_{\gamma,\beta}\left(x\right)}{\left(1-\nabla_{\gamma,\beta}\left(x\right)\right)^{1-\alpha}} \times\sum_{l_1,l_2=0}^{+\infty}\frac{\left(-1\right)^{l_1+l_2}\left(l_1+1\right)^{l_2}\Gamma\left(\theta\right)}{l_1!\,l_2!\Gamma\left(\theta-l_1\right)}\frac{\left[1-\nabla_{\gamma,\beta}\left(x\right)\right]^{(2l_2+1)\alpha}}{\left\{1-\left[1-\nabla_{\gamma,\beta}\left(x\right)\right]^{\alpha}\right\}^{2l_2+3}}. \tag{7}$$

Consider the series expansion

$$\left(1-\frac{\zeta_1}{\zeta_2}\right)^{-\zeta_3} = \sum_{l_3=0}^{+\infty}\frac{\Gamma\left(\zeta_3+l_3\right)}{l_3!\Gamma\left(\zeta_3\right)}\left(\frac{\zeta_1}{\zeta_2}\right)^{l_3}|_{\left|\frac{\zeta_1}{\zeta_2}\right|<1,\ \zeta_3>0}. \tag{8}$$

Applying the expansion in (8) to (7) for the term $\left[1-\left(1-\nabla_{\gamma,\beta}\left(x\right)\right)^{\alpha}\right]^{2l_2+3}$, equation (7) becomes

$$f_{\underline{\Psi}}\left(x\right) = \sum_{l_2,l_3=0}^{+\infty}\varsigma_{l_2,l_3}\,\pi_{\alpha^{\cdot}}\left(x\right)|_{(\alpha^{\cdot}=(2l_2+1)\alpha+l_3+1)}, \tag{9}$$

where

$$\varsigma_{l_2,l_3} = \frac{2\theta\left(-1\right)^{l_2}\Gamma\left(\theta\right)\Gamma\left(2l_2+l_3+3\right)}{l_2!l_3!\Gamma\left(2l_2+3\right)\alpha^{\cdot}}\sum_{l_1=0}^{+\infty}\frac{\left(-1\right)^{l_1}\left(l_1+1\right)^{l_2}}{l_1!\,\Gamma\left(\theta-l_1\right)}$$

and

$$\pi_{\alpha^{\cdot}}\left(x\right) = \alpha^{\cdot}g_{\gamma,\beta}\left(x\right)\left[G_{\gamma,\beta}\left(x\right)\right]^{\alpha^{\cdot}-1}.$$

Equation (9) reveals that the density of X can be expressed as a linear mixture of EC densities. So, several mathematical properties of the new family can be obtained by knowing those of the EC distribution. Similarly, the CDF of the BXEC family can also be expressed as a mixture of EC CDFs given by

$$F_{\underline{\Psi}}\left(x\right) = \sum_{l_2,l_3=0}^{+\infty}\varsigma_{l_2,l_3}\,\Pi_{\alpha^{\cdot}}\left(x\right) \tag{10}$$

where $\Pi_{\alpha^{\cdot}}\left(x\right) = \left[G_{\gamma,\beta}\left(x\right)\right]^{\alpha^{\cdot}}$ is the CDF of the EC family with power parameter $\alpha^{\cdot}$.

3. Mathematical and statistical properties

3.1. **Moments and generating function.** Following Dey et al. (2017), we can extract two theorems (see Appendix A). Based on Theorem 1, the r^{th} ordinary moment of the BXEC model can then be expressed as

$$\mu_r' = \mathbb{E}\left[X^r\right] = \alpha^{\cdot}\beta\sum_{l_2,l_3,\rho,q=0}^{+\infty}\varsigma_{l_2,l_3}\,\alpha_q^{\cdot}\left(\frac{r}{\beta}\right)\alpha_q^{\cdot}\left(\frac{r}{\beta}+\rho\right)\frac{\left(-1\right)^{\frac{2r}{\beta}+\rho}}{\gamma^{\frac{2r}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+r\right]}. \tag{11}$$

In particular,

$$\mu_1' = \mathbb{E}\left[X\right] = \alpha^{\cdot}\beta\sum_{l_2,l_3,\rho,q=0}^{+\infty}\varsigma_{l_2,l_3}\,\alpha_q^{\cdot}\left(\frac{1}{\beta}\right)\alpha_q^{\cdot}\left(\frac{1}{\beta}+\rho\right)\frac{\left(-1\right)^{\frac{2}{\beta}+\rho}}{\gamma^{\frac{2}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+1\right]},$$

$$\mu_2' = \mathbb{E}\left[X^2\right] = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \varsigma_{l_2,l_3}\,\alpha^{\cdot}_q\left(\frac{2}{\beta}\right)\alpha^{\cdot}_q\left(\frac{2}{\beta}+\rho\right)\frac{(-1)^{\frac{2}{\beta}+\rho}}{\gamma^{\frac{2}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+2\right]},$$

$$\mu_3' = \mathbb{E}\left[X^3\right] = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \varsigma_{l_2,l_3}\,\alpha^{\cdot}_q\left(\frac{3}{\beta}\right)\alpha^{\cdot}_q\left(\frac{3}{\beta}+\rho\right)\frac{(-1)^{\frac{6}{\beta}+\rho}}{\gamma^{\frac{6}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+3\right]},$$

and

$$\mu_4' = \mathbb{E}\left[X^4\right] = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \varsigma_{l_2,l_3}\,\alpha^{\cdot}_q\left(\frac{4}{\beta}\right)\alpha^{\cdot}_q\left(\frac{4}{\beta}+\rho\right)\frac{(-1)^{\frac{8}{\beta}+\rho}}{\gamma^{\frac{8}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+4\right]}.$$

The variance ($\mathrm{V}(Y)$), cumulants, n^{th} central moment, skewness ($\mathrm{S}(Y)$), kurtosis ($\mathrm{K}(Y)$) and Index of dispersion of the variance to mean ratio ($\mathrm{ID}(Y)$) measures can be calculated from the ordinary moments using well-known relationships.

3.2. **Conditional moments.** For the increasing failure rate models, it is also of interest to know what $\mathbb{E}\left(X^r|X>x\right)$ is. It can be easily seen that

$$\mathbb{E}\left(X^r|X>x\right) = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\,\alpha^{\cdot}_\rho\left(\frac{r}{\beta}\right)\alpha^{\cdot}_q\left(\frac{r}{\beta}+\rho\right)(-1)^{\frac{2r}{\beta}+\rho}\left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{2r}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+r\right]\left[1-\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha^{\cdot}}\right]}. \tag{12}$$

In particular,

$$\mathbb{E}\left(X|X>x\right) = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\,\alpha^{\cdot}_\rho\left(\frac{1}{\beta}\right)\alpha^{\cdot}_q\left(\frac{1}{\beta}+\rho\right)(-1)^{\frac{2}{\beta}+\rho}\left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{2}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+1\right]\left[1-\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha^{\cdot}}\right]},$$

$$\mathbb{E}\left(X^2|X>x\right) = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\,\alpha^{\cdot}_\rho\left(\frac{2}{\beta}\right)\alpha^{\cdot}_q\left(\frac{2}{\beta}+\rho\right)(-1)^{\frac{4}{\beta}+\rho}\left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{4}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+2\right]\left[1-\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha^{\cdot}}\right]},$$

$$\mathbb{E}\left(X^3|X>x\right) = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\,\alpha^{\cdot}_\rho\left(\frac{3}{\beta}\right)\alpha^{\cdot}_q\left(\frac{3}{\beta}+\rho\right)(-1)^{\frac{3}{\beta}+\rho}\left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{3}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+3\right]\left[1-\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha^{\cdot}}\right]},$$

and

$$\mathbb{E}\left(X^4|X>x\right) = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\,\alpha^{\cdot}_\rho\left(\frac{4}{\beta}\right)\alpha^{\cdot}_q\left(\frac{4}{\beta}+\rho\right)(-1)^{\frac{8}{\beta}+\rho}\left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{8}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+4\right]\left[1-\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha^{\cdot}}\right]}.$$

3.3. **Mean residual life.** The mean residual life (MRL) is the expected remaining life, $X-x$, given that the item has survived to time x. Thus, in life testing situations, the expected additional lifetime given that a component has survived until time x is called the MRL. Since the MRL function is the expected remaining life, x must be subtracted, yielding

$$M_1 = \mathbb{E}\left(X-x|X>x\right) = \frac{1}{1-F_{\underline{\Psi}}(x)}\left[\int_x^{+\infty} y f_{\underline{\Psi}}(y)\,dy\right]-x.$$

Then using (12), we get

$$M_1 = \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\, \alpha^{\cdot}_{\rho}\left(\frac{1}{\beta}\right) \alpha^{\cdot}_{q}\left(\frac{1}{\beta}+\rho\right) (-1)^{\frac{2}{\beta}+\rho} \left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{2}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+1\right]\left[1-\left(1-\nabla_{\gamma,\beta}(x)\right)^{\alpha^{\cdot}}\right]} - x.$$

3.4. **Mean past lifetime.** In a real life situation, where systems often are not monitored continuously, one might be interested in getting inference more about the history of the system, for example, when the individual components have failed. Assume now that a component with lifetime X has failed at or some time before x, $x \geq 0$. Consider the conditional random variable $x - X|X \leq x$. This conditional random variable shows, in fact, the time elapsed from the failure of the component given that its lifetime is less than or equal to x. Hence, the mean past lifetime (MPL) of the component can be defined as

$$\mathfrak{M}_1 = \mathbb{E}\left(x - X|X \leq x\right) = x - \frac{1}{F_{\underline{\Psi}}(x)} \int_0^t y f_{\underline{\Psi}}(y)\, dy.$$

Then using (11) and (12), we get

$$\mathfrak{M}_1 = x - \alpha^{\cdot}\beta \sum_{l_2,l_3,\rho,q=0}^{+\infty} \frac{\varsigma_{l_2,l_3}\, \alpha^{\cdot}_{\rho}\left(\frac{1}{\beta}\right) \alpha^{\cdot}_{q}\left(\frac{1}{\beta}+\rho\right) (-1)^{\frac{2}{\beta}+\rho}}{\gamma^{\frac{2}{\beta}+\rho}\left[\beta\left(\alpha^{\cdot}+\rho+q\right)+1\right]} \left(1-\nabla_{\gamma,\beta}(x)\right)^{\frac{\beta(\rho+q)+1}{\beta}}.$$

3.5. **Numerical analysis for mean, V(X), S(X), K(X) and DisIx(X).** Table 1 gives Numerical analysis for the mean, V(X), S(X), K(X) and DisIx(X). Based on Table 1, we note that: 1-The skewness of the BXEC distribution can range in the interval (-0.1151715, 27.99215). 2-The spread for the BXEC kurtosis is much larger ranging from 0.674322 to 853.3517. 3-ID(X) can be "between 0 and 1" and "more than 1".

Table 1: Mean, variance, skewness and kurtosis.

θ	α	γ	β	**E**(X)	V(X)	S(X)	K(X)	ID(X)
0.001	0.25	1.15	0.5	0.004704	0.0065858	22.86888	602.2104	1.400110
0.01				0.046359	0.0634222	7.113755	59.83784	1.368082
0.05				0.217748	0.2696108	2.950300	11.75074	1.238177
0.1				0.404606	0.4452129	1.888049	5.880733	1.100361
0.5				1.280745	0.6754974	0.263415	2.166524	0.527425
1				1.752028	0.5140454	−0.1151715	2.44533	0.293400
5				2.571438	0.1700379	−0.1185289	2.977865	0.066126
50				3.191300	0.0498866	0.2802537	3.108527	0.015632
150				3.382794	0.0330405	0.4045376	3.244357	0.009767
500				3.552790	0.0227708	0.5052380	3.396144	0.006409
0.5	0.001	1	0.25	0.0161660	0.099236	22.90835	569.8220	6.138579
	0.25			1.6983920	3.633604	1.388534	4.517459	2.139438
	0.5			0.3211543	0.172874	1.879163	6.956402	0.538289
	0.75			0.1052558	0.021496	2.143353	8.516228	0.204223
	1			0.0449386	0.004308	2.331145	9.772082	0.095856
	1.5			0.0125201	0.000376	2.623065	10.14921	0.030045
1.5	1.5	0.5	0.5	0.037479	0.00089046	1.1555660	0.674322	0.02375894
		1		0.149247	0.00542029	0.4019048	2.784748	0.03631768
		5		0.729749	0.02153580	−0.3031202	2.94796	0.02951124
		10		1.049247	0.02424775	−0.4039516	3.119624	0.02310967
		20		1.377546	0.02477488	−0.4555275	3.223266	0.01798479
		50		1.807956	0.02377759	−0.4866825	3.291349	0.01315164
		150		2.305916	0.02169227	−0.4999334	3.322090	0.00940722
		500		2.823258	0.01930193	−0.5031953	3.330685	0.00683676
1.75	0.25	2	0.05	0.010810	0.065244	27.99215	853.3517	6.035829
			0.10	0.055959	0.375494	12.16822	158.7941	6.710135
			0.15	0.241225	1.698862	5.68209	35.08007	7.042654
			0.35	5.303015	2.012257	−0.1814966	2.806119	0.379455
			0.5	3.188738	0.381287	−0.4362321	3.145548	0.119573
			0.65	2.431359	0.136769	−0.5923569	3.475648	0.056252

4. Characterizations of the BXEC distribution

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions one of which is based on the truncated moments. This section is devoted to the characterizations of the BXEC distribution based on: (i) a simple relationship between two truncated moments and (ii) the hazard function.

4.1. **Characterizations based on two truncated moments.** This subsection deals with the characterizations of BXEC distribution in terms of a simple relationship between two truncated moments. We will employ Theorem of Glänzel (1987) (see Appendix A). As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 4.1.1. Let $X : \Omega \rightarrow (0, \infty)$ be a continuous random variable and let $q_1(x) = [A(x)]^{-1}$ and

$$q_2(x) = q_1(x) \exp\left[\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right] \, |_{x>0}.$$

Then X has pdf (4) if and only if the function η defined in Theorem 3 is of the form

$$\eta(x) = \frac{1}{2} \exp\left[\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right] |_{x>0},$$

where

$$A(x) = \frac{\left[1 - \exp\left\{\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right\}\right]^{2\alpha-1} \exp\left[-\left\{\left[1 - \exp\left\{\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right\}\right]^{-\alpha} - 1\right\}^{-2}\right]}{\left\{\left[1 - \left[1 - \exp\left\{\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right\}\right]^{\alpha}\right\}^{3} \left\{1 - \exp\left[-\left\{\left[1 - \exp\left\{\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right\}\right]^{-\alpha} - 1\right\}^{-2}\right]\right\}^{1-\theta}}.$$

Proof. If X has pdf (4), then

$$\left(1 - F_{\underline{\Psi}}(x)\right) E\left[q_1(X) \mid X \geq x\right] = 2\theta\alpha \exp\left[\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right] |_{x>0},$$

and

$$\left(1 - F_{\underline{\Psi}}(x)\right) E\left[q_2(X) \mid X \geq x\right] = \theta\alpha \exp\left[2\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right] |_{x>0},$$

and hence

$$\eta(x) = \frac{1}{2} \exp\left[\gamma\left(1 - \exp\left(x^{\beta}\right)\right)\right] |_{x>0}0.$$

We also have

$$\eta(x)\,q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\exp\left[\gamma\left(1-\exp\left(x^{\beta}\right)\right)\right] < 0|_{x>0}.$$

Conversely, if η is of the above form, then

$$s'(x) = \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \gamma\beta x^{\beta-1}\exp\left(x^{\beta}\right)|_{x>0},$$

and

$$s(x) = \gamma\exp\left(x^{\beta}\right).$$

Now, according to Theorem 3, X has density (4).

Corollary 4.1.1. Suppose X is a continuous random variable. Let $q_1(x)$ be as in Proposition 4.1.1. Then X has density (4) if and only if there exist functions q_2 and η defined in Theorem 3 for which the following first order differential equation holds

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \gamma\beta x^{\beta-1}\exp\left(x^{\beta}\right)|_{x>0}.$$

Corollary 4.1.2. The differential equation in Corollary 4.1.1 has the following general solution

$$\eta(x) = \exp\left[-\gamma\left(1-\exp\left(x^{\beta}\right)\right)\right]\left[-\int \gamma\beta x^{\beta-1}\exp\left(x^{\beta}\right)\exp\left[\gamma\left(1-\exp\left(x^{\beta}\right)\right)\right](q_1(x))^{-1}q_2(x) + D\right],$$

where D is a constant. A set of functions satisfying the above differential equation is given in Proposition 4.1.1 with $D = 0$. Clearly, there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 3.

4.2. **Characterization based on hazard function.** The hazard function, h_F, of a twice differentiable distribution function, F with the pdf f, satisfies the following trivial differential equation

$$\frac{f'(y)}{f(y)} = \frac{h_F'(y)}{h_F(y)} - h_F(y).$$

The following proposition establishes a non-trivial characterization of BXEC distribution, for the case $\theta = 1$, based on the hazard function.

Proposition 4.2.1. Suppose X is a continuous random variable. Then, X has density (4), for $\theta = 1$, if and only if its hazard function $h_{F_{\underline{\Psi}}}(x)$ satisfies the following first order differential equation

$$h'_{F_{\underline{\Psi}}}(x) - \beta x^{\beta-1}h_{F_{\underline{\Psi}}}(x)$$
$$= 2\alpha\gamma\beta\exp\left(x^{\beta}\right)\frac{d}{dx}\left\{\frac{x^{\beta-1}\{\exp[\gamma\left(1-\exp\left(x^{\beta}\right)\right)]\}[1-\exp\left\{\gamma\left(1-\exp\left(x^{\beta}\right)\right)\right\}]^{2\alpha-1}}{\left\{[1-[1-\exp\{\gamma\left(1-\exp\left(x^{\beta}\right)\right)\}]^{\alpha}\right\}^{3}}\right\}|_{x>0}.$$

Proof. Is straightforward and hence omitted. See Hamedani et al. (2018a, 2018b, 2019 and 2021) for more details.

5. Estimation methods

5.1. **The ML method.** The method of maximum likelihood is the most frequently used method of parameter estimation. Its success stems from its many desirable properties including consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. Let $x_1, \ldots, x_n$ be a random sample of size n from (4), then the log-likelihood function of (4) without constant terms is given by

$$\begin{aligned}\ell(\underline{\mathbf{\Psi}}) &= n\log 2 + n\log\theta + n\log\alpha + n\log\gamma + n\log\beta \\ &+ (\beta-1)\sum_{i=1}^{n}\log x_{[i:n]} + \sum_{i=1}^{n} x_i^{\beta} + \gamma\sum_{i=1}^{n}\left[1-\exp\left(x_i^{\beta}\right)\right] \\ &- (1-\theta)\sum_{i=1}^{n}\left\{1-\exp\left[-\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x_{[i:n]})\right]\right\} - \sum_{i=1}^{n}\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x_{[i:n]}) \\ &+ (2\alpha-1)\sum_{i=1}^{n}\left(1-\nabla_{\gamma,\beta}\left(x_{[i:n]}\right)\right) + \sum_{i=1}^{n}\left[1-\left(1-\nabla_{\gamma,\beta}\left(x_{[i:n]}\right)\right)^{\alpha}\right].\end{aligned}$$

The components of the score vector, $\mathbf{U}(\underline{\mathbf{\Psi}}) = \frac{\partial\ell(\underline{\Psi})}{\partial\underline{\Psi}} = (\mathbf{U}(\theta), \mathbf{U}(\alpha), \mathbf{U}(\gamma), \mathbf{U}(\beta))^{\intercal}$. The ML estimates (MLEs) $\widehat{\theta}, \widehat{\alpha}, \widehat{\gamma}$ and $\widehat{\beta}$ of θ, α, γ and β are obtained by solving the following nonlinear systems of equations

$$\frac{\partial}{\partial\theta}\ell(\underline{\mathbf{\Psi}}) = 0, \frac{\partial}{\partial\alpha}\ell(\underline{\mathbf{\Psi}}) = 0, \frac{\partial}{\partial\gamma}\ell(\underline{\mathbf{\Psi}}) = 0, \frac{\partial}{\partial\beta}\ell(\underline{\mathbf{\Psi}}) = 0.$$

5.2. **The WLS method.** The WLS estimates (WLSE) are obtained by minimizing the function $\mathbf{WLSE}(\underline{\mathbf{\Psi}})$ WRT θ, α, γ and β

$$\mathbf{WLSE}(\underline{\mathbf{\Psi}}) = \sum_{i=1}^{n} W_{(i,n)}\left[F_{\underline{\Psi}}\left(x_{[i:n]}\right) - \Upsilon_{(i,n)}\right]^2,$$

where

$$\Upsilon_{(i,n)} = \frac{\varsigma}{n+1},$$

and

$$W_{(i,n)} = \left[(1+n)^2(2+n)\right] / \left[i(1+n-i)\right].$$

The WLSEs are obtained by solving

$$0 = \sum_{i=1}^{n} W_{(i,n)}\left(\left\{1-\exp\left[-\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x_{[i:n]})\right]\right\}^{\theta} - \Upsilon_{(i,n)}\right)\nabla_{(\theta)}\left(x_{[i:n]};\underline{\mathbf{\Psi}}\right),$$

$$0 = \sum_{i=1}^{n} W_{(i,n)}\left(\left\{1-\exp\left[-\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x_{[i:n]})\right]\right\}^{\theta} - \Upsilon_{(i,n)}\right)\nabla_{(\alpha)}\left(x_{[i:n]};\underline{\mathbf{\Psi}}\right),$$

$$0 = \sum_{i=1}^{n} W_{(i,n)}\left(\left\{1-\exp\left[-\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x_{[i:n]})\right]\right\}^{\theta} - \Upsilon_{(i,n)}\right)\nabla_{(\gamma)}\left(x_{[i:n]};\underline{\mathbf{\Psi}}\right),$$

and

$$0=\sum_{i=1}^{n}W_{(i,n)}\left(\left\{1-\exp\left[-\mathbf{O}_{\alpha,\gamma,\beta}^{-2}(x_{[i:n]})\right]\right\}^{\theta}-\Upsilon_{(i,n)}\right)\nabla_{(\beta)}\left(x_{[i:n]};\underline{\mathbf{\Psi}}\right),$$

where

$$\nabla_{(\cdot)}\left(x_{[i:n]};\underline{\mathbf{\Psi}}\right)=\partial F_{\underline{\Psi}}\left(x_{[i:n]}\right)/\partial\cdot.$$

5.3. **The AD method.** The AD estimates (ADE) are obtained by minimizing the function

$$\mathrm{ADE}\left(\underline{\mathbf{\Psi}}\right)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\begin{array}{c}\log F_{(\underline{\Psi})}(x_{[i:n]})\\+\log\left[1-F_{(\underline{\Psi})}(x_{[-i+1+n:n]})\right]\end{array}\right\}.$$

The parameter estimates follow by solving the nonlinear equations

$$\begin{aligned}0&=\partial\left[\mathbf{AD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\theta,\\0&=\partial\left[\mathbf{AD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\alpha,\\0&=\partial\left[\mathbf{AD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\gamma,\end{aligned}$$

and

$$0=\partial\left[\mathbf{AD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\beta.$$

5.4. **The RTAD method.** The RTAD estimates (RTADE) are obtained by minimizing

$$\mathrm{RTADE}\left(\underline{\mathbf{\Psi}}\right)=\frac{1}{2}n-2\sum_{i=1}^{n}F_{(\underline{\Psi})}(x_{[i:n]})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[1-F_{(\underline{\Psi})}(x_{[-i+1+n:n]})\right]\right\}.$$

The estimates follow by solving the nonlinear equations

$$\begin{aligned}0&=\partial\left[\mathrm{RTAD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\theta,\\0&=\partial\left[\mathrm{RTAD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\alpha,\\0&=\partial\left[\mathrm{RTAD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\gamma,\end{aligned}$$

and

$$0=\partial\left[\mathrm{RTAD}_{\left(x_{[i:n]},x_{[-\varsigma+1+n:n]}\right)}\left(\underline{\mathbf{\Psi}}\right)\right]/\partial\beta.$$

6. Simulation for comparing estimation methods

The estimation persuaders of the BXEC model is performed under the ML, AD, WLS and RTADE methods. Simulation studies are performed in order to compare and assess the above mentioned estimation methods. The simulation studies are based on $N=1000$ generated data sets from the BXEC version where $n=20,50,100,200,\ 300,500$ and $\theta=2,\alpha=0.7,\gamma=0.6$ and $\beta=0.2$. The performance of the different estimators are compared in terms of the average of its bias and root mean square error (RMSE). Table 2 and Table 3 list the simulation results. From Table 2 and Table 3, it is noted that the $\mathrm{RMSE}_{(\underline{\Psi})}$ tend to zero as n$\rightarrow\infty$ and the bias tends to zero as n$\rightarrow\infty$.

The Bias_θ under the ML method stared with 0.11064 for $n = 20$ and reached $\simeq 0$ for n =500. The Bias_α under the ML method stared with 0.00083 for $n = 20$ and reached $\simeq 0$ for n =500. The Bias_γ under the ML method stared with 0.00665 for $n = 20$ and reached 0.00020 for n =500. The BIAS_β under the ML method stared with 0.00260 for $n = 20$ and reached 0.00003 for n =500. The Bias_θ under the WLS method stared with 0.38705 for $n = 20$ and reached 0.03783 for n =500. The Bias_α under the WLS method stared with 0.02353 for $n = 20$ and reached 0.00148 for n =500. The Bias_γ under the WLS method stared with 0.01697 for $n = 20$ and reached 0.00050 for n =500. The BIAS_β under the WLS method stared with 0.01535 for $n = 20$ and reached 0.00153 for n =500. The Bias_θ under the ADE method stared with 0.07513 for $n = 20$ and reached 0.00675 for n =500. The Bias_α under the ADE method stared with 0.00303 for $n = 20$ and reached 0.00050 for n =500. The Bias_γ under the ADE method stared with 0.00021 for $n = 20$ and reached 0.00038 for n =500. The BIAS_β under the ADE method stared with 0.00220 for $n = 20$ and reached 0.00030 for n =500. The Bias_θ under the RTAD method stared with 0.15966 for $n = 20$ and reached 0.00695 for n =500. The Bias_α under the RTAD method stared with 0.00545 for $n = 20$ and reached $\simeq 0$ for n =500. The Bias_γ under the RTAD method stared with 0.00197 for $n = 20$ and reached $\simeq 0$ for n =500. The BIAS_β under the RTAD method stared with 0.00443 for $n = 20$ and reached $\simeq 0$ for n =500. For all parameters under all estimation methods the RMSE started with small values and ended with a very small value as n$\rightarrow \infty$. The values of the Dabs and the Dmax also decreases n$\rightarrow \infty$. The values of the difference between the Dabs and the Dmax also decreases n$\rightarrow \infty$.

Table 2: Simulation results.

	n	Bias_θ	Bias_α	Bias_γ
ML	20	0.11064	0.00083	0.00665
WLS		0.38705	0.02353	0.01697
ADE		0.07513	0.00303	0.00021
RTAD		0.15966	0.00545	0.00197
ML	50	0.04575	0.00050	0.00290
WLS		0.17185	0.00906	0.00552
AD		0.02678	0.00044	0.00092
RTAD		0.05382	0.00108	0.00041
ML	100	0.02966	0.00064	0.00064
WLS		0.11129	0.00560	0.00304
ADE		0.02024	0.00074	0.00002
RTADE		0.03021	0.00078	000002
ML	200	0.01770	0.00061	0.00005
WLS		0.06027	0.00276	0.00132
AD		0.00256	0.00015	0.00042
RTAD		0.00874	0.00013	0.00016
MLE	300	0.00377	0.00004	0.00017
WLS		0.05551	0.00272	0.001443
ADE		0.00847	0.00054	0.00034
RTADE		0.01446	0.00072	0.00046
ML	500	0.00047	0.00009	0.00020
WLS		0.03783	0.00148	0.00050
AD		0.00675	0.00050	0.00038
RTAD		0.00695	0.00018	0.0000004

Table 3: Simulation results.

	n	$BIAS_\beta$	$RMSE_\theta$	$RMSE_\alpha$	$RMSE_\gamma$	$RMSE_\beta$	Dabs	Dmax
ML	20	0.00260	0.51751	0.04488	0.04389	0.02541	0.01012	0.01765
WLS		0.01535	0.75360	0.05136	0.04496	0.03107	0.13726	0.19605
AD		0.00220	0.51977	0.04416	0.04313	0.02471	0.01936	0.02770
RTAD		0.00443	0.70782	0.04474	0.04146	0.02783	0.04081	0.05825
ML	50	0.00103	0.30783	0.02815	0.02688	0.01553	0.00382	0.00690
WLS		0.00679	0.37844	0.03059	0.02752	0.01811	0.05823	0.08267
AD		0.00063	0.32719	0.02935	0.02859	0.01619	0.00430	0.00650
RTAD		0.00127	0.39436	0.02920	0.02713	0.01755	0.01086	0.01577
ML	100	0.00097	0.20805	0.02029	0.01895	0.01114	0.00619	0.00910
WLS		0.00450	0.25537	0.02212	0.02009	0.01281	0.03723	0.05277
AD		0.00064	0.22365	0.02043	0.01987	0.01117	0.00519	0.00743
RTAD		0.00080	0.26729	0.02055	0.01915	0.01218	0.00687	0.00987
ML	200	0.00065	0.14244	0.01377	0.01287	0.07644	0.00467	0.00672
WLS		0.00242	0.16259	0.01443	0.01321	0.00841	0.01938	0.02758
AD		0.00003	0.15217	0.01413	0.01377	0.00772	0.00054	0.00086
RTAD		0.00015	0.17741	0.01406	0.01313	0.00829	0.00145	0.00214
MLE	300	0.00005	0.11686	0.011422	0.01064	0.00636	0.00044	0.00071
WLS		0.00231	0.13620	0.01204	0.01091	0.00709	0.01858	0.02643
ADE		0.00033	0.12715	0.01177	0.01141	0.00645	0.00312	0.00442
RTADE		0.00050	0.14905	0.01169	0.01087	0.00693	0.00472	0.00669
ML	500	0.00003	0.08851	0.00878	0.00820	0.004776	0.00042	0.00062
WLS		0.00153	0.10132	0.00922	0.00845	0.00532	0.01140	0.01626
AD		0.00030	0.09572	0.00889	0.00863	0.00485	0.00281	0.00397
RTAD		0.00020	0.11233	0.00898	0.00840	0.00526	0.00164	0.00237

7. Applications for comparing methods under uncensored data

For comparing the four classical methods, two application to real data set are presented. The first subsection is related to studying the data set (Gross and Clark (1975)) on the relief times of twenty patients receiving an analgesic: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. The second subsection is related to studying the minimum flow data which was presented by Cordeiro and Castro (2011) that include 38 observations. The data set is the following: 43.86, 44.97, 46.27, 51.29, 61.19, 61.20, 67.80, 69.00, 71.84, 77.31,85.39, 86.59, 86.66, 88.16, 96.03, 102.00, 108.29, 113.00, 115.14, 116.71, 126.86, 127.00, 127.14, 127.29, 128.00, 134.14, 136.14, 140.43, 146.43, 146.43,148.00, 148.43, 150.86, 151.29, 151.43, 156.14, 163.00, 186.43.

7.1. **Analyzing the uncensored relief times data.** For comparing the four estimation methods under uncensored relief times data, we consider the Cramér-Von Mises (W^*) and the Anderson-Darling (A^*) statistics. Table 4 lists the different estimators as well as W^* and A^* statistics. From Table 4, the ML method is the best method among all estimation techniques with W^* =0.07043 and A^* =0.41643. However, the AD performes well with W^* =0.1148 and A^* =0.68175. The other two methods can be used in some particular cases.

Table 4: Comparing methods under uncensored relief times data.

Method	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\gamma}$	$\widehat{\beta}$	W^*	A^*
ML	**20.88487**	**6.70450**	**1.47856**	**0.09324**	**0.07043**	**0.41643**
WLS	4.48989	1.35147	0.44386	0.37917	0.20960	1.23874
AD	6.32523	2.08544	0.74377	0.19106	0.11485	0.68175
RTAD	2.89731	0.81886	0.27949	0.33959	0.16896	0.99760

7.2. **Analyzing the minimum flow data.** For comparing the four estimation methods under uncensored minimum flow data, , we consider the W^* and the A^* statistics. Table 4 lists the different estimators as well as W^* and A^* statistics. From Table 5, the WLS method is the best method among all estimation techniques with W^* =0.06896 and A^* =0.46616. However, all other methods performe well.

Table 5: Comparing methods under uncensored minimum flow data.

Method	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\gamma}$	$\widehat{\beta}$	W^*	A^*
ML	1.07468	0.85091	0.00735	0.30655	0.07033	0.47072
WLS	**1.20261**	**0.90472**	**0.00811**	**0.30645**	**0.06896**	**0.46616**
AD	1.20715	0.92018	0.01436	0.27936	0.08041	0.51378
RTAD	1.39542	0.95907	0.01855	0.26968	0.08431	0.53262

8. Conclusion

In this paper, we derived a new four-parameter Chen model based on the the Burr type X generator. Some statistical properties of the new model such as a linear representation, moments, moment generating function, conditional moments, mean residual life and mean past lifetime are derived. Numerical analysis for mean, variance, skewness and kurtosis is presented.Some characterizations of the novel distribution based on: (i) a simple relationship between two truncated moments and (ii) the hazard function are presented. Different classical estimation methods under uncensored schemes are considered, such as the maximum likelihood, Anderson-Darling, weighted least squares and right-tail Anderson–Darling methods are considered. Simulation studies are performed in order to compare and assess the above-mentioned estimation methods. The applicability of the four classical methods is assessed based on two application to real data sets.

As future potential works, we can apply many new useful goodness-of-fit tests for right censoring distributional validation such as the Nikulin-Rao-Robson goodness-of-fit test, Bagdonavicius-Nikulin goodness-of-fit test, modified Nikulin-Rao-Robson goodness-of-fit test and modified Bagdonavicius-Nikulin goodness-of-fit test to the new model as performed by Goual et al. (2019, 2020), Mansour et al. (2020a-f), Yadav et al. (2020), Goual and Yousof (2020), Ibrahim et al (2021) and Yousof et al. (2021b), among others.

Appendix A

Theorem 1. Let X be a RV with the EC distribution. Then using the transformation $t = \left[G_{\alpha,\gamma,\beta}(x)\right]^{\frac{1}{\alpha}}$, the r^{th} ordinary moment of X is given by

$$\mu'_r = \mathbb{E}\left[X^r\right] = \alpha\beta \sum_{\rho,q=0}^{+\infty} \alpha_\rho\left(\frac{r}{\beta}\right)\alpha_q\left(\frac{r}{\beta}+\rho\right)\frac{(-1)^{\frac{2r}{\beta}+\rho}}{\gamma^{\frac{2r}{\beta}+\rho}\left[\beta\left(\alpha+\rho+q\right)+r\right]},$$

where $\alpha_\rho\left(\frac{r}{\beta}\right)$ is the coefficient of $\left[\frac{1}{\gamma}\log\left(1-t\right)\right]^{\frac{2r}{\beta}+\rho}$ in the expansion of

$$\left\{\sum_{j_1=1}^{+\infty}\frac{1}{j_1}\left[\frac{1}{\gamma}\log\left(1-t\right)\right]\right\}^{\frac{r}{\beta}}$$

and $\alpha_q\left(\frac{r}{\beta}+\rho\right)$ is the coefficient of $t^{\rho+q+\frac{r}{\beta}}$ in the expansion of

$$\left(\sum_{j_2=1}^{+\infty}\frac{t^{j_2}}{j_2}\right)^{\frac{r}{\beta}+\rho}$$

(see Balakrishnan and Cohen (2014) for more details).

Theorem 2. Let X be a RV with the EC distribution. Then, the r^{th} conditional moment can be derived as

$$\mathbb{E}\left(X^r|X>x\right)=\alpha\beta\sum_{\rho,q=0}^{+\infty}\alpha_\rho\left(\frac{r}{\beta}\right)\alpha_q\left(\frac{r}{\beta}+\rho\right)\frac{(-1)^{\frac{2r}{\beta}+\rho}\left(\nabla_{\gamma,\beta}(x)\right)}{\gamma^{\frac{2r}{\beta}+\rho}\left[\beta\left(\alpha+\rho+q\right)+r\right]\left\{1-\left[1-\nabla_{\gamma,\beta}(x)\right]^{\alpha}\right\}}$$

Theorem 3. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[a,b]$ be an interval for some $d<b$ $(a=-\infty,\ b=\infty$ might as well be allowed$)$. Let $X:\Omega\rightarrow H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X)\ \mid X\geq x\right]=\mathbf{E}\left[q_1(X)\ \mid X\geq x\right]\eta(x),\quad x\in H,$$

is defined with some real function ξ. Assume that $q_1,q_2\in C^1(H)$, $\eta\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1,q_2 and η , particularly

$$F(x)=\int_a^x C\left|\frac{\eta'(u)}{\eta(u)\,q_1(u)-q_2(u)}\right|\exp\left(-s(u)\right)\ du\ ,$$

where the function s is a solution of the differential equation $s'=\frac{\eta'\,q_1}{\eta q_1-q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

Note: The goal is to have the function $\eta(x)$ as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel, 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n} , q_{2n} and η_n $(n\in\mathbb{N})$ satisfy the conditions of Theorem 3 and let $q_{1n}\rightarrow q_1$, $q_{2n}\rightarrow q_2$ for some continuously differentiable real functions q_1 and q_2 . Let, finally, X be a random variable with distribution F . Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if η_n converges to η, where

$$\eta(x)=\frac{E\left[q_2(X)\ \mid X\geq x\right]}{E\left[q_1(X)\ \mid X\geq x\right]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1 , q_2 and η , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha\rightarrow\infty$.

A further consequence of the stability property of Theorem 3 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions.

For such purpose, the functions q_1, q_2 and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

In some cases, one can take $q_1(x) \equiv 1$, which reduces the condition of Theorem 3 to $\mathbf{E}[q_2(X) \mid X \geq x] = \eta(x), \quad x \in H$. We, however, believe that employing three functions q_1 , q_2 and η will enhance the domain of applicability of Theorem 3.

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