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On the T-X Class of Topp Leone-G Family of Distributions: Statistical Properties and Applications

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Abstract. This article investigates the T-X class of Topp Leone- G family of distributions. Some members of the new family are discussed. The exponential-Topp Leone-exponential distribution (ETLED) which is one of the members of the family is derived and some of its properties which include central and non-central moments, quantiles, incomplete moments, conditional moments, mean deviation, Bonferroni and Lorenz curves, survival and hazard functions, moment generating function, characteristic function and R`enyi entropy are established. The probability density function (pdf) of order statistics of the model is obtained and the parameter estimation is addressed with the maximum likelihood method (MLE). Three real data sets are used to demonstrate its application and the results are compared with two other models in the literature.

1. Introduction

In recent times, the focus of many researchers has been drawn to the significance of introducing additional parameters into the existing probability models. This has brought about a huge breakthrough in the analysis of some complex data arising from various disciplines including financial management, insurance, economics and reliability analysis. Many methods have been proposed by different authors over the decades which include the exponentiated- G family of distributions introduced by Lehmann (1953) which became popular in the last two decades by Gupta et al. (1998) and Gupta and Kundu (1999, 2001 and 2002). The cdf (cumulative distribution function) of his type I is given by

$F(x) = G^{\alpha}(x),\ \alpha > 0$,

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where G is the cdf (cumulative distribution function) of any random variable. Suppose X is a continuous random variable, its corresponding pdf is

$$f(x) = \alpha g(x)G^{\alpha-1}(x) .$$

The cdf of his type II is defined by

$$F(x) = 1-(1-G(x))^{\alpha},\ \alpha>0 .$$

Its corresponding pdf is

$$f(x) = \alpha g(x)\big(1-G(x)\big)^{\alpha-1}.$$

Eugene et al (2002) proposed beta-G family of distributions with the cdf defined as

$F(x) = \frac{1}{B(\alpha,\beta)}\int_0^{G(x)} x^{\alpha-1}(1-x)^{\beta-1}dx$ and its pdf is

$$f(x) = \frac{g(x)}{B(\alpha,\beta)}G^{\alpha-1}(x)(1-G(x))^{\beta-1}, \alpha>0, \beta>0,$$

where $B(\alpha,\beta)$ is the beta function. Some of the members in the literature include beta Frechet distribution by Nadarajah andGupta (2004), the beta-Weibull distribution by Famoye et al. (2005), the beta-Pareto distribution by Akinsete et al. (2008) and other generalizations.

Jones (2009) and Cordeiro and de Castro (2011) obtained Kumaraswamy-G family of distributions. Its cdf which is obtained from the distribution of Kumaraswamy (1980) is

$$F(x) = 1-(1-G^{\alpha}(x))^{\beta}, \alpha>0,\ \beta>0$$

and its pdf is

$$f(x) = \alpha\beta g(x)G^{\alpha-1}(x)\big(1-G^{\alpha}(x)\big)^{\beta-1}.$$

Shaw and Buckley (2009) proposed quadratic rank transmutation map (QRTM) and the technique has been applied by many authors including Aryal and Tsokos (2011) on the transmuted Weibull distribution, Merovci (2014) on transmuted generalized Rayleigh distribution, Rahman et al. (2018) on general transmuted family of distributions and others. The cdf of transmuted distribution is defined as

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), \quad |\lambda| \le 1$$

with the corresponding pdf defined by

$$f(x) = g(x)[1+\lambda-2\lambda G(x).$$

Alzaatreh et al. (2013b) suggested T-X family of distributions which is the extension of beta-G family of distributions and it is obtained by replacing beta random variable with any non-negative continuous random variable T while random variables T and X are termed "transformed" and "transformer" respectively. Its cdf is defined as

$$V(x) = \int_a^{W(F(x))} r(t)dt = R\{W(F(x))\} \quad (1)$$

with corresponding pdf

$$v(x) = \left\{\frac{d}{dx} W(F(x))\right\} r\{W(F(x))\}, \quad (2)$$

where $r(t)$ is the pdf of random variable $T \in [a, b]$ such that $-\infty \leq a < b \leq \infty$ and $W(F(x))$ is a function of cdf $F(x)$ for any random variable X that satisfies the following conditions:

(i) $W(F(x)) \in [a, b]$

(ii) $W(F(x))$ is differentiable and monotonically non-decreasing

(iii) $W(F(x)) \to a$ as $x \to -\infty$ and

(iv) $W(F(x)) \to b$ as $x \to \infty$

Aljarrah et al. (2014) extended the work of Alzaatreh et al. (2013b) by introducing a wider class of W (.) function defined as

$W: (0,1) \to (a, b)$ for that $-\infty \leq a < b \leq \infty$ is right continuous and non-decreasing function such that

$\lim_{\gamma \to 0+} W: (\gamma) = a$ and $\lim_{\gamma \to 1-} W: (\gamma) = b$.

Therefore, $V(x), x \in (-\infty, \infty)$ is a distribution function that satisfies the required conditions of a distribution function:

$V(x)$ is a non-decreasing

$V(x)$ is right continuous and

$V(x) \to 0$ as $x \to -\infty$ and $V(x) \to 1$ as $x \to \infty$.

Alzaghal et al. (2013) considered exponentiated T-X family of distributions. Alzomarani et al. (2016) suggested Topp Leone-G family of distributions with the cdf defined as

$$F_{TLG}(x) = (G(x))^a (2 - G(x))^a, \; x \epsilon R, \; a > 0 \quad (3)$$

and its corresponding pdf given by

$$f_{TLG}(x) = 2ag(x)\overline{G}(x)(G(x))^{a-1}(2 - G(x))^{a-1}, \quad (4)$$

where $G(x)$ and $g(x)$ are the cdf and pdf of any continuous random variable X, $\overline{G}(x) = 1 - G(x)$ and a is the shape parameter. This was derived from Topp Leone (1955) distribution with the cdf

$F_{TL}(x) = x^a(2-x)^a,\ 0 \leq x \leq 1,\ a > 0.$

Jayakumar and Babu (2017) studied T-X transmuted family of distributions. Reyad et al. (2019) considered the exponentiated generalized Topp Leone –G family of distributions while Ibrahim et al. (2020) worked on Topp Leone exponentiated-G family of distributions.

In this paper, a new class of Topp Leone-G family of distributions is investigated with the introduction of additional parameter which makes the distributions from this family more flexible. Section 2 of this paper presents T-X Topp Leone-G family of distributions and some of its properties. Some members of the new family with different T distributions are discussed in section 3. In Section 4, some members of exponential Topp Leone-G family are obtained with graphical illustrations. Section 5 contains exponential Topp Leone exponential distribution (ETLED) and some of its properties which include moments, moment generating function, hazard function and order statistics. The application of ETLED is presented in section 6 and conclusion in section 7.

2. T-X Topp Leone-G family of distributions (TXTLG)

Suppose $R(t)$ and $r(t)$ are the cdf and pdf of non-negative continuous random variable T defined on $[0,\infty)$ and $F(x)$ denotes the cdf of Topp Leone-G family of distributions as defined in (3). Let $W\big(F(x)\big)$ be the hazard function of Topp Leone -G family defined as

$$W\big(F(x)\big) = -\ln\left[1-(G(x))^a(2-G(x))^a\right].$$

Using (1) and (2), the cdf of T-X Topp Leone –G family of distributions is

$$V_{TXTLG}(x) = \int_0^{-\ln\left[1-(G(x))^a(2-G(x))^a\right]} r(t)dt = R\{-\ln\left[1-(G(x))^a(2-G(x))^a\right]\} \qquad (5)$$

with the corresponding pdf

$$v_{TXTLG}(x) = \frac{2ag(x)\bar{G}(x)\big(G(x)\big)^{a-1}\big(2-G(x)\big)^{a-1}}{1-\big(G(x)\big)^a\big(2-G(x)\big)^a} r\{-\ln\left[1-(G(x))^a(2-G(x))^a\right]\}. \qquad (6)$$

Its survival function is

$$s_{TXTLG}(x) = 1 - V_{TXTLG}(x) = 1 - R\{-\ln\left[1-(G(x))^a(2-G(x))^a\right]\}$$

and its hazard function is

$$h_{TXTLG}(x) = \frac{v_{TXTLG}(x)}{1-V_{TXTLG}(x)} = \frac{2ag(x)\bar{G}(x)\big(G(x)\big)^{a-1}\big(2-G(x)\big)^{a-1} r\Big\{-\ln\Big[1-\big(G(x)\big)^a\big(2-G(x)\big)^a\Big]\Big\}}{\Big\{1-R\Big[-\ln\Big(1-\big(G(x)\big)^a\big(2-G(x)\big)^a\Big)\Big]\Big\}\Big[1-\big(G(x)\big)^a\big(2-G(x)\big)^a\Big]}.$$

3. Some members of T-X Topp Leone-G family of distributions with different T distributions.

Some members of TXTLG are discussed in this section for different distributions of random variable T. These members include

3.1 Gompertz-Topp Leone-G family of distributions (GTLGD)

If a continuous random variable T follows a two parameter Gompertz distribution having its cdf and pdf given by $R(t) = 1 - e^{-\beta(e^{kt}-1)}$ and $r(t) = \beta k e^{kt} e^{-\beta(e^{kt}-1)}, t > 0,\ \beta > 0, k > 0,$ where β and k are the parameters, the cdf of GTLGD is

$$V_{GTLGD}(x) = 1 - e^{-\beta\left\{\left[1-(G(x))^{a}(2-G(x))^{a}\right]^{-k}-1\right\}}$$

and its corresponding pdf is

$$v_{GTLGD}(x) = \frac{2a\beta k g(x)\overline{G}(x)\left(G(x)\right)^{a-1}\left(2-G(x)\right)^{a-1}}{\left[1-(G(x))^{a}(2-G(x))^{a}\right]^{k+1} e^{\beta\{[1-(G(x))^{a}(2-G(x))^{a}]^{-k}-1\}}}.$$

3.2 Rayleigh-Topp Leone-G family of distributions (RTLGD)

If a continuous random variable T follows Rayleigh distribution having its cdf and pdf given by $R(t) = 1 - e^{-\frac{t^2}{2\sigma^2}}$ and $r(t) = \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}, t > 0, \sigma > 0$, where σ is the parameter, the cdf of RTLGD is

$$V_{RTLGD}(x) = 1 - e^{-\frac{1}{2\sigma^2}\{\ln[1-(G(x))^{a}(2-G(x))^{a}]\}^{2}}$$

and its corresponding pdf is

$$v_{RTLGD}(x) = \frac{-2a g(x)\overline{G}(x)\left(G(x)\right)^{a-1}\left(2-G(x)\right)^{a-1}\ln[1-(G(x))^{a}(2-G(x))^{a}}{\sigma^2\left[1-(G(x))^{a}(2-G(x))^{a}\right] e^{\frac{1}{2\sigma^2}\{\ln[1-(G(x))^{a}(2-G(x))^{a}]\}^{2}}}.$$

3.3 Lomax-Topp Leone-G family of distributions (LTLGD)

If a continuous random variable T follows a two parameter Lomax distribution having its cdf and pdf given by $R(t) = 1 - (1+bt)^{-\lambda}$ and $r(t) = \frac{b\lambda}{(1+bt)^{\lambda+1}}, t > 0,\ \lambda > 0,\ b > 0$, where b and λ are the parameters , the cdf of LTLGD is

$$V(x)_{LTLGD} = 1 - \{1 - b\ln[1-(G(x))^{a}(2-G(x))^{a}]\}^{-\lambda}$$

and its corresponding pdf is

$$v_{LTLGD}(x) = \frac{2ab\lambda g(x)\overline{G}(x)\left(G(x)\right)^{a-1}\left(2-G(x)\right)^{a-1}}{\left[1-(G(x))^{a}(2-G(x))^{a}\right]\{1-b\ln[1-(G(x))^{a}(2-G(x))^{a}]\}^{\lambda+1}}.$$

3.4 Weibull-Topp Leone-G family of distributions (WTLGD)

If a continuous random variable T follows a two parameter Weibull distribution having its cdf and pdf given by $R(t)=1-e^{-(\frac{t}{m})^{\theta}}$ and $r(t)=\frac{\theta t^{\theta-1}}{m^{\theta}}e^{-(\frac{t}{m})^{\theta}}, t>0,\ \theta>0,\ m>0$, where m and θ are the parameters, the cdf of WTLGD is

$$V_{WTLGD}(x)=1-e^{-\frac{1}{m^{\theta}}\left\{-\ln\left[1-(G(x))^{a}(2-G(x))^{a}\right]\right\}^{\theta}}$$

and its corresponding pdf is

$$v_{WTLGD}(x)=\frac{2a\theta g(x)\overline{G}(x)(G(x))^{a-1}(2-G(x))^{a-1}\left\{-\ln\left[1-(G(x))^{a}(2-G(x))^{a}\right]\right\}^{\theta+1}}{m^{\theta}\left[1-(G(x))^{a}(2-G(x))^{a}\right]e^{\frac{1}{m^{\theta}}\left\{-\ln\left[1-(G(x))^{a}(2-G(x))^{a}\right]\right\}^{\theta}}}$$

3.5 Exponential-Topp Leone-G family of distributions (ETLGD)

If a continuous random variable T follows an exponential distribution having its cdf and pdf given by $R(t)=1-e^{-ct}$ and $r(t)=ce^{-ct},\ t>0,\ c>0$ where c is the parameter, the cdf of ETLGD is

$$V_{ETLGD}(x)=1-\left[1-(G(x))^{a}(2-G(x))^{a}\right]^{c}$$

and its corresponding pdf is

$$v_{ETLGD}(x)=2acg(x)\overline{G}(x)(G(x))^{a-1}(2-G(x))^{a-1}\left[1-(G(x))^{a}(2-G(x))^{a}\right]^{c-1}.$$

4. Some members of exponential-Topp Leone-G family of distributions (ETLGD).

Here, some members of ETLGD are obtained and the graphical illustrations of their cdfs and pdfs are displayed in Figure 1 to 4.

4.1 Exponential-Topp Leone-Lomax distribution (ETLLD).

If a continuous random variable X follows a two parameter Lomax distribution having its cdf and pdf given by $G(x)=1-(1+kx)^{-p}$ and $g(x)=\frac{pk}{(1+kx)^{p+1}}, x>0,\ p>0,\ k>0,$ where k and β are the parameters, the cdf of ETLLD is

$$V_{ETLLD}(x)=1-\left[1-\left(\frac{(1+kx)^{2p}-1}{(1+kx)^{2p}}\right)^{a}\right]^{c}$$

and its corresponding pdf is

$$v_{ETLLD}(x)=\frac{2acpk\left((1+kx)^{2p}-1\right)^{a-1}\left[(1+kx)^{2ap}-\left((1+kx)^{2p}-1\right)^{a}\right]^{c-1}}{(1+kx)^{2acp+1}}.$$

The graphs of cdf and pdf of ETLLD for different selected values of parameters a, c, p and k are displayed in Figure 1.

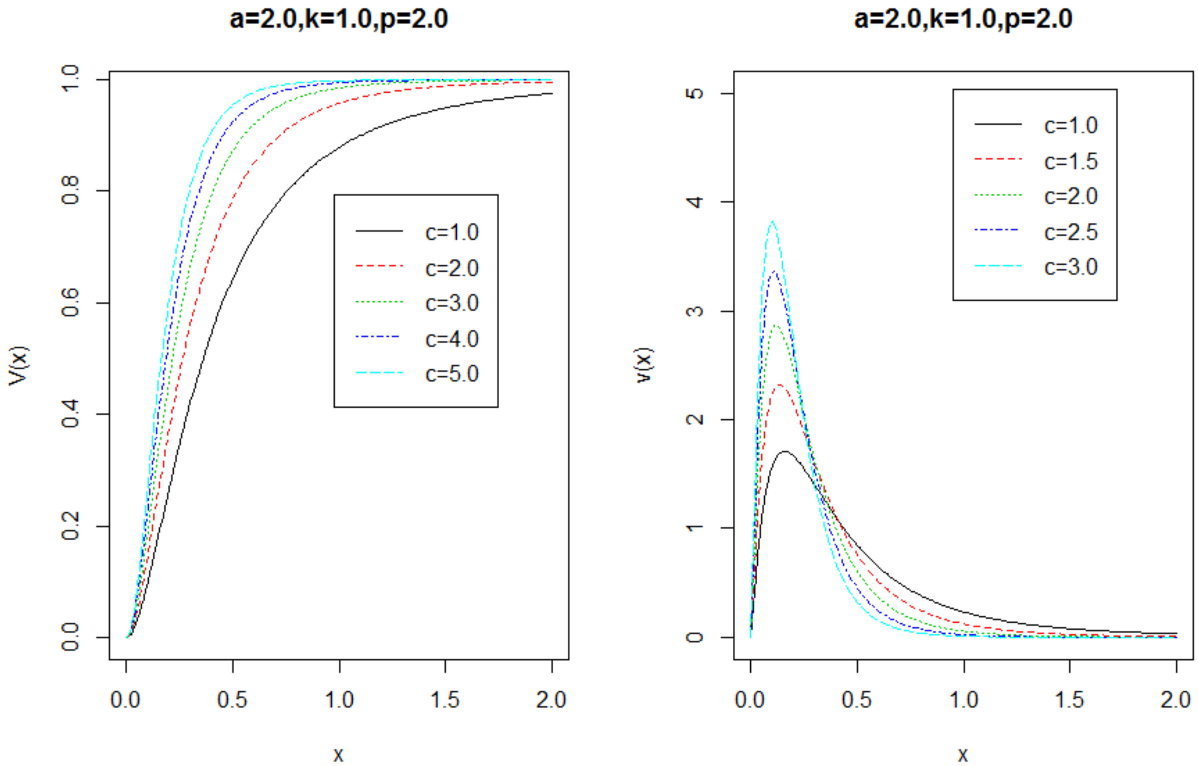

Figure 1: The graphs of cdf and pdf of ETLLD

4.2 Exponential-Topp Leone-uniform distribution (ETLUD).

If a continuous random variable X has uniform distribution having its cdf and pdf given by $G(x) = \frac{x}{b}$ and $g(x) = \frac{1}{b}$, $0 < x < b$, where b is the parameter, the cdf of ETLUD is

$$V_{ETLUD}(x) = 1 - \left[1 - \left(\frac{x}{b}\right)^{a}\left(2 - \frac{x}{b}\right)^{a}\right]^{c}$$

and its corresponding pdf is

$$v_{ETLUD}(x) = \frac{2ac}{b}\left(1 - \frac{x}{b}\right)\left(\frac{x}{b}\right)^{a-1}\left(2 - \frac{x}{b}\right)^{a-1}\left[1 - \left(\frac{x}{b}\right)^{a}\left(2 - \frac{x}{b}\right)^{a}\right]^{c-1}.$$

The graphs of cdf and pdf of ETLUD for different selected values of parameters a, b and c are displayed in Figure 2.

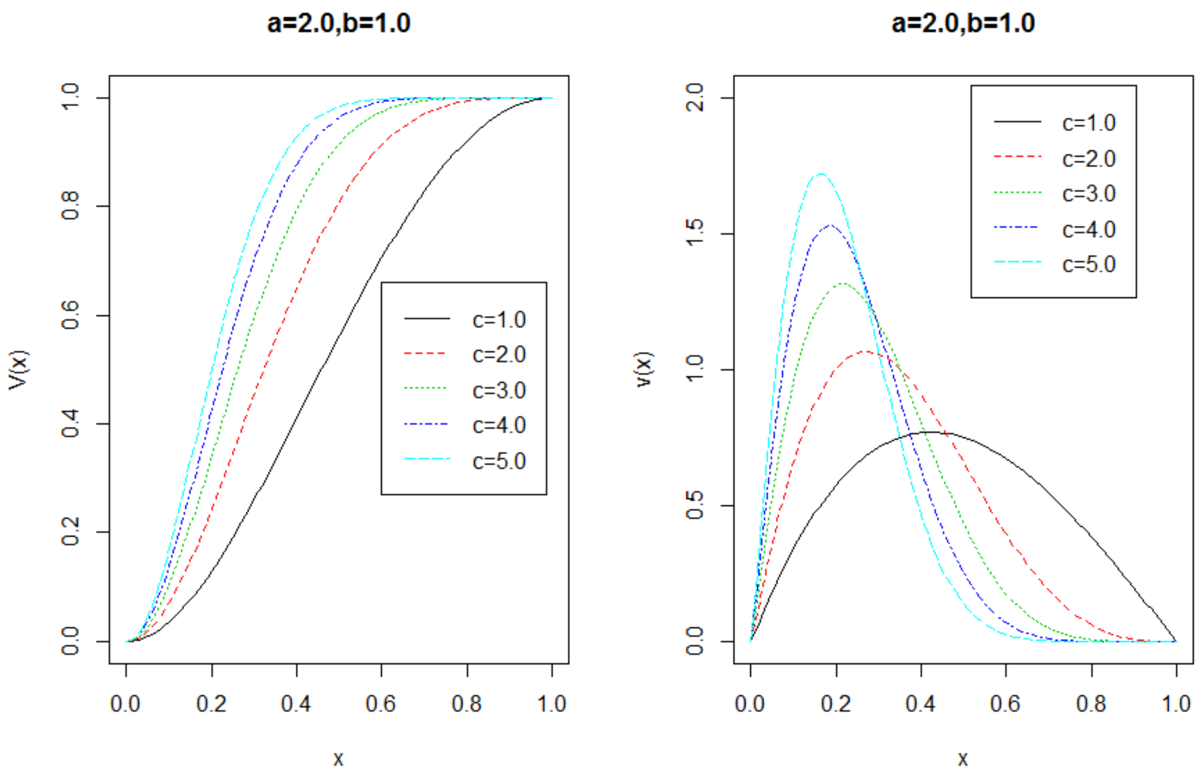

Figure 2: The graphs of cdf and pdf of ETLUD

4.3 Exponential-Topp Leone-Rayleigh distribution (ETLRD).

If a continuous random variable X has Rayleigh distribution having its cdf and pdf given by $G(x) = 1 - e^{-\frac{x^2}{2p^2}}$ and $g(x) = \frac{x}{p^2} e^{-\frac{x^2}{2p^2}},\ x > 0,\ p > 0$, where p is the parameter, the cdf of ETLRD is

$$V_{ETLRD}(x) = 1 - \left[1 - \left(1 - e^{-\frac{x^2}{p^2}}\right)^a\right]^c$$

and its corresponding pdf is

$$v_{ETLRD}(x) = \frac{2ac}{p^2} x e^{-\frac{x^2}{p^2}} \left(1 - e^{-\frac{x^2}{p^2}}\right)^{a-1} \left[1 - \left(1 - e^{-\frac{x^2}{p^2}}\right)^a\right]^{c-1}.$$

The graphs of cdf and pdf of ETLRD for different selected values of parameters a, c and p are displayed in Figure 3.

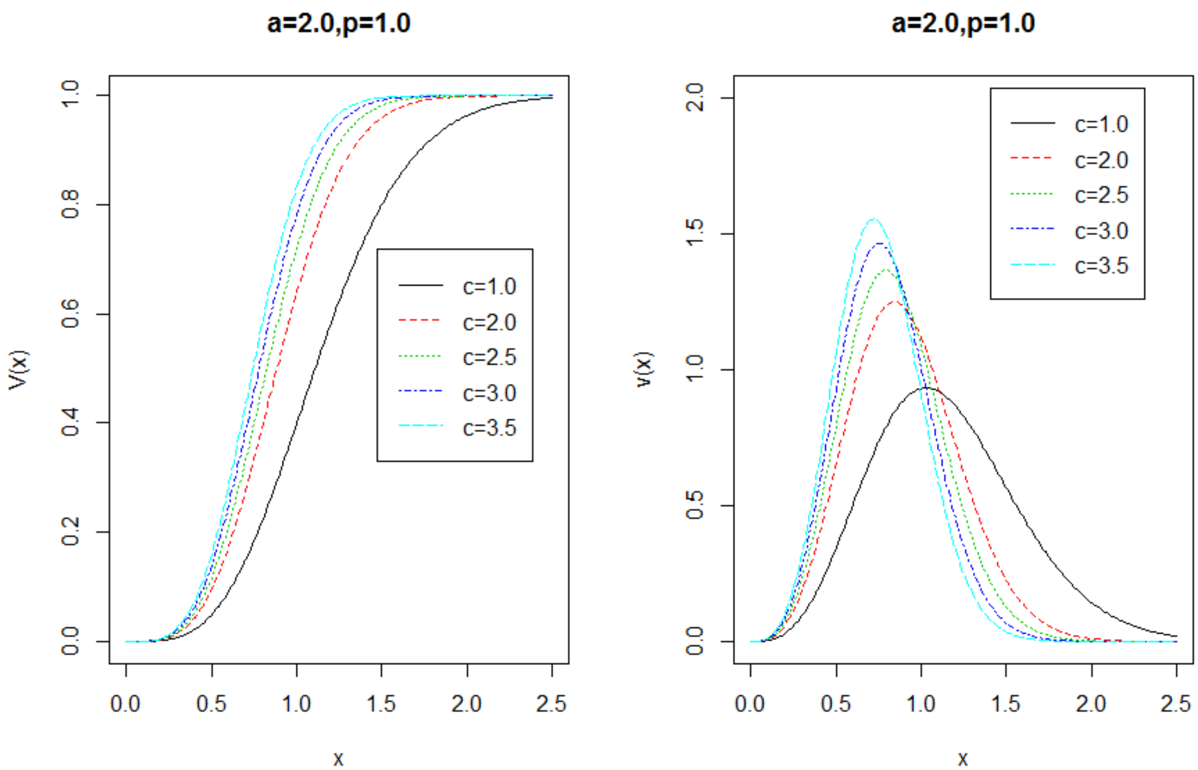

Figure 3: The graphs of cdf and pdf of ETLRD

4.4 Exponential-Topp Leone-exponential distribution (ETLED).

If a continuous random variable X has exponential distribution having its cdf and pdf given by $G(x) = 1 - e^{-mx}$ and $g(x) = me^{-mx}$, $x > 0$, $m > 0$, where m is the parameter. The cdf of ETLRD is

$$V_{ETLED}(x) = 1 - [1 - (1 - e^{-2mx})^a]^c \tag{7}$$

and its corresponding pdf is

$$v_{ETLED}(x) = 2acme^{-2mx}(1 - e^{-2mx})^{a-1}[1 - (1 - e^{-2mx})^a]^{c-1}. \tag{8}$$

The graphs of cdf and pdf of ETLED for different selected values of parameters a, c and m are displayed in Figure 4.

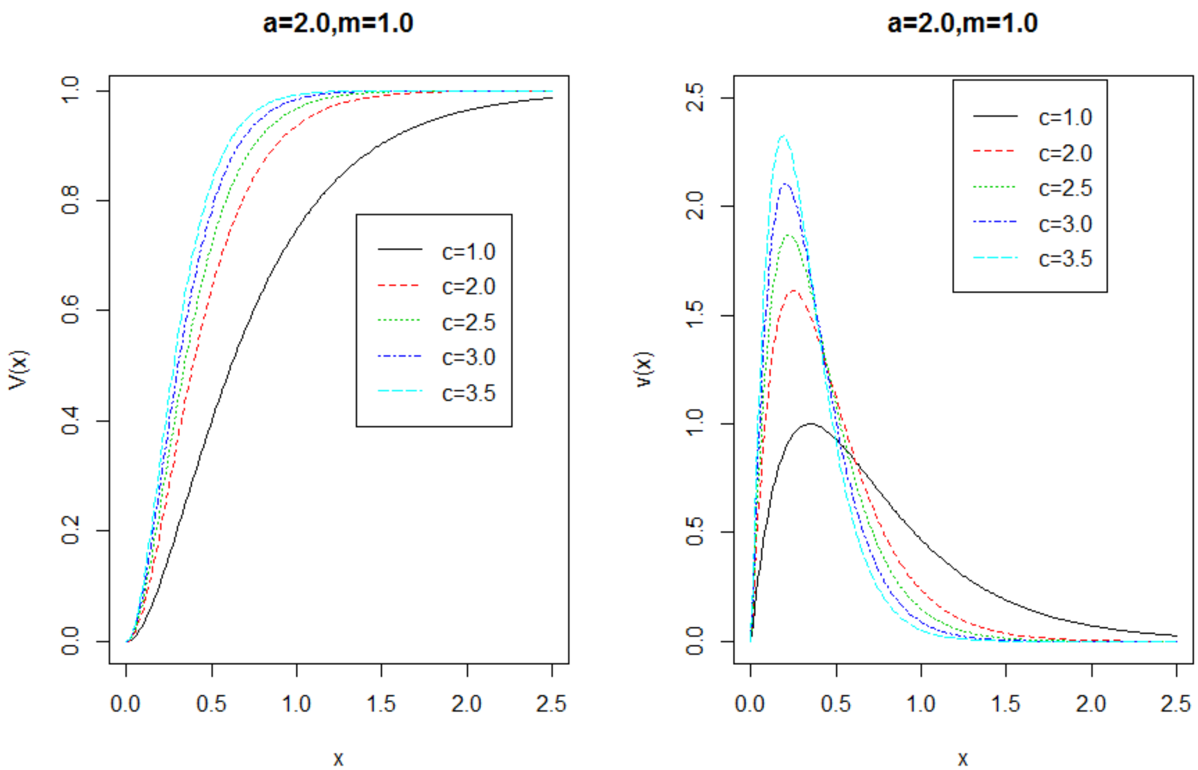

Figure 4: The graphs of cdf and pdf of ETLED

5. Some properties of exponential-Topp Leone-exponential distribution (ETLED)

5.1 Moments

The pth non-central moment of ETLED is

$$\mathrm{E}[X^p] = 2acm \int_0^\infty x^p e^{-2mx}(1-e^{-2mx})^{a-1}[1-(1-e^{-2mx})^a]^{c-1}dx. \tag{9}$$

Using series expansion, (9) becomes,

$$\mathrm{E}[X^p] = 2acm \sum_{q=0}^\infty \sum_{r=0}^\infty \binom{c-1}{q}\binom{a(q+1)-1}{r}(-1)^{q+r}\int_0^\infty x^p e^{-2m(r+1)x}\,dx.$$

On integration,

$$\mathrm{E}[X^p] = \frac{ac\Gamma(p+1)}{(2m)^p}\sum_{q=0}^\infty \sum_{r=0}^\infty \binom{c-1}{q}\binom{a(q+1)-1}{r}(-1)^{q+r}(r+1)^{-(p+1)}. \tag{10}$$

The mean of ETLED is obtained when $p=1$ in (10) and it is given as

$$\mathrm{E}[\mathrm{X}] = \frac{ac}{2m}\sum_{q=0}^\infty \sum_{r=0}^\infty \binom{c-1}{q}\binom{a(q+1)-1}{r}(-1)^{q+r}(r+1)^{-2}. \tag{11}$$

The sth central moment of ETLED is

$$E[X-\mu]^s = \sum_{p=0}^{s} \binom{s}{p} (-\mu)^{s-p} E[X^p] \quad (12)$$

Where $E[X^p]$ is given in (10) and μ is the mean of ETLED. On substitution,

$$E[X-\mu]^s = 2acm \sum_{p=0}^{s} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{s}{p} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} (-\mu)^{s-p} \frac{\Gamma(p+1)}{(2m(r+1))^{p+1}}. \quad (13)$$

By setting $s=2$ in (13), the variance of ETLED is obtained as

$$E[X-\mu]^2 = 2acm \sum_{p=0}^{2} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{2}{p} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} (-\mu)^{2-p} \frac{\Gamma(p+1)}{(2m(r+1))^{p+1}}.$$

Other higher moments such as kurtosis and skewness of ETLED can be derived from (13).

5.2 Quantiles

The kth quantile W(k) of ETLED is defined as

$$1-[1-(1-e^{-2mx_k})^a]^c = k. \quad (14)$$

On solving for x_k in (14),

$$W(k) = x_k = -\frac{1}{2m} ln\left[1-\left(1-(1-k)^{\frac{1}{c}}\right)\right]^{\frac{1}{a}}. \quad (15)$$

The median M is obtained by setting $k=0.5$ in (15) to have

$$M = x_{0.5} = -\frac{1}{2m} ln\left[1-\left(1-(0.5)^{\frac{1}{c}}\right)\right]^{\frac{1}{a}}. \quad (16)$$

5.3 Incomplete Moments

The pth incomplete moment of ETLED is defined as

$$J_p(z) = \int_0^z f(x)dx = 2acm \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} \int_0^z x^p e^{-2m(r+1)x} dx. \quad (17)$$

But the integrand in (17) is a lower incomplete gamma. This implies that

$$J_p(z) = \frac{ac}{(2m)^p} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} \frac{\gamma(p+1,[2m(r+1)]z)}{(r+1)^{p+1}}. \quad (18)$$

The first incomplete moment is obtained by setting p=1 in (18) to have

$$J_1(z) = \frac{ac}{2m} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} \frac{\gamma(2,[2m(r+1)]z)}{(r+1)^2}. \quad (19)$$

5.4 Moment generating function and characteristic function

The moment generating function of ETLED is

$$M_X(t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} E[X^p] = 2acm \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} \frac{t^p \Gamma(p+1)}{p!(2m(r+1))^{p+1}}.$$

Its corresponding characteristic function is

$$Q_X(it) = 2acm \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} \frac{(it)^p \Gamma(p+1)}{p!(2m(r+1))^{p+1}}$$

5.5 Random number generation

Random numbers can be generated from ETLED by using the method of inversion. This implies that

$$V_{ETLED}(x) = u \ , \tag{20}$$

where random variable $u \sim U(0,1)$ i.e. u is uniformly distributed on zero and unity. This implies

$$1 - [1 - (1 - e^{-2mx}\ )^{a}]^{c} = u. \tag{21}$$

Solving for x in (21) gives

$$x = -\frac{1}{2m} ln\left[1 - \left(1 - (1-u)^{\frac{1}{c}}\right)\right]^{\frac{1}{a}}. \tag{22}$$

Random numbers can be generated from (22) when the values of parameters a, c and m are known.

5.6 Mean Deviation

The mean deviation about the mean of ETLED is defined by

$$\delta_1(x) = \int_0^{\infty} |x - \mu| f(x)dx = 2\mu_1' F(\mu_1') - 2J_1(\mu_1')$$

and the mean deviation about the median M of ETLED is given by

$$\delta_2(x) = \int_0^{\infty} |x - M| f(x)dx = 2\mu_1' F(\mu_1') - 2J_1(\mu_1'),$$

Where μ_1' is the first non-central moment of ETLED given in (11), $F(\mu_1')$ can be obtained from the cdf of ETLED and $J_1(\mu_1')$ can be calculated from the first incomplete moment in (19) .

5.7 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves of ETLED are defined by

$$B(\pi) = \frac{J_1(b)}{\pi(\mu_1')} \text{ and } L(\pi) = \frac{J_1(b)}{\mu_1'}$$

respectively, where $b = W(\pi)$ which can be determined from the quantile function in (15) and $J_1(b)$ can be obtained from (19).

5.8 Conditional Moments

The pth conditional moment of ETLED is defined as

$$E[X^p/X > t] = \frac{1}{\overline{F}(t)} \int_t^{\infty} x^p f(x)dx.$$

$$= 2acm \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r} \int_t^{\infty} x^p e^{-2m(r+1)x}\, dx. \tag{23}$$

where $\overline{F}(t) = 1 - F(t)$. The integrand in (23) is an upper incomplete gamma. Therefore,

$$E[X^p/X > t] = \frac{ac}{(2m)^p\ \bar{F}(t)} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{c-1}{q} \binom{a(q+1)-1}{r} (-1)^{q+r}\ \frac{\Gamma(p+1)-\gamma(p+1,[2m(r+1)]z)}{(r+1)^{p+1}}. \quad (24)$$

The mean residual lifetimes of ETLED which is given by $E[X^p/X > t] - t$ can be determined from the conditional moment in (24).

5.9 R\`enyi Entropy

The R\`enyi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta} log\ [I(\delta)]\ ,$$

where $\delta > 0$ and $\delta \neq 1$

If $f(x)$ is the pdf of ETLED then $I(\delta)$ is given by

$$I(\delta) = \int_0^{\infty} f^{\delta}(x) dx$$

$$= (2acm)^{\delta} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{\delta(c-1)}{q} \binom{a(q+\delta)-\delta}{r} (-1)^{q+r} \int_0^{\infty} x^p e^{-2m(r+\delta)x}\, dx\ . \quad (25)$$

By integrating the integrand in (25),

$$I(\delta) = (2acm)^{\delta} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{\delta(c-1)}{q} \binom{a(q+\delta)-\delta}{r} (-1)^{q+r} [2m(r+\delta)]^{-1}. \quad (26)$$

Therefore, the Re\`nyi entropy of ETLED is

$$I_R(\delta) = \frac{1}{1-\delta} log \left[(2acm)^{\delta} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \binom{\delta(c-1)}{q} \binom{a(q+\delta)-\delta}{r} (-1)^{q+r} [2m(r+\delta)]^{-1} \right]. \quad (27)$$

5.10 Reliability Analysis

The survival function of ETLED is

$$s_{ETLED}(x) = 1 - V_{ETLED}(x) = [1 - (1 - e^{-2mx}\ )^a]^c$$

and its hazard function is

$$h_{ETLED}(x) = \frac{v_{ETLED}(x)}{1-V_{ETLED}(x)} = 2acme^{-2mx}(1 - e^{-2mx})^{a-1}[1 - (1 - e^{-2mx})^a]^{-1}.$$

The graphs of survival and hazard function of ETLED are displayed in Figure 5.

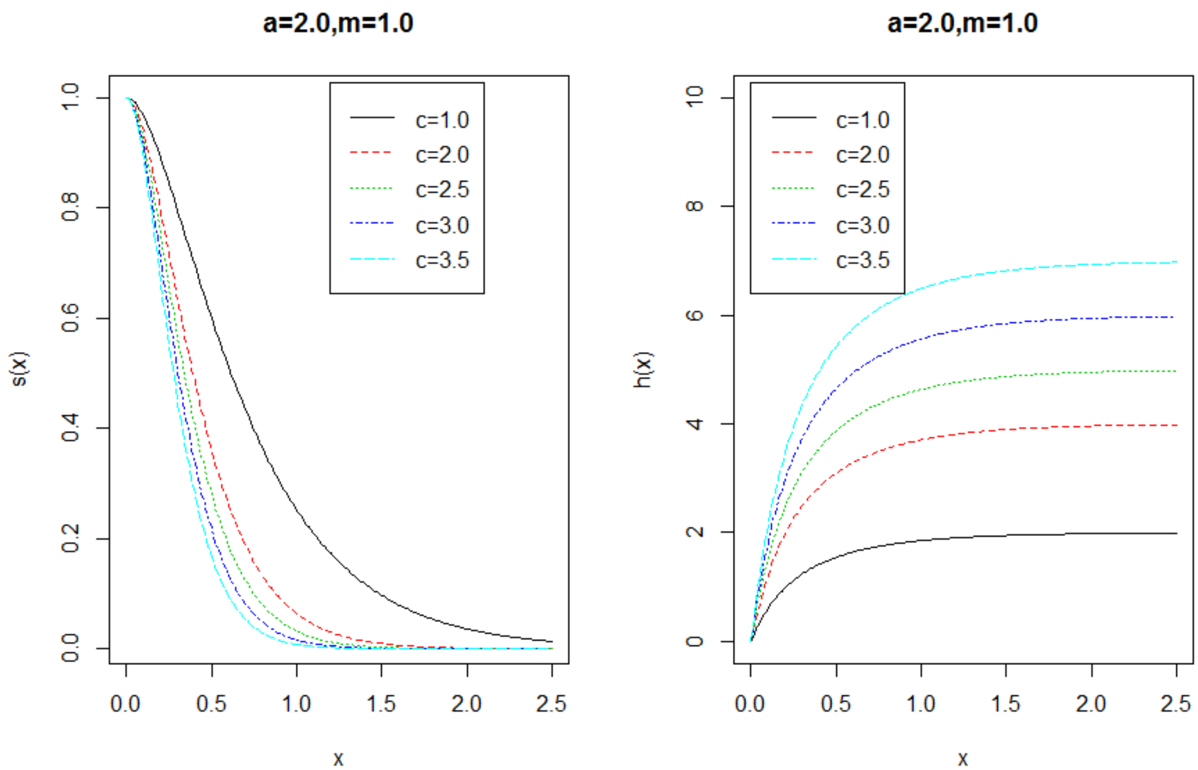

Figure 5: The survival and hazard function of ETLED

5.11 Order Statistics

Let $X_1, X_2 \ldots X_n$ be independent random variables with size n. The arrangement $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ is called order statistics. The pdf of order statistics $X_{(j)}$ of ETLED, where $j = 1,2,3, \ldots, n$, is given by David (1970) as

$$v_{ETLED\,X_{(j)}}(x) = \frac{1}{B(j,n-j+1)}[V_{ETLED}(x)]^{j-1}[1 - V_{ETLED}(x)]^{n-j}\, v_{ETLED}(x). \tag{28}$$

On substitution,

$$v_{ETLED\,X_{(j)}}(x) = \frac{2acme^{-2mx}}{B(j,n-j+1)}(1 - e^{-2mx})^{a-1}[1 - (1 - e^{-2mx})^{a}]^{c(n-j+1)-1}\{1 - [1 - (1 - e^{-2mx})^{a}]^{c}\}^{j-1}.$$

On expansion,

$$v_{ETLED\,X_{(j)}}(x) = \frac{2acme^{-2mx}}{B(j,n-j+1)}\sum_{i=0}^{j-1}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\binom{j-1}{i}\binom{c(n-j+2)-1}{l}\binom{a(l+1)-1}{w}(-1)^{i+l+w}e^{-2mwx}. \tag{29}$$

The pdf of the smallest order statistics is obtained by setting $j = 1$ in (29) to give

$$v_{ETLED\,X_{(1)}}(x) = 2acmne^{-2mx}\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}\binom{c(n+1)-1}{l}\binom{a(l+1)-1}{w}(-1)^{i+l+w}e^{-2mwx}$$

while pdf of the largest order statistics is obtained by setting $j = n$ in (29) and the result is

$$v_{ETLED}{}_{X_{(n)}}(x) = 2acmne^{-2mx} \sum_{i=0}^{n-1} \sum_{l=0}^{\infty} \sum_{w=0}^{\infty} \binom{n-1}{i} \binom{2c-1}{l} \binom{a(l+1)-1}{w} (-1)^{i+l+w} e^{-2mwx}.$$

5.12 Parameter Estimation

Suppose sample $x_1, x_2 \dots x_n$ of size n is drawn from ETLED with pdf $v_{ETLED}(x)$ given in (8), its likelihood function is

$$L(x; a, c, m) = (2acm)^n e^{-2m \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1 - e^{-2mx_i})^{a-1} [1 - (1 - e^{-2mx_i})^a]^{c-1}.$$

Its log likelihood function is

$$lnL(x; a, c, m) = \text{nln}(2acm) - 2m \sum_{i=1}^{n} x_i + (a-1) \sum_{i=1}^{n} ln(1 - e^{-2mx_i}) + (c-1) \sum_{i=1}^{n} ln[1 - (1 - e^{-2mx_i})^a]. \quad (30)$$

The estimate of each inherent parameter is obtained by differentiating (30) with respect to each parameter and equates it to zero. The following non-linear system of equations is obtained.

$$\frac{\partial lnL(x,a,c,m)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} ln(1 - e^{-2mx_i}) - (c-1) \sum_{i=1}^{n} \frac{(1-e^{-2mx_i})^a \ln(1-e^{-2mx_i})}{[1-(1-e^{-2mx_i})^a]} = 0,$$

$$\frac{\partial lnL(x,a,c,m)}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n} ln[1 - (1 - e^{-2mx_i})^a] = 0$$

and

$$\frac{\partial lnL(x, a, c, m)}{\partial m} = \frac{n}{m} - 2 \sum_{i=1}^{n} x_i + 2(a-1) \sum_{i=1}^{n} \frac{x_i e^{-2mx_i}}{(1 - e^{-2mx_i})} - 2a(c-1) \sum_{i=1}^{n} \frac{x_i e^{-2mx_i} (1 - e^{-2mx_i})^{a-1}}{[1 - (1 - e^{-2mx_i})^a]} = 0.$$

The log-likelihood function in (30) can be maximized using non-linear optimization algorithms such as Particle Swarm Optimization (PSO), Broyden-Fletcher-Goldfarb-Shanno (BFGS), Nelder-Mead (NM), Conjugate Gradient (CG) and Simulated-Annealing (SANN) in Adequacy model which can be implemented in R package.

6. Applications

In this section, three real data sets are presented to illustrate the application of the exponential-Topp Leone-exponential distribution (ETLED). The first data is obtained from Gupta and Kundu (2010). It is a strength data originally considered by Badar and Priest (1982). The data which represent the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. The single fibers data set of 10 mm in gauge lengths with sample size 63 is considered in this work. The second and third data sets are extracted from Tahir et al. (2015). They represent the failure and service times for a particular model. The data consist of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that had not failed at the time of observation. The unit for measurement is 1000 h. The second and third data sets have 84 and 63 observations respectively.

6.1 The strength of single carbon fibers data

The data on the strength of single fibers of 10 mm in gauge lengths with 63 observations is presented as follows:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

6.2 The failure times of 84 Aircraft Windshield data

The data on the failure times of 84 Aircraft Windshield is presented as follows:

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

6.3 The service times of 63 Aircraft Windshield data

The data on the service times of 63 Aircraft Windshield is presented as follows:

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 150.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

The maximum likelihood using SANN method is implemented in R to obtain the maximum likelihood estimates with the corresponding standard errors of the inherent parameters in the fitted distributions which are reported alongside with the negative maximum log-likelihood (-LL), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov Smirnov (K-S) and the corresponding p-value for each data set as shown in Table 1, 2 and 3. The results in Table1, 2 and 3 reveal that the exponential-Topp Leone-exponential distribution (ETLED) has the smallest -LL value, AIC, BIC and K-S statistics followed by Topp Leone-exponential distribution (TLED) considered by Al-Shomrani et al. (2016) and exponential distribution (ED), which indicate that ETLED gives the best fit for all the data sets. Figure 6 displays the estimated pdfs of the strength of single carbon fibers data while Figure 7 and Figure 8 display the estimated pdfs for the failure times and service times of the Aircraft Windshield data respectively.

Table1. The MLEs with corresponding standard errors (in parentheses) and statistics for the strength of single carbon fibers data

Model	Estimates	-LL	AIC	BIC	K-S	p-value
ETLED	$\hat{a} = 13.651(3.990)$ $\hat{c} = 9.382\ (4.566)$ $\hat{m} = 0.286(0.064)$	58.250	122.501	128.930	0.076	0.861
TLED	$\hat{a} = 25.261(6.015)$ $\hat{m} = 0.606(0.046)$	66.728	137.457	141.743	0.142	0.159
ED	$\hat{m} = 0.327(0.041)$	133.446	268.892	271.035	0.486	2.378e-13

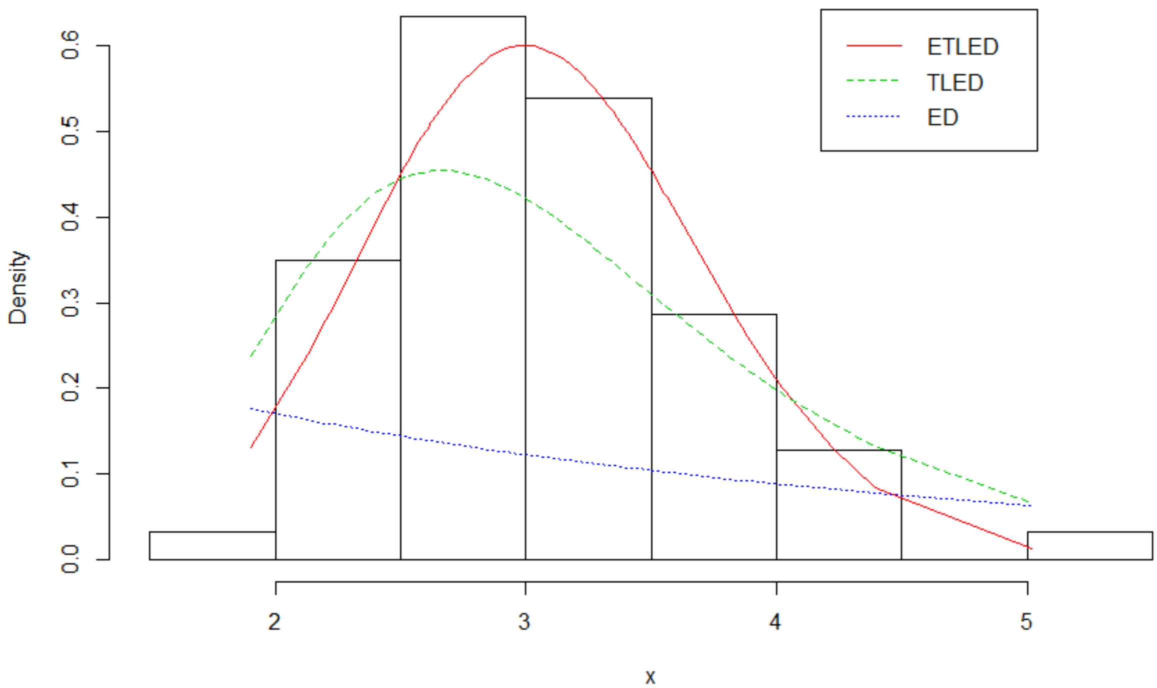

Figure 6: The fitted pdfs for the strength of single carbon fibers data

Table 2: The MLEs with corresponding standard errors (in parentheses) and statistics for the failure times of 84 Aircraft Windshield data

Model	Estimates	-LL	AIC	BIC	K-S	p-value
ETLED	$\hat{a} = 2.982(0.893)$ $\hat{c} = 6.826(17.802)$ $\hat{m} = 0.129(0.192)$	133.396	272.792	280.085	0.078	0.682
TLED	$\hat{a} = 3.558(0.611)$ $\hat{m} = 0.379(0.038)$	139.841	283.681	288.543	0.121	0.172
ED	$\hat{m} = 0.391(0.043)$	162.877	327.754	330.185	0.303	4.085e-07

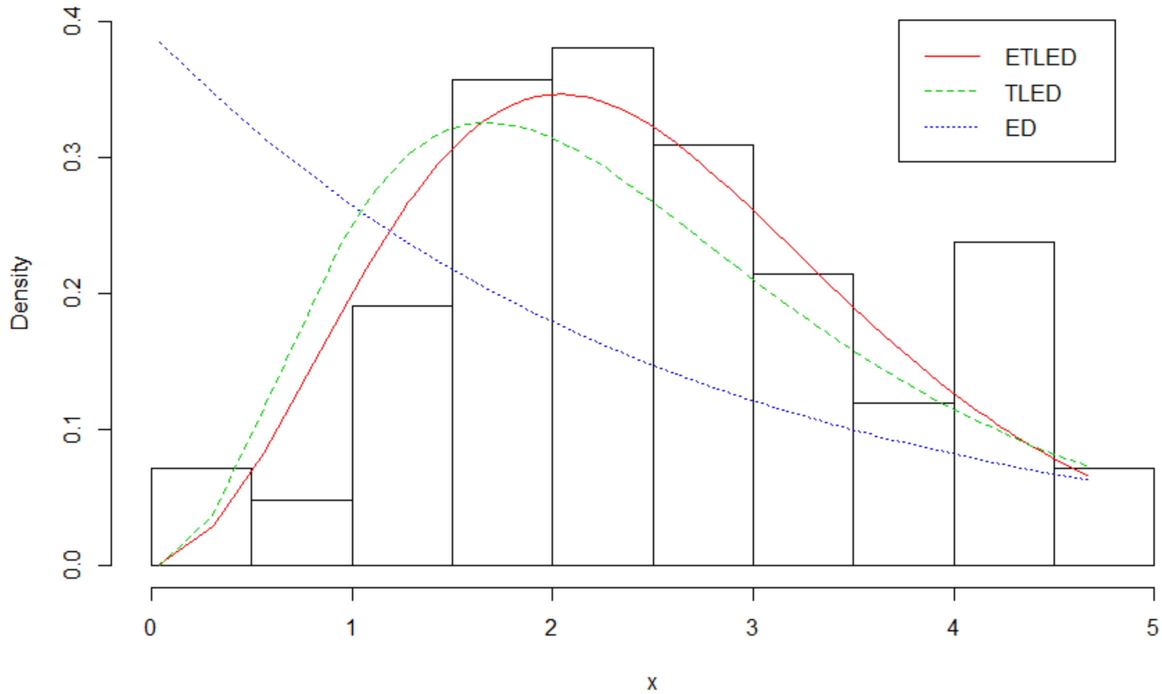

Figure 7: The fitted pdfs for the failure times of 84 Aircraft Windshield data

Table 3: The MLEs with corresponding standard errors (in parentheses) and statistics for the service times of 63 Aircraft Windshield data

Model	Estimates	-LL	AIC	BIC	K-S	p-value
ETLED	$\hat{a} = 1.818(0.201)$ $\hat{c} = 3.660(---)$ $\hat{m} = 0.136(---)$	101.972	209.943	216.372	0.132	0.204
TLED	$\hat{a} = 1.893(0.340)$ $\hat{m} = 0.345(0.047)$	103.547	211.094	215.380	0.143	0.138
ED	$\hat{m} = 0.480(0.060)$	109.299	220.597	222.740	0.208	7.291e-03

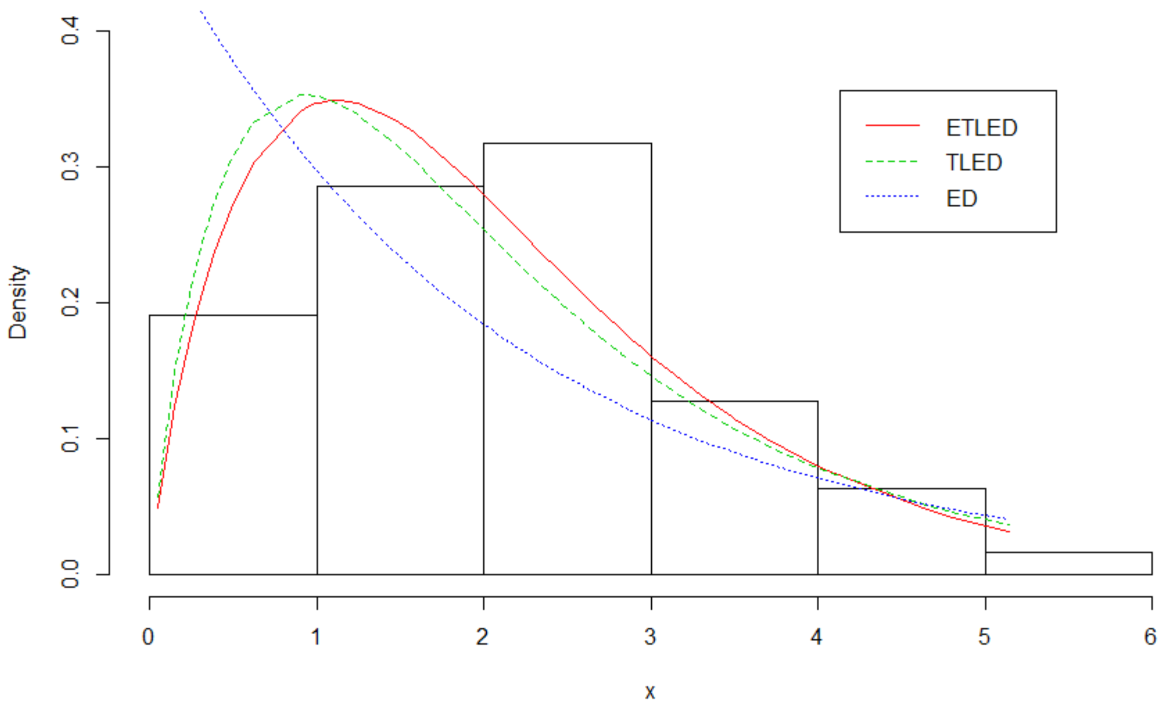

Figure 8: The fitted pdfs for the service times of 63 Aircraft Windshield data

7. Conclusion

A new family of Topp Leone-G family which is called T-X Topp Leone-G family of distributions with some of its properties is discussed. Some members of the family for different T distributions are derived. One of the members of exponential-Topp Leone-G family which is exponential-Topp Leone-exponential distribution (ETLED) is studied with some of its properties such as moments, quantiles, incomplete moments, conditional moments, mean deviation, Bonferroni and Lorenz curves, survival and hazard functions, moment generating function, characteristic function and R`enyi entropy are established. The model is applied to three data sets and the results are compared with Topp Leone-exponential distribution (TLED) and exponential distribution (ED). It is established that ETLED provides better fit than TLED and ED.

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