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A Four-Parameter Generalization of the Chen Distribution Family

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Abstract. As an addition to the known continuous distribution families, a four-parameter model of significant flexibility is introduced for the purpose of modeling positive random variables. After the Weibull-Chen {Weibull} Type I model is introduced, some standard numerical properties are explored, including those involved in maximum likelihood estimation. We compare the performance of the model with similar models with application to two real data sets and demonstrate the improved fit.

1. Introduction

The modeling of random processes depends on both the flexibility of a model to accurately reflect the observed aspects of the data distribution and a high degree of certainty in model estimates. Variables in real populations, or resulting from real processes, rarely exhibit the exact behavior of simple statistical distributions, and parameter estimates often come with high uncertainty. Valuable traits for a distribution family model include the ability to accomodate multiple probability density and hazard rate profiles, multimodality, and combinations of skewness and tail density. Another highly important trait is parsimony; while many distributions have been studied which possess the ability mentioned above, they become unwieldy due to having five or more parameters.

1.1. **The Chen Distribution Family.** We begin by introducing the two-parameter Chen model and provide recent examples of applications of this model.

The Chen distribution family, first analyzed in [1], is, in its standard form, a two-parameter family with the following density.

$$F(x \mid \lambda, \theta) = 1 - e^{-\lambda(e^{x^{\theta}} - 1)} \tag{1}$$

The model has been used for many applications involving censored data, such as graft survival times in months of renal transplant patients [2], survival times of chemotherapy patients [3], and life of electrodes under a high stress voltage endurance life test [4].

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1.2. **T-R{Y} Distribution Families.** In [5], the T-X method was introduced. New families of continuous distributions are formed from a combination of a transformer probability density $r(t)$; a transformed cumulative density $F(x), x \in (a,b), -\infty \leq a < b \leq \infty$; and a transform $W(u)$ which is differentiable, increasing on $(0,1)$, and $W((0,1)) = (a,b)$.

Then

$$G(x) = \int_a^{W(F(x))} r(t)dt \tag{2}$$

is a valid cdf on the support of $F(x)$.

In [6] it was noted that the transform $W(u)$ can be produced as the quantile function of a continuous random variable Y, because of which the method of distribution generation was relabeled T-$X\{Y\}$ (or in some instances T-$R\{Y\}$), where T is the transformer variable, X is the transformed variable, and Y is the variable which quantile function serves as the transform.

The aim of this article is to produce a new continuous probability distribution via the T-$X\{Y\}$ method with Weibull random variables as transformer and transform, producing a distribution which generalizes the Chen distribution. Basic properties such as moments, entropies, and derivatives for maximum likelihood estimation are provided. A copula and multivariate model are derived from the new model. The model is fitted to two real data sets, and the superior fit of the new model is demonstrated against several competitor models.

2. The New Chen Model

We apply (2) to (1), with $r(t)$ and $W(u)$ being the probability density and quantile function of Weibull(λ, θ) and Weibull(1,1) distributions, respectively. The resulting cumulative and probability densities are as follow.

$$F(x \mid \alpha, \gamma, \lambda, \theta) = 1 - e^{-\lambda(e^{(\gamma x)^\alpha}-1)^\theta} \tag{3}$$

$$f(x \mid \alpha, \gamma, \lambda, \theta) = \alpha\gamma\lambda\theta(\gamma x)^{\alpha-1}e^{(\gamma x)^\alpha}(e^{(\gamma x)^\alpha} - 1)^{\theta-1}e^{-\lambda(e^{(\gamma x)^\alpha}-1)^\theta} \tag{4}$$

Note that the Chen distribution is a special case when $\theta = 1$.

The probability density can take a variety of shapes, such as decreasing, right-skewed, left-skewed, and even a right-skew with tail mode, as evinced in 1.

The hazard rate is the following.

$$h(x \mid \alpha, \gamma, \lambda, \theta) = \alpha\gamma\lambda\theta(\gamma x)^{\alpha-1}e^{(\gamma x)^\alpha}(e^{(\gamma x)^\alpha} - 1)^{\theta-1} \tag{5}$$

In 2, one may observe hazard rates with increasing, J-shaped, decreasing, and bathtub-shaped profiles. Finally, the quantile function of the new generalized Chen family is given below.

$$Q(u \mid \alpha, \gamma, \lambda, \theta) = \frac{1}{\gamma}\log\left(\left(\frac{-1}{\lambda}\log(1-u)\right)^{\frac{1}{\theta}} + 1\right)^{\frac{1}{\alpha}} \tag{6}$$

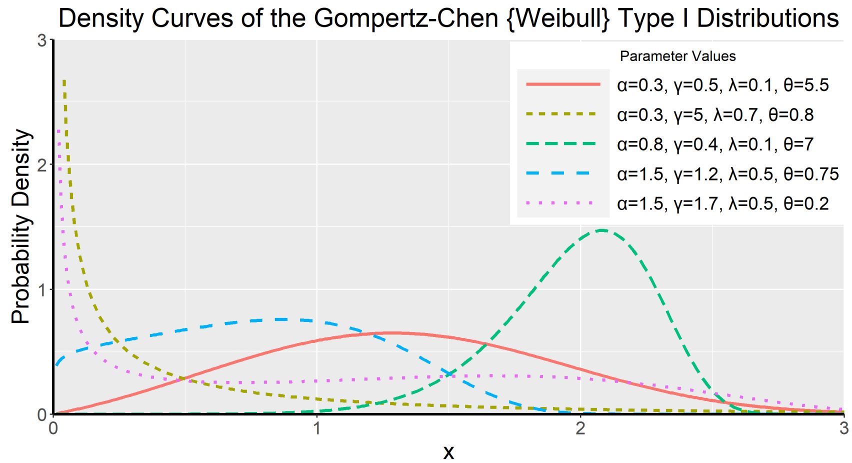

FIGURE 1. Some probability density curves of the Weibull-Chen {Weibull} Type I family

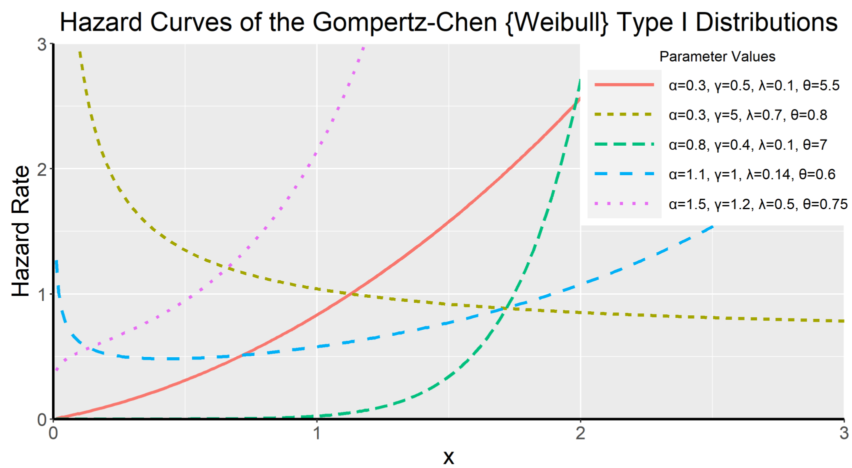

FIGURE 2. Some hazard rate curves of the Weibull-Chen {Weibull} Type I family

2.1. **Series Expansion.** In this subsection, we will derive a series expansion for the cumulative density function. We begin with the fact, expressed in the previous section, that a cumulative density family that can be expressed as Weibull-X{Weibull} Type I has the form

$$G(x \mid \underline{\beta}, \lambda, \theta) = 1 - e^{-\lambda(-\log(\overline{F}(x|\underline{\beta})))^{\theta}}$$

for x in the support of $F(\cdot)$. Given that $-\log(\overline{F}(x \mid \underline{\beta}))$ has the series expansion

$$-\log(\overline{F}(x \mid \underline{\beta})) = \sum_{n=1}^{\infty} \frac{(-1)^n(\overline{F}(x \mid \underline{\beta}) - 1)^n}{n} = F(x \mid \underline{\beta}) \sum_{n=0}^{\infty} \frac{F(x \mid \underline{\beta})^n}{n+1}$$

we can express $(-\log(\overline{F}(x \mid \underline{\beta})))^m$ as

$$(-\log(\overline{F}(x \mid \underline{\beta})))^m = F(x \mid \underline{\beta})^m \sum_{k=0}^{\infty} c_k F(x \mid \underline{\beta})^k \tag{7}$$

where $c_0 = 1$ and $c_k = \frac{1}{k}\sum_{j=1}^{k}(jm-k+j)\frac{c_{k-j}}{j+1}$.

Then the expansion $z^\theta = \sum_{m=0}^{\infty}\binom{\theta}{m}(z-1)^m$ can be used to express the exponent as a series.

$$(-\log(\overline{F}(x \mid \underline{\beta})))^\theta = \sum_{m=0}^{\infty}\binom{\theta}{m}(-\log(\overline{F}(x \mid \underline{\beta}))-1)^m = \sum_{m=0}^{\infty}\binom{\theta}{m}d_m F(x \mid \underline{\beta})^m \tag{8}$$

where $d_0 = 1$ and $d_m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}c_{m-k}$.

Defining $t_{m,n,\theta} = \frac{1}{m}\sum_{k=0}^{m}(kn-m+k)\binom{\theta}{k}d_k t_{m-k,n,\theta}$ and $t_{0,n,\theta} = 1$, the survival function of a Weibull-X {Weibull} Type I distribution can be expressed as the following series expansion.

$$\begin{aligned}\underline{G}(x \mid \underline{\beta},\lambda,\theta) &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\sum_{m=0}^{\infty}t_{m,n,\theta}F(x \mid \underline{\beta})^m \\ &= \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}T_{n,\theta}F(x \mid \underline{\beta})^n\end{aligned} \tag{9}$$

where $T_{n,\theta} = \sum_{k=0}^{\infty}t_{n,k,\theta}$. Therefore, a Weibull-Chen {Weibull} Type I survival function can also be expressed as a weighted sum of Chen(α,γ,n) cumulative densities.

2.2. **Moments.** The r$^{\text{th}}$ moment of a Chen (α,γ,ϕ) distribution is given by

$$\mu_r = \frac{re}{\alpha\gamma}\Gamma\left(\frac{r}{\alpha}\right)\mathrm{E}_1^{\frac{r}{\alpha}-1}(\phi) \tag{10}$$

Using the series expansion of (9), we can write the series expansion of the r^{th} moment of $X \sim$ W-C{W}I$(\alpha,\gamma,\lambda,\theta)$ as

$$\mathbb{E}[X^r] = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}T_{n,\theta}\frac{re}{\alpha\gamma}\Gamma\left(\frac{r}{\alpha}\right)\mathrm{E}_1^{\frac{r}{\alpha}-1}(n) \tag{11}$$

We can also find moments in terms of some other known functions and integrals. Consider first that if $X \sim W-C\{W\}I(\alpha,\gamma,\lambda,\theta)$, then $Y = (\phi X)^\beta \sim W-C\{W\}I(\alpha/\beta,(\gamma/\phi)^\beta,\lambda,\theta)$. Specifically, $Z = (\gamma X)^\alpha \sim W-C\{W\}I(1,1,\lambda,\theta)$. Then $\mathbb{E}[X^r] = \frac{1}{\gamma^r}\mathbb{E}\left[Z^{\frac{r}{\alpha}}\right]$.

If $\theta = 1$,

$$\mathbb{E}[X^r] = \frac{re^\lambda}{\gamma^r}\Gamma\left(\frac{r}{\alpha}\right)\mathrm{E}_1^{\frac{r}{\alpha}-1}(\lambda) \tag{12}$$

In general, a series expansion of natural logarithm provides a series expansion for the general form of $\mathbb{E}[X]$.

$$\mathbb{E}[X^r] = \frac{1}{\gamma^r}\lambda\int_0^\infty\left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{v^{n/\theta}}{n}\right)^{\frac{r}{\alpha}}e^{-\lambda v}dv \tag{13}$$

If $\frac{r}{\alpha}$ is an integer, the above formula can be written as

$$\mathbb{E}[X^r] = \frac{1}{\gamma^r}\lambda\sum_{n=0}^{\infty}c_n\frac{\Gamma\left(\frac{n+\frac{r}{\alpha}}{\theta}+1\right)}{\lambda^{\frac{n+\frac{r}{\alpha}}{\theta}+1}} \tag{14}$$

where $c_0 = 1$ and $c_n = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{jr}{\alpha}-n+j\right)\frac{(-1)^j}{j+1}c_{n-j}$.

Finally, some bounds can be placed on the expected value of γX, provided in 1.

Table 1. Upper and lower bounds for the expected value of a scaled variable, based on whether or not α and θ, respectively, are less than 1

α	θ	**Lower Bound**	**Upper Bound**
< 1	≥ 1	$\mathbb{E}[(\gamma X)^\alpha] + \alpha^{\frac{-1}{\alpha-1}} - \alpha^{\frac{-\alpha}{\alpha-1}}$	$\frac{e^\lambda}{\theta}\Gamma(\frac{1}{\alpha})\mathrm{E}_1^{\frac{1}{\alpha}-1}(\lambda)$
< 1	< 1	$\mathbb{E}[(\gamma X)^\alpha] + \alpha^{\frac{-1}{\alpha-1}} - \alpha^{\frac{-\alpha}{\alpha-1}}$	$\frac{e^\lambda}{\alpha\theta}\Gamma(\frac{1}{\alpha\theta})\mathrm{E}_1^{\frac{1}{\alpha\theta}-1}(\lambda)$
≥ 1	≥ 1	$\frac{e^\lambda}{\alpha}\Gamma(\frac{1}{\alpha\theta})\mathrm{E}_1^{\frac{1}{\alpha\theta}-1}(\lambda)$	$\mathbb{E}[(\gamma X)^\alpha] + \alpha^{\frac{-1}{\alpha-1}} - \alpha^{\frac{-\alpha}{\alpha-1}}$
≥ 1	< 1	$\frac{e^\lambda}{\alpha}\Gamma(\frac{1}{\alpha})\mathrm{E}_1^{\frac{1}{\alpha}-1}(\lambda) - \frac{1}{\alpha}$	$\mathbb{E}[(\gamma X)^\alpha] + \alpha^{\frac{-1}{\alpha-1}} - \alpha^{\frac{-\alpha}{\alpha-1}}$

2.3. **Probability Weighted Moments.** Probability weighted moments were introduced in [7] as the following.

$$M_{i,j,k} = \mathbb{E}[X^i F(x)^j (1 - F(x))^k] \tag{15}$$

In the case of the Weibull-Chen {Weibull} Type I family, a familiar series expansion allows us to express these moments in terms of the moments of X.

$$M_{i,j,k} = \sum_{n=0}^{\infty} \frac{\binom{j}{n}(-1)^n}{n+k+1} \mathbb{E}[{}_{n+t+1}X^r] \tag{16}$$

where ${}_{n+t+1}X \sim$ W-C{W}I$(\alpha, \gamma, (n+t+1)\lambda, \theta)$.

2.4. **Entropy.** The Shannon Entropy, defined for a continuous random variable as $-\int_{-\infty}^{\infty} \log(f(x))f(x)dx$, is as follows.

$$H(X) = -\log\alpha - \alpha\log\gamma - \log\lambda - \log\theta + \frac{\theta-1}{\theta}(\log\lambda + \gamma_E) - (\alpha-1)\mathbb{E}[Y] + 1. \tag{17}$$

where γ_E is the Euler-Mascheroni constant and $Y = \log X$, that is,

$$F_Y(y) = 1 - e^{-\lambda\left(e^{(\gamma^\alpha e^{\alpha y}} - 1\right)^\theta}. \tag{18}$$

2.5. **Kullback-Leibler Divergence.** It was shown in [8] that, if Y is a variable of type T_1-R $\{T_2\}$I, where T_1 and T_2 are of family T with parameters $\underline{\beta}_1$ and $\underline{\beta}_2$, respectively, then the Kullback-Leibler divergence of Y against the distribution R, having the same respective parameters, is $KL(h \mid f) = \mathbb{E}[\log(g(T_1 \mid \underline{\beta}_1)) - \log(g(T_1 \mid \underline{\beta}_2))]$, where $g(\cdot \mid \underline{\beta})$ is the probability density of T.

The Kullback-Leibler divergence of a Weibull-Chen {Weibull} I $(\alpha, \gamma, \lambda, \theta)$ distribution versus that of a Chen$(\gamma, \alpha, 1)$ distribution is

$$KL(f_1, f_2) = \log(\theta_1\lambda_1) - \log(\theta_2\lambda_2) + \frac{\theta_1 - \theta_2}{\theta_1}(\log\lambda_1 + \gamma_E) - 1 + \frac{\lambda_2\Gamma\left(\frac{\theta_2}{\theta_1}+1\right)}{\lambda_1^{\frac{\theta_2}{\theta_1}+1}} \tag{19}$$

2.6. **Order Statistics.** For the Weibull-Chen{Weibull}I distribution with parameters $(\alpha,,\gamma,\lambda,\theta)$, the order statistics have probability density

$$f_{i:n}(x) = \binom{n}{i} i\alpha\gamma\lambda\theta(\gamma x)^{\alpha-1}e^{(\gamma x)^\alpha}(e^{(\gamma x)^\alpha}-1)^{\theta-1}e^{-(n+1-i)\lambda(e^{(\gamma x)^\alpha}-1)^\theta}(1-e^{-\lambda(e^{(\gamma x)^\alpha}-1)^\theta})^{i-1} \quad (20)$$

2.7. **Reliability.** Let $X_1 \sim$ Weibull-Chen{Weibull}I$(\alpha_1,\gamma_1,\lambda_1,\theta_1)$, $X_2 \sim$ Weibull-Chen{Weibull}I$(\alpha_2,\gamma_2,\lambda_2,\theta_2)$, and $X_1 \perp X_2$.

When the two variables differ only by λ, the stress-strength reliability is found to be

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}.$$

When the two variables are equal in terms of α and γ, the stress-strength reliability is found to be

$$P(X_1 < X_2) = \sum_{n=0}^{\infty} \frac{(-\lambda_2)^n}{n!}\lambda_1^{-n\frac{\theta_2}{\theta_1}}\Gamma\left(1+\frac{n}{\theta_1}\right)$$

In general, the stress-strength reliability does not have closed form, but it can be given a series expansion via the following.

$$P(X_1 < X_2) = \sum_{n=0}^{\infty} p_n(\alpha_2,\gamma_2,\theta_2)\mathbb{E}[X_1^{n\alpha_2}]$$

$$p_n(\alpha_2,\gamma_2,\theta_2) = \sum_{m=0}^{\infty} \frac{(-\lambda_2)^n}{n!} d_m(\alpha_2,\gamma_2,\theta_2)$$

$$d_m(\alpha_2,\gamma_2,\theta_2) = \sum_{k=0}^{\infty} \binom{\theta_2}{k} c_{m,k}$$

$$c_{0,k} = (-1)^k; c_{m,k} = \frac{(-1)^k}{m}\sum_{j=1} m(jk-m+j)a_j c_{n-j,k}$$

$$a_0 = -1; a_j = \frac{(\gamma_2)^{j\alpha_2}}{j!}$$

2.8. **Maximum Likelihood Estimation.** In this subsection, we present the derivatives of the likelihood function for uncensored, iid data, for the purposes of maximum likelihood estimation. The likelihood function is simply found, as usual, as the product of the model probability density evaluated at each datum. We will begin with the log-likelihood.

The log-likelihood function of the Weibull-Chen{Weibull} I distribution, with parameters $(\alpha,\gamma,\lambda,\theta)$, for an iid sample, is as follows.

$$\ell(\alpha,\gamma,\lambda,\theta) = n(\log(\alpha)+\alpha\log(\gamma)+\log(\lambda)+\log(\theta)) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) \quad (21)$$
$$+\sum_{i=1}^{n}(\gamma x_i)^\alpha + (\theta-1)\sum_{i=1}^{n}\log(e^{(\gamma x_i)^\alpha}-1) - \lambda\sum_{i=1}^{n}(e^{(\gamma x_i)^\alpha}-1)^\theta$$

The simultaneous estimate of the four parameters can be found by setting the following four derivatives equal to 0 and solving the resulting equations.

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + n\log\gamma + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}(\gamma x_i)^{\alpha}\log(\gamma x_i) \\
&+(\theta-1)\sum_{i=1}^{n}\frac{e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha}\log(\gamma x_i)}{e^{(\gamma x_i)^{\alpha}}-1} \\
&-\lambda\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-1}e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha}\log(\gamma x_i) \qquad (22)\\
\frac{\partial \ell}{\partial \gamma} &= \frac{n\alpha}{\gamma} + \alpha\sum_{i=1}^{n}(\gamma x_i)^{\alpha-1}x_i + (\theta-1)\sum_{i=1}^{n}\frac{e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha-1}x_i}{e^{(\gamma x_i)^{\alpha}}-1} \\
&-\lambda\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-1}e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha-1}x_i \\
\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta} \\
\frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} + \sum_{i=1}^{r}\log(e^{(\gamma x_i)^{\alpha}}-1) - \lambda\sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta}\log(e^{(\gamma x_i)^{\alpha}}-1)
\end{aligned}$$

As parameter inference relies on knowledge of the joint variability of the estimates, we include also the second derivatives of the log-likelihood.

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha^2} &= -\frac{n}{\alpha^2} + \sum_{i=1}^{n}(\gamma x_i)^{\alpha}\log^2(\gamma x_i) - (\theta-1)\sum_{i=1}^{n}\frac{\log^2(\gamma x_i)(\gamma x_i)^{\alpha}e^{(\gamma x_i)^{\alpha}}(1+(\gamma x_i)^{\alpha}-e^{(\gamma x_i)^{\alpha}})}{(e^{(\gamma x_i)^{\alpha}}-1)^2} \\
&-\lambda\theta\sum_{i=1}^{n}(\gamma x_i)^{\alpha}e^{(\gamma x_i)^{\alpha}}\log^2(\gamma x_i)(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-2}((\gamma x_i)^{\alpha}(\theta e^{(\gamma x_i)^{\alpha}}-1)+e^{(\gamma x_i)^{\alpha}}-1) \\
\frac{\partial^2 \ell}{\partial \alpha\partial \gamma} &= \frac{n}{\gamma} + \sum_{i=1}^{n}x_i(\gamma x_i)^{\alpha-1}(\alpha\log(\gamma x_i)+1) \qquad (23)\\
&+(\theta-1)\sum_{i=1}^{n}\frac{x_i(\gamma x_i)^{\alpha-1}e^{(\gamma x_i)^{\alpha}}(e^{(\gamma x_i)^{\alpha}}-\alpha((\gamma x_i)^{\alpha}-e^{(\gamma x_i)^{\alpha}}+1)\log(\gamma x_i)-1)}{(e^{(\gamma x_i)^{\alpha}}-1)^2} \\
&-\lambda\theta\sum_{i=1}^{r}x_i e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha-1}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-2}(\alpha\log(\gamma x_i)(e^{(\gamma x_i)^{\alpha}}(\theta(\gamma x_i)^{\alpha}+1)-(\gamma x_i)^{\alpha}-1)+e^{(\gamma x_i)^{\alpha}}-1)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \alpha\partial \lambda} &= -\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-1}e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha}\log(\gamma x_i) \\
\frac{\partial^2 \ell}{\partial \alpha\partial \theta} &= \sum_{i=1}^{n}\frac{e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha}\log(\gamma x_i)}{e^{(\gamma x_i)^{\alpha}}-1} \\
&-\lambda\sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-1}e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha}\log(\gamma x_i) \\
&-\lambda\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^{\alpha}}-1)^{\theta-1}\log(e^{(\gamma x_i)^{\alpha}}-1)e^{(\gamma x_i)^{\alpha}}(\gamma x_i)^{\alpha}\log(\gamma x_i)
\end{aligned}$$

$$\frac{\partial^2\ell}{\partial\gamma^2} = -\frac{n\alpha}{\gamma^2} + \alpha(\alpha-1)\sum_{i=1}^{n}(\gamma x_i)^{\alpha-2}x_i^2$$

$$-(\theta-1)\sum_{i=1}^{n}\frac{e^{(\gamma x_i)^\alpha}x_i^2(\gamma x_i)^{\alpha-2}(\alpha((\gamma x_i)^\alpha - e^{(\gamma x_i)^\alpha}+1)+e^{(\gamma x_i)^\alpha}-1)}{(e^{(\gamma x_i)^\alpha}-1)^2}$$

$$-\lambda\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^\alpha}-1)^{\theta-2}e^{(\gamma x_i)^\alpha}(\gamma x_i)^{\alpha-2}x_i^2(\alpha(e^{(\gamma x_i)^\alpha}(\theta(\gamma x_i)^\alpha+1)-(\gamma x_i)^\alpha-1)-e^{(\gamma x_i)^\alpha}+1)$$

$$\frac{\partial^2\ell}{\partial\gamma\partial\lambda} = -\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^\alpha}-1)^{\theta-1}e^{(\gamma x_i)^\alpha}(\gamma x_i)^{\alpha-1}x_i$$

$$\frac{\partial^2\ell}{\partial\gamma\partial\theta} = \sum_{i=1}^{n}\frac{e^{(\gamma x_i)^\alpha}(\gamma x_i)^{\alpha-1}x_i}{e^{(\gamma x_i)^\alpha}-1}$$

$$-\lambda\sum_{i=1}^{n}(e^{(\gamma x_i)^\alpha}-1)^{\theta-1}e^{(\gamma x_i)^\alpha}(\gamma x_i)^{\alpha-1}x_i - \lambda\theta\sum_{i=1}^{n}(e^{(\gamma x_i)^\alpha}-1)^{\theta-1}\log(e^{(\gamma x_i)^\alpha}-1)e^{(\gamma x_i)^\alpha}(\gamma x_i)^{\alpha-1}x_i$$

$$\frac{\partial^2\ell}{\partial\lambda^2} = -\frac{n}{\lambda^2}$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\theta} = -\sum_{i=1}^{n}(e^{(\gamma x_i)^\alpha}-1)^{\theta}\log(e^{(\gamma x_i)^\alpha}-1)$$

$$\frac{\partial^2\ell}{\partial\theta^2} = -\frac{n}{\theta^2} - \lambda\sum_{i=1}^{r}(e^{(\gamma x_i)^\alpha}-1)^{\theta}\log^2(e^{(\gamma x_i)^\alpha}-1)$$

2.9. **A Copula and New Bivariate Models.** The survival function of the new model is applied to the construction of an Archimedean copula as provided in [9]. That is, if $S(x \mid \alpha,\gamma,\lambda,\theta)$ is the survival function of the Weibull-Chen {Weibull} Type I distribution, then the resulting copula is

$$C(u,v) = S(S^{-1}(u)+S^{-1}(v)) \tag{24}$$
$$= e^{-\lambda\left(e^{\left(\log\left(1+\left(\frac{-1}{\lambda}\log(u_1)\right)^{\frac{1}{\theta}}\right)^{\frac{1}{\alpha}}+\log\left(1+\left(\frac{-1}{\lambda}\log(u_2)\right)^{\frac{1}{\theta}}\right)^{\frac{1}{\alpha}}\right)^{\alpha}}-1\right)^{\theta}}$$

The copula is valid for $\alpha,\theta \le 1, \lambda \ge 1$.

A bivariate Weibull-Chen {Weibull} Type I joint cdf may be produced by applying the cumulative densities of two WCWI distributions to the copula:

$$F(x_1,x_2) = e^{-\lambda\left(e^{(g(x_1|\alpha,\lambda,\theta,\alpha_1,\lambda_1,\gamma_1,\theta_1)+g(x_2|\alpha,\lambda,\theta,\alpha_2,\lambda_2,\gamma_2,\theta_2))^\alpha}-1\right)^\theta}; x_1,x_2>0 \tag{25}$$

where $g(x_i \mid \alpha,\lambda,\theta,\alpha_i,\lambda_i,\gamma_i,\theta_i) = \log\left(1+\left(\frac{-1}{\lambda}\log\left(1-e^{-\lambda_i\left(e^{(\gamma_i x_i)^{\alpha_i}}-1\right)^{\theta_i}}\right)\right)^{\frac{1}{\theta}}\right)^{\frac{1}{\alpha}}$. Similarly, the joint survival function of a different bivariate Weibull-Chen {Weibull} Type I model may be produced by applying the survival functions of two WCWI distributions to the copula:

$$S(x_1, x_2) = e^{-\lambda\left(e^{\left(\log\left(1+\left(\frac{\lambda_1}{\lambda}\left(e^{(\gamma_1 x_1)^{\alpha_1}}-1\right)^{\frac{\theta_1}{\theta}}\right)\right)^{\frac{1}{\alpha}}+\log\left(1+\left(\frac{\lambda_2}{\lambda}\left(e^{(\gamma_2 x_2)^{\alpha_2}}-1\right)^{\frac{\theta_2}{\theta}}\right)\right)^{\frac{1}{\alpha}}\right)^{\alpha}}-1\right)^{\theta}}; x_1, x_2 > 0 \tag{26}$$

In the special case of $\lambda_1 = \lambda_2 = \lambda$ and $\theta_1 = \theta_2 = \theta$, (26) reduces to

$$S(x_1, x_2) = e^{-\lambda(e^{((\gamma_1 x_1)^{\alpha_1/\alpha}+(\gamma_2 x_2)^{\alpha_2/\alpha})^{\alpha}}-1)^{\theta}}; x_1, x_2 > 0 \tag{27}$$

3. Application to Data

Data on the relief time in hours of 30 arthritis patients, reported in [10]), and survival times in years of chemotherapy patients, as reported in [11], were fit to the Weibull-Chen{Weibull} Type I model (abbreviated WCW1 in the following tables), and several fit statistics were calculated. The Chen model was also used, in order to compare to the WCWI distribution, as well as the Exponentiated Chen and Exponentiated Weibull models, using maximum likelihood via the function `optim` in `R`. The cumulative density of the Exponentiated Chen is parameterized as

$$F(x \mid \alpha, \beta, \gamma, \lambda) = \left(1 - e^{-\lambda(e^{(\gamma x)^{\alpha}}-1)}\right)^{\beta}$$

and the cdf of the Exponentiated Weibull as

$$F(x \mid \alpha, \beta, \gamma, \lambda) = \left(1 - e^{-(\gamma x)^{\alpha}}\right)^{\beta}$$

The renowned versality of the Exponentiated Weibull model makes its fit difficult to beat, and

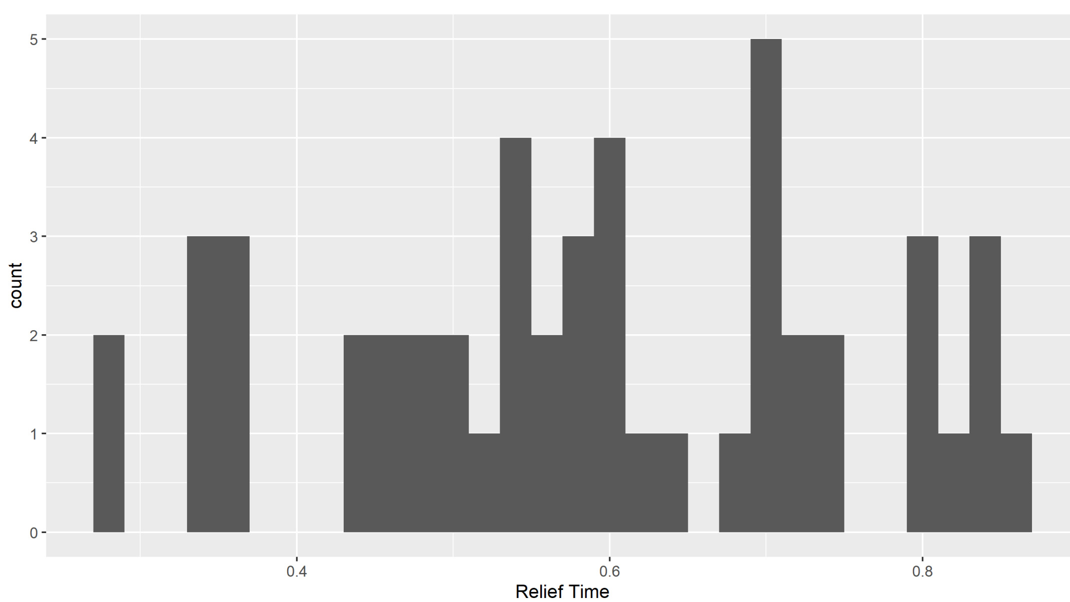

Figure 3. A histogram of the arthritis relief data

therefore this model makes a good foil for the new model. The Exponentiated Chen is also

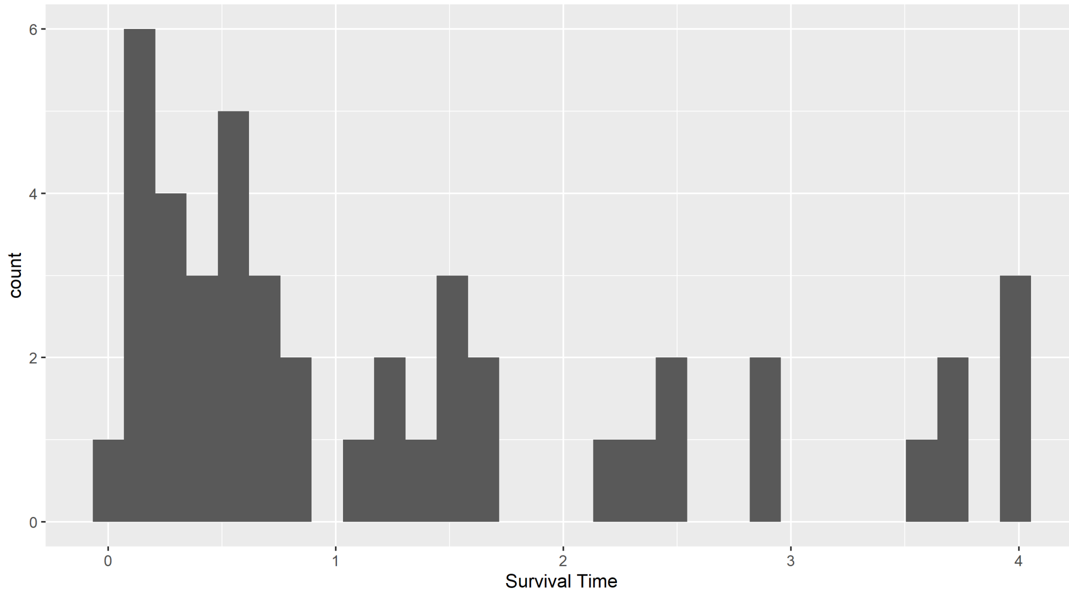

FIGURE 4. A histogram of the chemotherapy survival data

TABLE 2. Summary of results for the arthritis relief data

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\theta}$	AICc	BIC	K-S	A-D
W-C-W I	30.41	—	1.23	2.23	0.13	-	-	0.08	0.27
	(0.02)		(0.01)	(0.37)	(0.01)	39.34	32.54	(0.92)	(0.96)
Exp-	3.79	0.97	1.10	2.76	—	-	-	0.09	0.43
Chen	(3.68)	(1.21)	(0.37)	(5.06)		34.82	28.06	(0.81)	(0.82)
Chen	3.71	—	1.11	2.70	—	-	-	0.09	0.43
	(0.79)		(0.328)	(4.28)		37.19	31.98	(0.82)	(0.82)
Exp-	5.81	0.61	1.40	—	—	-	-	0.09	0.43
Weibull	(3.89)	(0.59)	(0.22)			36.60	31.40	(0.82)	(0.82)

worth studying as an alternative four-parameter generalization of the Chen distribution. The primary advantage of the new model will be in the simultaenous fitting of the left skew and platykurtosis of the arthritis relief data (shown in 6) and the platykurtosis in the right tail of the chemotherapy data (shown in 4).

Kolmogorov-Smirnov and Anderson-Darling tests reject none of the models as being well-fitting to the first data set. In terms of AICc and BIC, the new model is preferred to each of the other models, providing a significant improvement to the fit. An additional advantage of the fit of the new model over the Exponentiated Weibull is that the distinguishing parameter θ of the new model is found to be significantly different from the standard value of 1 by comparison with standard error, while the same cannot be said of neither the β parameter of the Exponentiated Weibull nor of the same parameter of Exponentiated Chen.

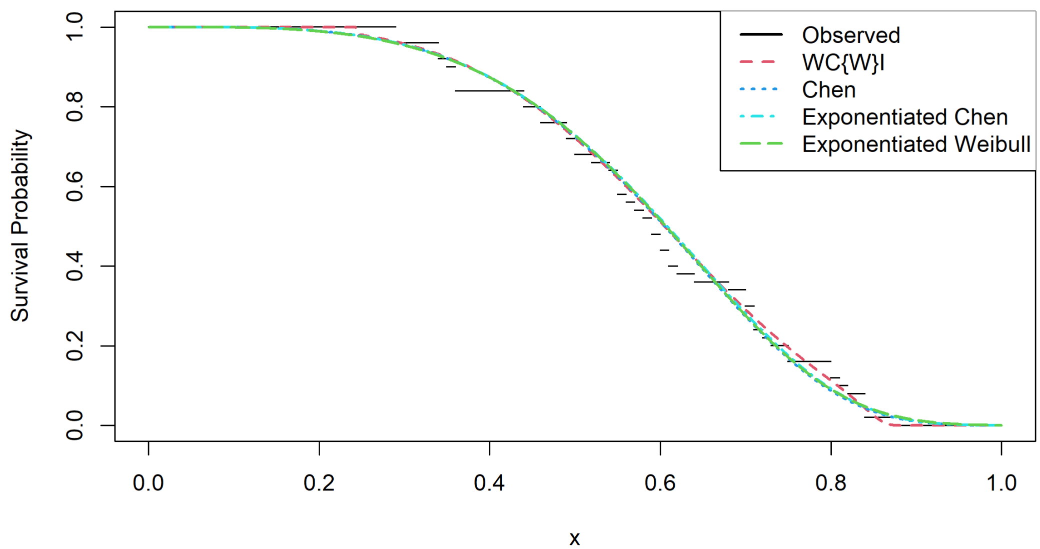

FIGURE 5. The empirical and theoretical survival curves for the arthritis relief data

TABLE 3. Summary of results for the chemotherapy data

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\theta}$	AICc	BIC	K-S	A-D
W-C-W I	8.44	—	0.31	1.99	0.11	116.04	122.27	0.10	0.49
	(0.003)		(0.002)	(0.30)	(0.01)			(0.75)	(0.75)
Exp-	7.85	0.07	0.27	0.21	—	124.30	130.53	0.18	1.72
Chen	(0.003)	(0.01)	(0.003)	(0.07)				(0.08)	(0.13)
Chen	0.98	—	0.12	5.36	—	122.55	127.38	0.11	0.48
	(0.19)		(0.21)	(12.17)				(0.56)	(0.76)
Exp-	0.81	1.61	1.16	—	—	122.67	127.51	0.10	0.48
Weibull	(0.52)	(1.96)	(1.52)					(0.72)	(0.76)

The results of model fitting to the chemotherapy data are very similar to the results of modeling the arthritis data: no model was rejected as fitting the data by KS and AD tests; AICc and BIC preferred the Weibull-Chen {Weibull} I model over the others, and the distinguishing parameter of the new model is significantly different from 1, while that of the Exponentiated Weibull was not found to be.

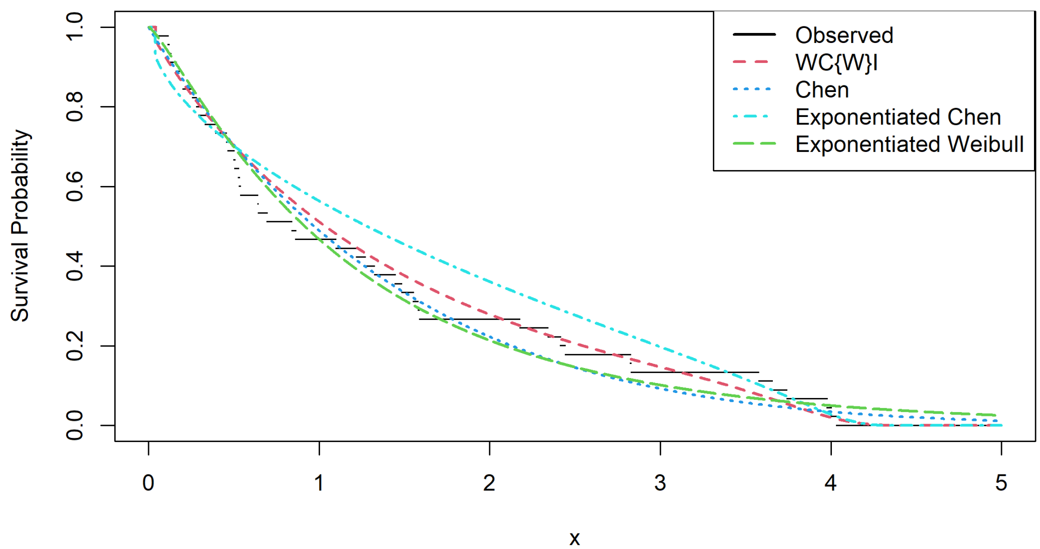

Figure 6. The empirical and theoretical survival curves for the chemotherapy data

4. Conclusion

A novel distribution was derived using the $T-R\{Y\}$ construction method. Basic properties such as density and hazard shapes, moments, and entropies were explored. Two real data sets displayed the enhanced ability of the new model to fit the behavior of random variables, beyond that of the special case of the Chen distribution, another four-parameter generalization of the Chen family, and the classic Exponentiated Weibull model. The properties of estimation and the application to data pertained to the case of independent, uncensored data; future works should focus on modeling of censored data with the new distribution family. In addition, a copula and two multivariate Weibull-Chen {Weibull} Type I distribution functions were derived; these can be applied to the modeling of bivariate data.

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