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Entropy-Based Trivariate q-Gaussian Model (TVq-GD): An Application to Financial Data

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Abstract. Understanding the characteristics of the random variables through their distributions is a really important way to understand the model. In this article, we employ the entropy maximazation method to construct a trivariate q-Gaussian distribution (TVq-GD). Maximum likelihood estimation (MLE) is used to estimate the parameters of the TVq-GD. We demonstrate the practical relevance of this distribution by applying it to a real-world financial dataset, specifically the NIFTY50 Index data, using metrics such as volatility, trading volume, and returns, as well as the soyabean complex and gold commodity markets. This application highlights the effectiveness of the trivariate q-Gaussian distribution in capturing complex dependencies and variations of financial data. A comparison is made with the trivariate Gaussian distribution and the Cauchy distribution to demonstrate the effectiveness of the proposed TVq-GD distribution.

1. Introduction

In the financial market, random variables are used to represent uncertainties, such as stock returns, volatility, and trading volume. In a univariate framework, researchers have closely analysed the characteristics of stock return distributions [1]. An investor seeks to understand the likelihood of gains; thus, the probability distribution is vital for measuring the risk of investing in common stocks. This also provides valuable insights into the processes that generate returns. Moreover, empirical research utilizes theoretical frameworks and econometric techniques to investigate the assumptions surrounding the distributions of stock returns, which need to be validated with the actual data. Stock returns have generally been assumed to follow a normal distribution. The assumption of normality is inconsistent with the real-world data, as demonstrated in [2], [3], [4]. In a comprehensive review of existing studies [5], it is suggested that a relationship between these random variables is essential, as it facilitates inference from them in research. It also provides valuable information on the dynamics of financial markets. Various research studies have been conducted in the bivariate framework to establish a relationship between trading volume and asset returns over the past several decades, as noted in [5], [6], [7],

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and [8]. It is widely believed that trading volume can signal potential price movements, making the relationship between them a subject of considerable interest. Investors have consistently focused on identifying relationships between stock prices and trading volume to assess stock information. Fluctuations in stock price and volume are sources to identify common characteristics, such as power laws. These characteristics are helpful in understanding the statistical behaviour of financial markets. However, the characteristics of three random variables have been less explored in the literature. Numerous studies have examined multivariate Gaussian and multivariate Student-t distributions; however, research on the q-exponential has been limited in this context. This has inspired us to undertake a study aimed at incorporating an additional variable into the bivariate framework. Researchers have extensively studied the connections between financial factors, including both fundamental and technical drivers. A key challenge in finance is to understand the distribution of price changes. A power-law distribution is among the probability distributions that have garnered significant attention recently. However, there are instances in which the power-law distribution dominates the conventional distribution. The maximum entropy principle [9] provides a framework for deriving various distributions [10], [11]. Consequently, entropy plays an important role in the construction of the heavy-tailed distribution. Specifically, the maximum entropy principle technique is a viable approach to constructing the underlying distribution under appropriate conditions, [12].

1.1. **Contribution.** Given the significance of entropy in financial modelling, we develop an entropy-optimised trivariate q-Gaussian distribution (TVq-GD) to model the relationships among the three factors: trading volume, volatility, and return. We consider monthly NIFTY50 Index data, including closing prices, return volatility, and trading volumes, sourced from the NSE (National Stock Exchange) Sensex records for the period from January 2023 to December 2025. The data is obtained from https://www.nseindia.com/. We also apply the proposed trivariate q-Gaussian distribution (TVq-GD) to two distinct commodity complexes: the Soybean complex (Soybean, Soy-Oil, Soymeal), sourced from https://www.cmegroup.com/markets/agriculture/oilseeds/soybean.html, and the Gold complex (CME, Shanghai, and Thai Futures), sourced from https://www.reuters.com/. We utilise daily log-returns derived from the mid-prices of futures contracts over the 2020-2025 period. A detailed analysis is presented in Section 4 of the application. We also discuss the closed-form CDF and marginal density of the proposed trivariate q-Gaussian distribution with its estimation technique and model selection criteria. Although previous studies have mainly examined relationships with one or two random variables, the proposed analysis extends this approach by integrating a third variable to improve evaluation. We evaluate the goodness of fit of the trivariate q-Gaussian model by employing a likelihood ratio test and compare it to the conventional trivariate models. To select the best-fitting model, we utilised information criteria, including AIC and BIC. Our findings indicate that the entropy-optimised trivariate q-Gaussian distribution provides a better fit compared to both the conventional trivariate Gaussian distribution and the Cauchy distribution.

1.2. **Outilile of Article.** The article is structured as follows. Section 2 provides a methodology for constructing the trivariate q-Gaussian distribution. Section 3 outlines the estimation methods and the information criteria utilised for the model. In Section 4, we explore the proposed density in the context of the financial data. A comparison is also made with baseline models. We present our conclusions and outline future studies in Section 5.

2. The Model

Consider a random variable S with support $\{s_1, s_2, \ldots, s_n\}$ and the probability distribution $\{p_1, p_2, \ldots, p_n\}$. The Shannon's entropy [13] is defined as

$$H(S) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{1}$$

In case of a continuous random variable S with pdf $f(s)$, entropy is defined as

$$H(S) = -\int_s f(s) \ln f(s)\, ds. \tag{2}$$

Numerous well-known distributions, including the exponential and Gaussian distributions, emerge as maximum-entropy distributions when subjected to various moment constraints [9]. In this article, we use the maximum entropy principle on the following generalization of Shannon's entropy [14],

$$M_q(f) = \frac{\int_S f^{2-q}(S) dS - 1}{q-1} \quad \text{for } q \neq 1, 0 < q < 2, \tag{3}$$

where f represents a continuous density function, and S can be a scalar, vector, or matrix, either real or complex. By maximising this measure subject to certain moment-like constraints, various real and complex distributions can be obtained.

2.1. **Optimization under Constraints.** We consider the maximization of $M_q(f)$, given in Eq.(3), under the following constraints

$$\int_S f(S) dS = 1, \tag{4}$$

$$\int_S (S-\mu)^T \Sigma^{-1} (S-\mu) f(S) dS = C, \tag{5}$$

where μ and Σ are the mean and the variance-covariance matrix. To optimize $M_q(f)$, let the Lagrangian is given as

$$\mathcal{L} = \frac{\int_S f^{2-q}(S) dS - 1}{q-1} + \lambda_1 \Big(1 - \int_S f(S) dS\Big) + \lambda_2 \Big(\int_S (S-\mu)^T \Sigma^{-1} (S-\mu) f(S) dS - C\Big), \tag{6}$$

where, λ_1 and λ_2 are Lagrange's multipliers. We now use the Euler-Lagrange method $\mathcal{L}(f) = \int \mathcal{L}(S, f, f') dS$, that is,

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dS}\Big(\frac{\partial \mathcal{L}}{\partial f'}\Big) = 0, \tag{7}$$

Note that our Lagrangean term is free from f', this implies

$$\frac{\partial}{\partial f}\left[\frac{\int_S f^{2-q}(S)dS-1}{q-1}+\lambda_1\Big(1-\int_S f(S)dS\Big)+\lambda_2\Big(\int_S (S-\mu)^T\Sigma^{-1}(S-\mu)f(S)dS-C\Big)\right]=0$$

$$\frac{2-q}{1-q}f^{1-q}-\lambda_1 f+\lambda_2(S-\mu)^T\Sigma^{-1}(S-\mu)f=0,$$

this gives,

$$f^{1-q}=\frac{(1-q)\lambda_1}{2-q}\left[1-\frac{\lambda_2}{\lambda_1}(S-\mu)^T\Sigma^{-1}(S-\mu)\right].$$

For simplicity, we use $\frac{\lambda_2}{\lambda_1}=1-q$. Thus,

$$f=\left(\frac{(1-q)\lambda_1}{2-q}\right)^{\frac{1}{1-q}}\left[1-(1-q)(S-\mu)^T\Sigma^{-1}(S-\mu)\right]^{\frac{1}{1-q}}. \tag{8}$$

We now use different notation for f, for different intervals of q. Let

$$f_1(S)=c_1\left[1-(1-q)(S-\mu)^T\Sigma^{-1}(S-\mu)\right]^{\frac{1}{1-q}},\quad for\quad q<1, \tag{9}$$

$$f_2(S)=c_2\left[1+(q-1)(S-\mu)^T\Sigma^{-1}(S-\mu)\right]^{\frac{-1}{q-1}},\quad for\quad q>1, \tag{10}$$

For $\quad q\to 1$, let

$$f_3(S)=\lim_{q\to 1}\left\{c_1\left[1-(1-q)(S-\mu)^T\Sigma^{-1}(S-\mu)\right]^{\frac{1}{1-q}}\right\}.$$

Using the approximation, $\ln(1-x)\approx -x$, the density function becomes

$$f_3(S)=c_3e^{-(S-\mu)^T\Sigma^{-1}(S-\mu)}, \tag{11}$$

where, c_3 is constant and S follows multivariate Gaussian. Using the linear transformation $Z=S-\mu$ in $f_2(S)$, we get

$$f_4(Z)=c_4\left[1+(q-1)Z^T\Sigma^{-1}Z\right]^{\frac{-1}{q-1}},\quad q>1. \tag{12}$$

Consider the vector

$$Z=\begin{pmatrix}\frac{Z_1}{\sqrt{2}}\\ \frac{Z_2}{\sqrt{2}}\\ \frac{Z_3}{\sqrt{2}}\end{pmatrix}$$

and the matrix

$$\Sigma=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Equation (12) becomes

$$f_5(Z_1,Z_2,Z_3)=c_5\left[1+\frac{(q-1)}{2}(Z_1^2+Z_2^2+Z_3^2)\right]^{\frac{-1}{q-1}},\quad q>1, Z_i\in\mathbb{R}, \tag{13}$$

where c_5 is the normalization constant such that

$$\int_{\mathbb{R}^3} f_5(Z_1, Z_2, Z_3)\, dZ_1 dZ_2 dZ_3 = 1,$$

that is,

$$\int_{\mathbb{R}^3} c_5 \Bigg[1 + \frac{1}{2}(q-1)(Z_1^2 + Z_2^2 + Z_3^2)\Bigg]^{\frac{-1}{q-1}} = 1.$$

Using the transformations $Z_1 = r\sin\theta\cos\phi$, $Z_2 = r\sin\theta\sin\phi$, $Z_3 = r\cos\theta$, we get

$$c_5 \int_0^\infty \int_0^\pi \int_0^{2\pi} \Bigg[1 + \frac{(q-1)}{2}(r^2)\Bigg]^{\frac{-1}{q-1}} r^2 \sin\theta\, d\phi\, d\theta\, dr = 1.$$

Using some suitable substitutions, we get

$$c_5 = \left(\frac{q-1}{2\pi}\right)^{3/2} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} - \frac{3}{2}\right)}, \quad 1 < q < \frac{5}{3}.$$

Remark: It is important to note that the validity of this constant is confirmed by its behaviour in the limit $q \to 1$. Using the Stirling approximation for the Gamma function ratio, $\lim_{x\to\infty} x^{-a}\frac{\Gamma(x)}{\Gamma(x-a)} = 1$. Let $x = \frac{1}{q-1}$ and $a = \frac{3}{2}$. As $q \to 1^+$, $x \to \infty$:

$$\begin{aligned}
\lim_{q\to1} c_5 &= \lim_{q\to1} \left(\frac{1}{2\pi}\right)^{3/2} (q-1)^{3/2} \frac{\Gamma(\frac{1}{q-1})}{\Gamma(\frac{1}{q-1} - \frac{3}{2})} \\
&= \left(\frac{1}{2\pi}\right)^{3/2} \lim_{x\to\infty} x^{-3/2} \frac{\Gamma(x)}{\Gamma(x - 3/2)} \qquad (14)\\
&= \left(\frac{1}{2\pi}\right)^{3/2} \cdot 1 = (2\pi)^{-3/2}
\end{aligned}$$

This successfully recovers the standard Gaussian normalization constant, confirming the consistency of Eq. (13) within the domain $1 < q < 5/3$. Hence, the trivariate q-density function, represented by $f_q(Z_1, Z_2, Z_3)$, is defined as

$$f_q(Z_1, Z_2, Z_3) = c_5 \exp_q\left(-\frac{1}{2}(Z_1^2 + Z_2^2 + Z_3^2)\right), \quad Z_i \in \mathbb{R}, \quad 1 < q < \frac{5}{3}, \tag{15}$$

where, $exp_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}}$ is q-exponential function. For $q \to 1$, we get the standard Gaussian density function

$$f_q(Z_1, Z_2, Z_3) = \frac{1}{(2\pi)^{3/2}} \exp_q\left(-\frac{1}{2}(Z_1^2 + Z_2^2 + Z_3^2)\right), \quad Z_i \in \mathbb{R} \quad (i = 1, 2, 3). \tag{16}$$

Now, to transform the q-density function into the trivariate q-Gaussian distribution, we apply appropriate transformations. We introduce the random variables S, U, and W with the constant $\mu_i \in \mathbb{R}$, $\sigma_i > 0, i = 1, 2, 3$ and $-1 < \rho < 1$, such as

$$S = \sigma_1 Z_1 + \mu_1,$$

$$U = \sigma_2\Big[\rho_{12} Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}} Z_2\Big] + \mu_2,$$

and

$$W=\sigma_3\left[\rho_{13}Z_1+\frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}Z_2+\sqrt{1-\rho_{13}^2-\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}Z_3\right]+\mu_3.$$

We construct the joint density function $g(s,u,w)$ by using the above linear transformation from Z_1,Z_2,Z_3 to S,U,W. That is,

$$g_q(s,u,w)=f_q(Z_1,Z_2,Z_3)|J|, \tag{17}$$

where the Jacobian is

$$J=\begin{vmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ -\frac{\rho_{12}}{\sigma_1\sqrt{1-\rho_{12}^2}} & \frac{1}{\sigma_2\sqrt{1-\rho_{12}^2}} & 0 \\ \frac{1}{\sigma_1\cdot m}\cdot\frac{\rho_{12}\rho_{23}-\rho_{13}}{1-\rho_{12}^2} & \frac{1}{\sigma_2\cdot m}\cdot\frac{\rho_{12}\rho_{13}-\rho_{23}}{1-\rho_{12}^2} & \frac{1}{\sigma_3\cdot m}\end{vmatrix}.$$

Here, $m=\left(\sqrt{1-\rho_{13}^2-\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}\right)$. Thus,

$$|J|=\frac{1}{\sigma_1\sigma_2\sigma_3\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2-\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}}.$$

Using Z_1,Z_2,Z_3 in $f_q(Z_1,Z_2,Z_3)$, we get

$$f_q(Z_1,Z_2,Z_3)=\exp_q\left(-\frac{1}{2}Q(s,u,w)\right), \tag{18}$$

where,

$$\begin{aligned} Q(s,u,w)&=\frac{1}{1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23}} \\ &\times\Bigg[(1-\rho_{23}^2)\left(\frac{s-\mu_1}{\sigma_1}\right)^2+(1-\rho_{13}^2)\left(\frac{u-\mu_2}{\sigma_2}\right)^2+(1-\rho_{12}^2)\left(\frac{w-\mu_3}{\sigma_3}\right)^2 \\ &+2(\rho_{13}\rho_{23}-\rho_{12})\left(\frac{s-\mu_1}{\sigma_1}\right)\left(\frac{u-\mu_2}{\sigma_2}\right)+2(\rho_{12}\rho_{23}-\rho_{13})\left(\frac{s-\mu_1}{\sigma_1}\right)\left(\frac{w-\mu_3}{\sigma_3}\right) \\ &+2(\rho_{12}\rho_{13}-\rho_{23})\left(\frac{u-\mu_2}{\sigma_2}\right)\left(\frac{w-\mu_3}{\sigma_3}\right)\Bigg]. \end{aligned}$$

Definition 1 (**Trivariate q-Gaussian distribution**). *Let us consider the random variables S, U, and W having a q-trivariate normal distribution characterized by parameters μ_1, μ_2, μ_3 $\in \mathbf{R}$, σ_1, σ_2, $\sigma_3>0$, along with ρ_{12}, ρ_{13}, and $\rho_{23}\in[-1,1]$ and $1<q<\frac{5}{3}$. The joint trivariate q-Gaussian density function is*

$$g_q(s,u,w)=\frac{\left(\frac{q-1}{2\pi}\right)^{3/2}\frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)}}{\sigma_1\sigma_2\sigma_3\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2-\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}}\times\exp_q\left(-\frac{1}{2}Q(s,u,w)\right), \tag{19}$$

where,

$$Q(s,u,w) = \frac{1}{D}[PV^{-1}P^T],$$

and

$$D = 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23},$$

$$P = [s-\mu_1 \quad u-\mu_2 \quad w-\mu_3],$$

$$V = \begin{vmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_{12} & \sigma_1\sigma_3\rho_{13} \\ \sigma_1\sigma_2\rho_{12} & \sigma_2^2 & \sigma_2\sigma_3\rho_{23} \\ \sigma_1\sigma_3\rho_{13} & \sigma_2\sigma_3\rho_{23} & \sigma_3^2 \end{vmatrix}.$$

Note on Correlation Structure: It is crucial to distinguish between the scale matrix parameter V (or Σ) and the actual covariance matrix of the data. In the standard Gaussian case ($q \to 1$), the covariance matrix coincides with $\mathbf{\Sigma}$. However, for q-Gaussian distributions ($1 < q < 5/3$), the heavy-tailed nature implies that the second moments are scaled. Let $\mathbf{S} = (S,U,W)^T$. The covariance matrix $\mathrm{Cov}(\mathbf{S})$ is related to the scale matrix $\mathbf{\Sigma}$ (composed of parameters σ_i, ρ_{ij}) by:

$$\mathrm{Cov}(\mathbf{S}) = \frac{2}{5-3q}\mathbf{\Sigma}, \quad \text{for } 1 < q < 5/3. \tag{20}$$

Consequently, the transformation matrix used in Eq. (17) represents the Cholesky decomposition of the *Scale Matrix* $\mathbf{\Sigma}$, not the covariance matrix itself. When estimating parameters from empirical data, the sample covariance $\mathbf{C}_{sample}$ must be mapped to the scale parameters via:

$$\mathbf{\Sigma}_{est} = \left(\frac{5-3q}{2}\right)\mathbf{C}_{sample}$$

Therefore, the parameters ρ_{ij} represent scale dependencies, not direct Pearson correlations. This adjustment ensures that the heavy tails modelled by q do not artificially inflate the correlation estimates. The subsequent density and contour plots illustrate the graphical characteristics of the trivariate q-Gaussian distribution for various parameter values q, keeping one variable constant. The parameters (μ_1, μ_2, μ_3, σ_1, σ_2, σ_3, ρ_{12}, ρ_{13}, ρ_{23}, q) are fixed, as shown in Figs.1, 2.

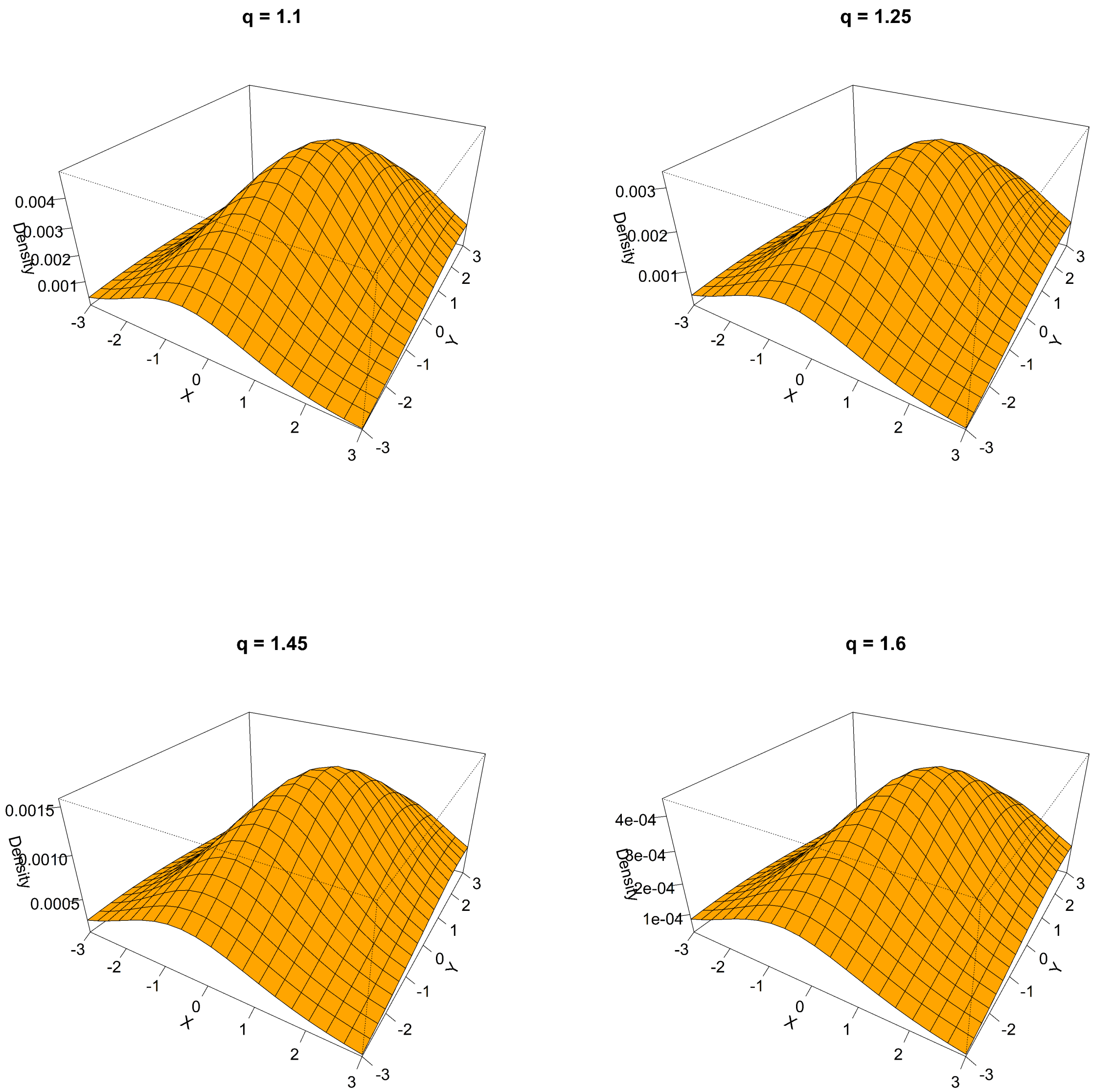

FIGURE 1. Illustrate of the density function for different values of q in the trivariate q-Gaussian distribution.

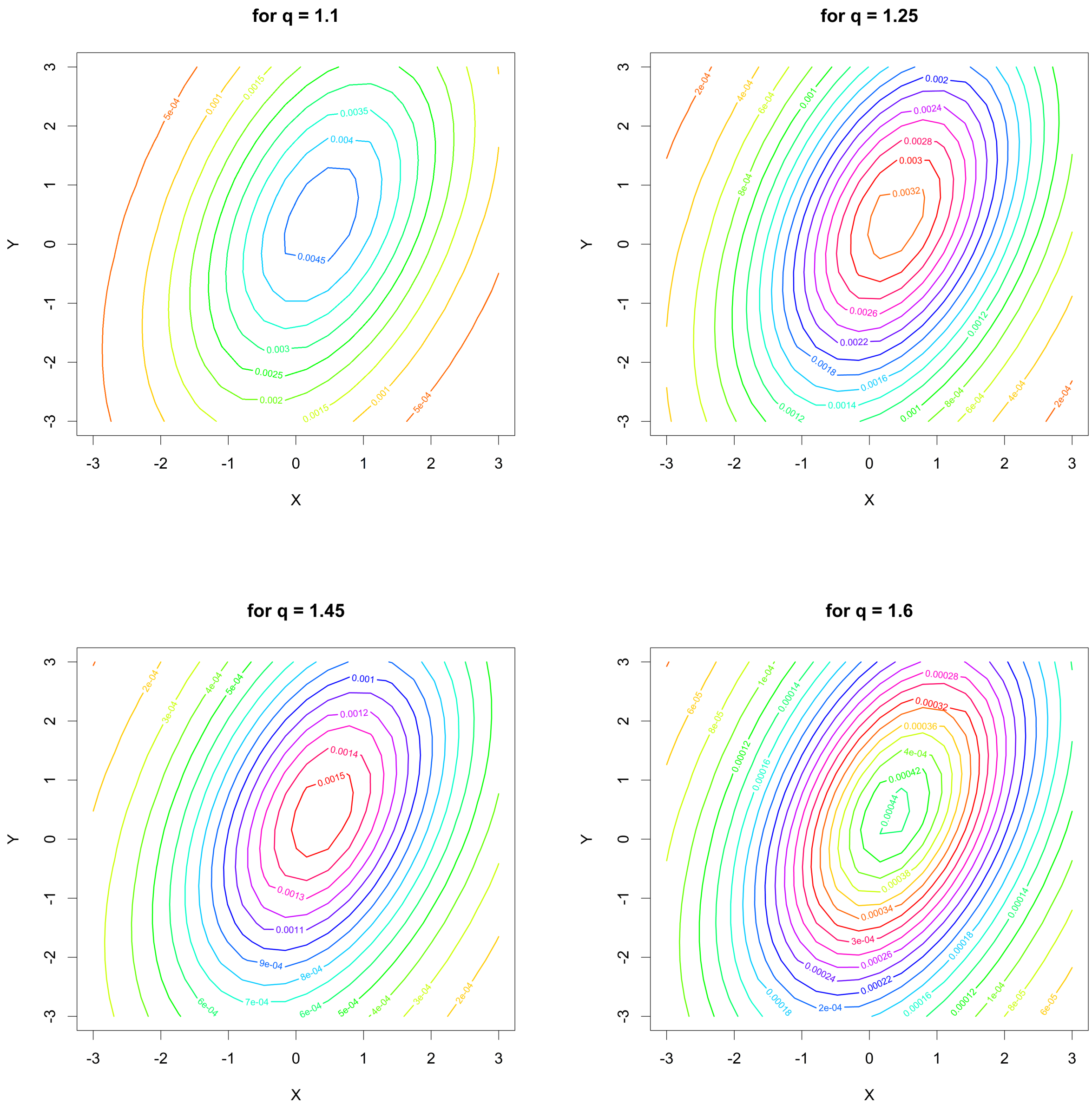

FIGURE 2. Contour graphs for different values of q in the trivariate q-Gaussian distribution.

We now obtain the marginal bivariate q-Gaussian density function using the trivariate q-Gaussian distribution.

Theorem 1. *Let S, U and W be the random variables follow a trivariate q-Gaussian distribution with parameters μ_1, μ_2, μ_3, σ_1, σ_2, σ_3, q, $-1 < \rho < 1$ and $1 < q < \frac{5}{3}$. The bivariate marginal* *q-Gaussian density is given by*

$$f_q(s,u) = \frac{(q-1)}{2\pi\sqrt{1-\rho_{12}^2}} \frac{\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{1}{q-1}+1\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)\Gamma\left(\frac{1}{q-1}+\frac{3}{2}\right)} \left[1 - \frac{(q-1)}{2(1-\rho_{12}^2)}Q(s,u)\right]^{\frac{1}{(q-1)}+\frac{1}{2}}. \tag{21}$$

Proof. Integrating the joint density $g_q(s,u,w)$ with respect to w, we obtain

$$f_q(s,u) = \int_{-\infty}^{\infty} \frac{c_5}{\sigma_1\sigma_2\sigma_3\sqrt{D}} \exp_q\left(-\frac{1}{2}Q(s,u,w)\right) dw,$$

or,

$$f_q(s,u) = \int_{-\infty}^{\infty} \frac{c_5}{\sigma_1\sigma_2\sigma_3\sqrt{D}} \left[1 + \frac{q-1}{2}Q(s,u,w)\right]^{-\frac{1}{q-1}} dw.$$

where,

$$c_5 = \left(\frac{q-1}{2\pi}\right)^{\frac{3}{2}} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)},$$

$$D = 1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23},$$

$$Q(s,u,w) = \frac{1}{D}\left[R_1(s-\mu_1)^2 + R_2(u-\mu_2)^2 + R_3(w-\mu_3)^2\right.$$
$$\left. +2R_4(s-\mu_1)(u-\mu_2) + 2R_5(w-\mu_3)(s-\mu_1) + 2R_6(u-\mu_2)(w-\mu_3)\right],$$

where,

$$R_1 = \frac{1-\rho_{23}^2}{\sigma_1^2}, \quad R_2 = \frac{1-\rho_{13}^2}{\sigma_2^2}, \quad R_3 = \frac{1-\rho_{12}^2}{\sigma_3^2},$$

$$R_4 = \frac{\rho_{13}\rho_{23}-\rho_{12}}{\sigma_1\sigma_2}, \quad R_5 = \frac{\rho_{12}\rho_{23}-\rho_{13}}{\sigma_1\sigma_3}, \quad R_6 = \frac{\rho_{12}\rho_{13}-\rho_{23}}{\sigma_2\sigma_3}.$$

Substitute $(w-\mu_3) = w_1$ in $Q(s,u,w)$,

$$Q(s,u,w) = \frac{1}{D}\left[R_3(w_1)^2 + R_1(s-\mu_1)^2 + R_2(u-\mu_2)^2\right.$$
$$\left. +2R_4(s-\mu_1)(u-\mu_2) + 2w_1(R_5(s-\mu_1) + 2R_6(u-\mu_2))\right].$$

Suppose

$$A = \frac{1}{D}\left[R_1(s-\mu_1)^2 + R_2(u-\mu_2)^2 + 2R_4(s-\mu_1)(u-\mu_2)\right],$$

$$B = \frac{2}{D}\left[(R_5(s-\mu_1) + R_6(u-\mu_2))\right],$$

$$C = \frac{1}{D}R_3.$$

We decompose the quadratic form $Q(s,u,w)$ into terms involving w and terms independent of w. Following the notation in Equation (19), we have:

$$Q(s,u,w) = Cw_1^2 + Bw_1 + A,$$

where $w_1 = (w-\mu_3)$ and A, B, C are defined previously. To integrate correctly, we complete the square for w_1 inside the bracket:

$$Cw_1^2 + Bw_1 + A = C\left(w_1 + \frac{B}{2C}\right)^2 + \left(A - \frac{B^2}{4C}\right).$$

Let $K = 1 + \frac{q-1}{2}\left(A - \frac{B^2}{4C}\right)$. This term K depends only on s and u. The integrand becomes:

$$\left[K + \frac{q-1}{2}C\left(w_1 + \frac{B}{2C}\right)^2\right]^{-\frac{1}{q-1}} = K^{-\frac{1}{q-1}}\left[1 + \frac{q-1}{2K}C\left(w_1 + \frac{B}{2C}\right)^2\right]^{-\frac{1}{q-1}}.$$

This factorization is crucial because the q-exponential is not separable. Now, we use the substitution variable t:

$$t = \left(w_1 + \frac{B}{2C}\right)\sqrt{\frac{C(q-1)}{2K}} \implies dw_1 = dt\sqrt{\frac{2K}{C(q-1)}}.$$

The integral transforms into a standard Beta function form:

$$\int_{-\infty}^{\infty}(1+t^2)^{-\frac{1}{q-1}}dt = \sqrt{\pi}\frac{\Gamma(\frac{1}{q-1} - \frac{1}{2})}{\Gamma(\frac{1}{q-1})}.$$

Substituting the Jacobian of the transformation and the value of the integral back into the expression for $f_q(s,u)$:

$$f_q(s,u) \propto K^{-\frac{1}{q-1}} \cdot K^{\frac{1}{2}} = K^{-\left(\frac{1}{q-1}-\frac{1}{2}\right)}.$$

Substituting back the value of $K = 1 + \frac{q-1}{2(1-\rho_{12}^2)}Q(s,u)$, we obtain the final form:

$$f_q(s,u) = \frac{(q-1)}{2\pi\sqrt{1-\rho_{12}^2}}\frac{\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{1}{q-1}+1\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)\Gamma\left(\frac{1}{q-1}+\frac{3}{2}\right)}\left[1 - \frac{(1-q)}{2(1-\rho_{12}^2)}Q(s,u)\right]^{\frac{1}{1-q}+\frac{1}{2}},$$

where $Q(s,u)$ corresponds to the standardized bivariate quadratic form. Similarly, we can write the other two bivariate densities as

$$f(u,w) = \frac{(q-1)}{2\pi\sqrt{1-\rho_{23}^2}}\frac{\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{1}{q-1}+1\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)\Gamma\left(\frac{1}{q-1}+\frac{3}{2}\right)}\left[1 - \frac{(1-q)}{2(1-\rho_{23}^2)}Q(u,w)\right]^{\frac{1}{1-q}+\frac{1}{2}}$$

$$f(s,w) = \frac{(q-1)}{2\pi\sqrt{1-\rho_{13}^2}}\frac{\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{1}{q-1}+1\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)\Gamma\left(\frac{1}{q-1}+\frac{3}{2}\right)}\left[1 - \frac{(1-q)}{2(1-\rho_{13}^2)}Q(s,w)\right]^{\frac{1}{1-q}+\frac{1}{2}}$$

□

3. Estimation and Model Selection

3.1. **Maximum likelihood estimation.** We explore a well-recognised approach for estimating parameters, the maximum likelihood estimation (MLE). Let $S_1, S_2, ..., S_n$ be a random sample of size n from a trivariate q-Gaussian distribution. The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} g_q(s_i, u_i, w_i), \tag{22}$$

where, $\theta = (\mu_1,\, \mu_2,\, \mu_3,\, \sigma_1,\, \sigma_2,\, \sigma_3,\, \rho_{12}\ ,\, \rho_{13},\, \rho_{23},\, q)$ is the set of parameters. Note that

$$L(\theta) = \prod_{i=1}^{n} \frac{\left(\frac{q-1}{2\pi}\right)^{3/2} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right)}}{\sigma_1\sigma_2\sigma_3\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2-\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}} \times \left[1+\frac{(q-1)}{2}Q(s_i,u_i,w_i)\right]^{\frac{-1}{q-1}}. \tag{23}$$

Then maximization of the likelihood function is given as

$$l(\theta) = \sum_{i=1}^{n} \ln(g_q(s_i,u_i,w_i)) \tag{24}$$

Applying logarithms $l(\theta) = \ln L(\theta)$, we get

$$l(\theta) = \frac{3n}{2}\ln\left(\frac{q-1}{2\pi}\right) + n\ln\Gamma\left(\frac{1}{q-1}\right) - n\ln\Gamma\left(\frac{1}{q-1}-\frac{3}{2}\right) - n\ln\sigma_1 - \tag{25}$$

$$n\ln\sigma_2 - n\ln\sigma_3 - \frac{n}{2}\ln(1-\rho_{12}^2) - \frac{n}{2}\ln\left(1-\rho_{13}^2-\frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}\right)$$

$$-\frac{1}{q-1}\sum_{i=1}^{n}\left[1+\frac{q-1}{2}Q(s_i,u_i,w_i)\right]. \tag{26}$$

We differentiate equation (25) with respect to each parameter
$\theta = (\mu_1,\mu_2,\mu_3,\sigma_1,\sigma_2,\sigma_3,\rho_{12},\rho_{13},\rho_{23},q)$ given as

$$\frac{\partial l}{\partial \mu_1} = \frac{-1}{\sigma_1(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \times \sum_{i=1}^{n}\Bigg[(1-\rho_{23}^2)\left(\frac{s_i-\mu_1}{\sigma_1}\right)$$

$$+(\rho_{13}\rho_{23}-\rho_{12})\left(\frac{u_i-\mu_2}{\sigma_2}\right)+(\rho_{12}\rho_{23}-\rho_{13})\left(\frac{w_i-\mu_3}{\sigma_3}\right)\Bigg],$$

$$\frac{\partial l}{\partial \mu_1} = \frac{-1}{\sigma_2(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \times \sum_{i=1}^{n}\Bigg[(1-\rho_{13}^2)\left(\frac{u_i-\mu_2}{\sigma_2}\right)$$

$$+(\rho_{13}\rho_{23}-\rho_{12})\left(\frac{s_i-\mu_1}{\sigma_1}\right)+(\rho_{12}\rho_{13}-\rho_{23})\left(\frac{w_i-\mu_3}{\sigma_3}\right)\Bigg],$$

$$\frac{\partial l}{\partial \mu_3} = \frac{-1}{\sigma_3(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \times \sum_{i=1}^{n}\Bigg[(1-\rho_{12}^2)\left(\frac{w_i-\mu_3}{\sigma_3}\right)$$

$$+(\rho_{12}\rho_{23}-\rho_{13})\left(\frac{s_i-\mu_1}{\sigma_1}\right)+(\rho_{12}\rho_{13}-\rho_{23})\left(\frac{u_i-\mu_2}{\sigma_2}\right)\Bigg],$$

$$\frac{\partial l}{\partial \sigma_1} = -\frac{n}{\sigma_1}+\frac{1}{(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \times \sum_{i=1}^{n}\Bigg[(1-\rho_{23}^2)\frac{(s_i-\mu_1)^2}{\sigma_1^3}$$

$$+(\rho_{13}\rho_{23}-\rho_{12})\left(\frac{s_i-\mu_1}{\sigma_1^2}\right)\left(\frac{u_i-\mu_2}{\sigma_2}\right)+(\rho_{13}\rho_{23}-\rho_{13})\left(\frac{s_i-\mu_1}{\sigma_1^2}\right)\left(\frac{w_i-\mu_3}{\sigma_3}\right)\Bigg],$$

$$\frac{\partial l}{\partial \sigma_2} = -\frac{n}{\sigma_2}+\frac{1}{(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})} \times \sum_{i=1}^{n}\Bigg[(1-\rho_{13}^2)\frac{(u_i-\mu_2)^2}{\sigma_2^3}$$

$$+(\rho_{13}\rho_{23}-\rho_{12})\left(\frac{s_i-\mu_1}{\sigma_1}\right)\left(\frac{u_i-\mu_2}{\sigma_2^2}\right)+(\rho_{12}\rho_{13}-\rho_{23})\left(\frac{u_i-\mu_1}{\sigma_2^2}\right)\left(\frac{w_i-\mu_3}{\sigma_3}\right)\Bigg],$$

$$\frac{\partial l}{\partial \sigma_3}=-\frac{n}{\sigma_3}+\frac{1}{(1-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2+2\rho_{12}\rho_{13}\rho_{23})}\times\sum_{i=1}^{n}\Bigg[(1-\rho_{12}^2)\frac{(w_i-\mu_3)^2}{\sigma_3^3}$$

$$+(\rho_{12}\rho_{23}-\rho_{13})\left(\frac{s_i-\mu_1}{\sigma_1}\right)\left(\frac{w_i-\mu_3}{\sigma_3^2}\right)+(\rho_{12}\rho_{13}-\rho_{23})\left(\frac{u_i-\mu_2}{\sigma_2}\right)\left(\frac{w_i-\mu_3}{\sigma_3^2}\right)\Bigg],$$

and so for $\frac{\partial l}{\partial \rho_{ij}}$, $\frac{\partial l}{\partial q}$, set the derivatives equal to zero. Since the closed-form solutions for the MLE equations do not generally exist, numerical optimization techniques are used to estimate the parameters. We evaluate the effectiveness of the maximum likelihood estimation (MLE) technique for estimating the parameters of the trivariate q-Gaussian distribution, and the calculation is carried out through numerical methods using a nonlinear quasi-Newton optimisation algorithm.

3.2. **Information Criterion.** The Information Criterion (IC) [15] is a popular tool for model selection. It helps find a balance between the model's fit to the data and its complexity, which is determined by the number of parameters. We have mainly discussed the AIC and the BIC. Where AIC is given as

$$\text{AIC} = 2k - 2\ln(\hat{L}),$$

and BIC is given as

$$\text{BIC} = k\ln(n) - 2\ln(\hat{L}),$$

where k represents the number of estimated parameters, n is the sample size and $\hat{L}$ represents the likelihood function, which is the estimated values of parameters.

4. Application

This section presents the applicability of the proposed trivariate q-Gaussian distribution for three datasets.

4.1. **Empirical Analysis of the Nifty 50 Index.** We initiate our empirical analysis by applying the proposed trivariate q-Gaussian distribution (TVq-GD) to the Nifty 50 index. We utilize monthly data comprising log-returns, realized volatility, and trading volume. Figure 3 shows all three components of the NIFTY50 index data. As a benchmark equity index, the Nifty 50 provides a fundamental testing ground for the model's ability to capture the stylized facts of financial time series—specifically structural dependence and leptokurtosis—without reference to the commodity markets analyzed later. We define the state vector as $X = (\text{Return}, \text{Volatility}, \ln(\text{Volume}))$. We estimate the parameters for the Trivariate Gaussian (TVGD), Trivariate Cauchy (TCD), and Trivariate q-Gaussian (TVq-GD) distributions using Maximum Likelihood Estimation (MLE).

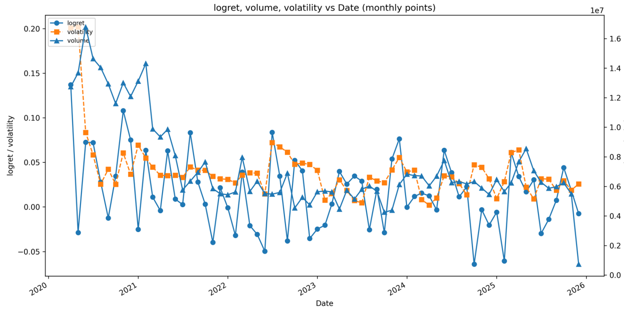

FIGURE 3. Returns , Volume and Volatility for NIFTY50 index data

Table 1 summarizes the estimation results. The standard Trivariate Gaussian model yields the poorest fit with an AIC of -486.20. The Trivariate Cauchy Distribution (TCD), which assumes a rigid heavy-tailed structure ($\nu = 1$), achieves an AIC of -511.60. This improvement of approximately 25 points over the Trivariate Gaussian model provides strong evidence that monthly equity data exhibits significant heavy tails that cannot be captured by Trivariate Gaussian assumptions. However, the proposed TVq-GD model provides the superior overall fit, achieving the lowest AIC of -534.58. The estimated entropic index is $q \approx 1.305$, which confirms significant non-extensivity in the Indian equity market. Notably, the q-Gaussian model outperforms the Cauchy distribution, suggesting that while the data is heavy-tailed, the specific tail thickness is better characterized by the adaptive q parameter than by the fixed Cauchy distribution. The Likelihood Ratio Test against the Gaussian null hypothesis yields a statistic of $\lambda = 50.37$, which is far greater than the critical value. This compels us to emphatically reject the null hypothesis, confirming that the q-trivariate Gaussian model offers a statistically significant improvement over traditional methods in modeling the return-volatility-volume relationship.

Model	**Parameter Estimates**	**Log-Lik.**	**AIC**	**BIC**
Trivariate Gaussian Distribution (TVGD)				
TVGD	$\mu = [0.016, 0.040, 15.75]$ $\sigma = [0.040, 0.033, 0.336]$ $\rho_{12} = 0.2849, \rho_{13} = 0.3437, \rho_{23} = 0.4591$	-252.10	-486.20	-466.09
Trivariate Cauchy Distribution (TCD)				
TCD	$\mu = [0.007, 0.031, 15.65]$ $\sigma = [0.028, 0.013, 0.191]$ $\rho_{12} = 0.1144, \rho_{13} = 0.3164, \rho_{23} = 0.2877$	-264.80	-511.60	-491.49
Trivariate q-Gaussian Distribution (TVq-GD)				
TVq-GD	$\mu = [0.010, 0.032, 15.68]$ $\sigma = [0.024, 0.012, 0.181]$ $\rho_{12} = 0.2130, \rho_{13} = 0.3275, \rho_{23} = 0.3100$ $\mathbf{q = 1.305056}$	**-277.29**	**-534.58**	**-512.23**

TABLE 1. Estimation results for the Nifty 50 Index. The TVq-GD outperforms both the trivaraite Gaussian and Cauchy benchmarks, indicating an optimal fit for the heavy-tailed equity data.

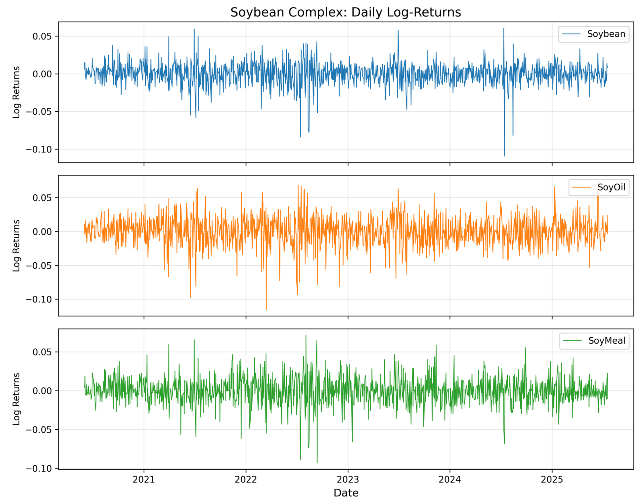

FIGURE 4. Daily log-returns of the Soybean Complex (Soybean, Soy-Oil, Soymeal).

4.2. **Empirical Analysis of Daily Futures Returns.** In this section, we apply the proposed trivariate q-Gaussian distribution (TVq-GD) to two distinct commodity complexes: the Soybean complex (comprising Soybean, Soy-Oil, and Soymeal) and the Gold complex (comprising CME, Shanghai, and Thai Futures). We utilize daily log-returns derived from the mid-prices of futures contracts over the last five years (2020-2025). This shift to daily frequency allows us to examine the structural dependence and tail behaviour of these assets over a longer horizon, minimising microstructure noise while capturing the stylised fact of leptokurtosis common in commodity markets. We benchmark the performance of the TVq-GD against two established baselines: the standard Trivariate Gaussian Distribution (TVGD) and the heavy-tailed Trivariate Cauchy Distribution (TCD). The parameters are estimated via Maximum Likelihood Estimation (MLE). To rigorously compare the models, we employ the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the Likelihood Ratio Test (LRT).

4.2.1. *The Soybean Complex.* The first dataset comprises the 1-month continuous futures returns for Soybean, Soy-Oil, and Soymeal. These commodities exhibit strong fundamental linkages due to the crushing process, which typically induces high correlation and volatility clustering. Figure 4 displays the daily log-returns for the three assets. Table 2 presents the estimation results. The standard Trivariate Gaussian model (TVGD) assumes a shape parameter of $q = 1$. However, the estimated entropic index for the TVq-GD is $q \approx 1.27$, a substantial deviation from unity. This indicates that the joint distribution of soybean returns possesses significantly heavier tails than predicted by the trivariate Gaussian distribution. The superiority of the q-Gaussian model is confirmed by the information criteria; the TVq-GD achieves an AIC of -22363.77 compared to -21655.33 for the Trivariate Gaussian model. Interestingly, the Trivariate Cauchy Distribution (TCD) also outperforms the Trivariate Gaussian model (AIC -21802.81), confirming the presence of heavy tails. However, the TVq-GD still outperforms the Cauchy model significantly, suggesting that the q-Gaussian distribution better adapts to the specific tail architecture of the data than the rigid Cauchy distribution. The Likelihood Ratio Test yields a statistic of 710.44 with a p-value effectively zero (1.6×10^{-156}), leading to the emphatic rejection of the null hypothesis that the data follows a Gaussian distribution.

Model	Parameter Estimates	Log-Lik.	AIC	BIC
Trivariate Gaussian Distribution (TVGD)				
TVGD	$\mu = [1.57, 5.47, -0.29] \times 10^{-4}$ $\sigma = [0.0145, 0.0213, 0.0172]$ $\rho_{12} = 0.5302, \rho_{13} = 0.6558, \rho_{23} = 0.1059$	-10836.67	-21655.33	-21608.88
Trivariate Cauchy Distribution (TCD)				
TCD	$\mu = [6.01, 17.91, -0.54] \times 10^{-4}$ $\sigma = [0.0088, 0.0142, 0.0107]$ $\rho_{12} = 0.5783, \rho_{13} = 0.6705, \rho_{23} = 0.0523$	-10910.40	-21802.81	-21756.36
Trivariate q-Gaussian Distribution (TVq-GD)				
TVq-GD	$\mu = [5.17, 14.25, 1.28] \times 10^{-4}$ $\sigma = [0.0082, 0.0131, 0.0101]$ $\rho_{12} = 0.5684, \rho_{13} = 0.6698, \rho_{23} = 0.0590$ $\mathbf{q = 1.271177}$	**-11191.88**	**-22363.77**	**-22312.16**

TABLE 2. Estimation results for the Soybean Complex. The TVq-GD yields the lowest AIC, indicating the best fit. While Cauchy outperforms the Trivariate Gaussian, it is still inferior to the q-Gaussian.

4.2.2. *The Gold Complex.* The second application analyzes the global gold market, incorporating CME Gold (USA), Shanghai Gold (China), and Thai Gold futures. This selection provides a comprehensive view of the precious metals market across three distinct liquidity centers. Figure 5 illustrates the synchronized volatility clusters across these geographically distinct markets, reflecting the strong integration of global bullion trading.

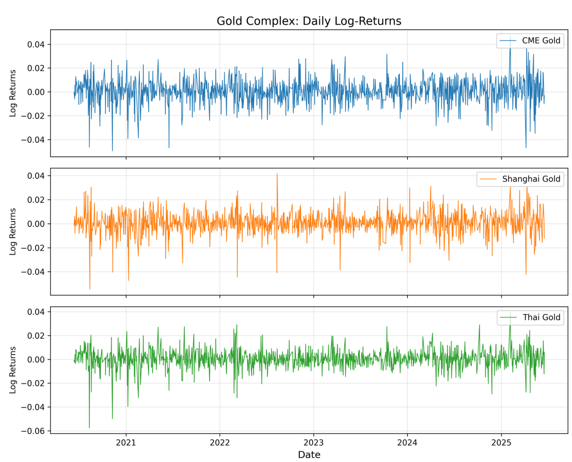

FIGURE 5. Daily log-returns of the Gold Complex (CME, Shanghai, Thai Futures).

The results in Table 3 reveal a remarkably strong correlation between CME and Shanghai markets ($\rho \approx 0.81$ across models). Similar to the agricultural commodities, the Gold complex exhibits non-Gaussian behaviour with an estimated entropic index of $q \approx 1.26$. Notably, in this case, the Cauchy distribution (TCD) performs worse than the standard Normal distribution (AIC -24663.89 vs. -24740.27). This implies that the Cauchy distribution overestimates the risk of extreme tails.

In contrast, the TVq-GD model captures the joint probability density far more accurately than both baselines, achieving an AIC of -25197.20, which is lower than that of the trivariate Gaussian model. The Likelihood Ratio statistic of 458.94 ($p < 10^{-100}$) confirms that the proposed distribution effectively models the systemic risk and heavy tails inherent in global futures markets without overestimating them as the Cauchy model does.

Model	Parameter Estimates	Log-Lik.	AIC	BIC
Trivariate Gaussian Distribution (TVGD)				
TVGD	$\mu = [5.66, 5.88, 6.09] \times 10^{-4}$ $\sigma = [0.0101, 0.0093, 0.0080]$ $\rho_{12} = 0.1861, \rho_{13} = 0.8264, \rho_{23} = 0.1559$	-12379.13	-24740.27	-24694.55
Trivariate Cauchy Distribution (TCD)				
TCD	$\mu = [4.87, 6.51, 5.53] \times 10^{-4}$ $\sigma = [0.0062, 0.0056, 0.0047]$ $\rho_{12} = 0.0933, \rho_{13} = 0.7994, \rho_{23} = 0.0619$	-12340.94	-24663.89	-24618.18
Trivariate q-Gaussian Distribution (TVq-GD)				
TVq-GD	$\mu = [6.31, 6.65, 6.33] \times 10^{-4}$ $\sigma = [0.0060, 0.0055, 0.0046]$ $\rho_{12} = 0.1277, \rho_{13} = 0.8097, \rho_{23} = 0.1001$ $\mathbf{q = 1.261104}$	**-12608.60**	**-25197.20**	**-25146.41**

Table 3. Estimation results for the Gold Futures Complex. The TVq-GD provides the best fit. Note that the Cauchy distribution fits worse than the Gaussian, highlighting the need for the adaptable q parameter.

5. Conclusion and future study

We constructed a trivariate q-Gaussian distribution using the maximum entropy principle to explore relationships among stocks, trading volume, volatility, and returns, and to analyse future returns in the soyaban complex and gold commodity markets. We identified several important features of the proposed distribution. Parameters of the trivariate q-Gaussian distribution were estimated using the maximum likelihood estimation (MLE) method. We demonstrated the relevance of the proposed distribution by comparing it with baseline models. It is evident from Tables 1, 2 and 3 that the trivariate q-Gaussian distribution offers a better fit for the underlying dataset than the trivariate Gaussian distribution and the trivariate Cauchy distribution. This enhanced fit implies that the trivariate q-Gaussian model is better able to represent the variability observed in the data. Future research can also focus on multifactor data analysis, examining intraday stock data alongside other economic data. It may also explore additional statistical properties for the proposed model.

Competing interests: The authors declare that there is no conflict of interest regarding the publication of this paper.

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