Eur. J. Stat. 6 (2026) 13
doi: 10.28924/ada/stat.6.13

Odd-Fréchet Log-Logistic Distribution with Statistical Properties and Applications

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Abstract. This manuscript introduces the Odd Fréchet Log-Logistic (OFrLL) distribution as a new extension of the Log-Logistic distribution within the Odd Fréchet G family. The proposed distribution's key functions, including the probability density function, cumulative distribution function and hazard function are presented along with graphical representations. We derive essential mathematical properties, reliability characteristics, generating functions and entropy measures for the OFrLL distribution. Parameter estimations are conducted using various classical estimation approaches, with a comprehensive simulation study assessing their performance. The practical applicability of the OFrLL distribution is demonstrated through real-world data on foreign direct investment in India and CO_2 emissions, validating the model's effectiveness.

1. Introduction

In many applied situations for diverse data modeling, choosing an appropriate probability distribution is a foremost concern. However, over the years numerous probability models have been broadly recommended for the analysis of data in several areas such as medical sciences, actuarial sciences, engineering, finance and insurance etc. Sometimes the existing probability distributions do not provide a good fit for more peaked and heavy-tailed data sets. Thus, there is a need to propose new probability distributions by adding one or more parameter(s). Log-Logistic (LL) distribution, also known as Fisk distribution in economics is a continuous probability distribution used in surival analysis for modeling time to event data, in economics for modeling income distribution and size distribution of human settlements and in reliability engineering describing the life length of products or systems. For the purpose of enhancing the capability and applicability of the LL distribution numerous researchers have put forward several generalizations of the distribution. [1] obtained transmuted LL distribution using Quadratic rank Transmutation. [2] developed a new class of LL distribution using Marshall Olkin transformation. [3] proposed four parameter Beta LL distributions. [4] introduced alpha power LL distribution. [5] introduced LL tangent distribution for analyzing COVID- 19 data. [6] proposed

Received: 9 Jan 2026.
Key words and phrases. mean residual; foreign direct investment; Aderson Darling estimation.

Kavya-Manoharan LL distribution. The SMPLog-logistic distribution was developed by [7] and applied to demographic and health datasets from India. [8] introduced Odd Fréchet Half-Logistic distribution. The Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of LL distribution are given by Eq. (1) and Eq. (2) respectively.

$$g(x,\theta)=\frac{\theta x^{\theta-1}}{(1+x^{\theta})^2},\quad x>0,\,\theta>0 \tag{1}$$

$$G(x,\theta)=\frac{x^{\theta}}{1+x^{\theta}},\quad x>0,\,\theta>0 \tag{2}$$

The foundation of the present study is the Odd-Fréchet generated (OFr-G) family introduced by [9] with CDF and PDF, respectively given as

$$F(x;\alpha,\theta)=\int_0^{\left[\frac{G(x;\theta)}{1-G(x;\theta)}\right]}\frac{\alpha}{x^{\alpha+1}}e^{-x^{-\alpha}}\,dx=\exp\left[-\left(\frac{1-G(x;\theta)}{G(x;\theta)}\right)^{\alpha}\right]\quad ,x>0,\theta>0 \tag{3}$$

$$f(x;\alpha,\theta)=\frac{\alpha g(x;\theta)\left[1-G(x;\theta)\right]^{\alpha-1}}{\left[G(x;\theta)\right]^{\alpha+1}}\exp\left[-\left(\frac{1-G(x;\theta)}{G(x;\theta)}\right)^{\alpha}\right]\quad ,x>0,\,\theta>0 \tag{4}$$

Where $g(x;\theta)$ is the PDF and $G(x;\theta)$ is the CDF of baseline distribution respectively.

In this article, we present a new extension of the LL distribution by incorporating OFr-G family of distributions and the new model so obtained is referred to as Odd-Fréchet Log-Logistic (OFrLL) distribution. The rationale behind espousing the OFr-G family of distributions lies in its capability to enhance the flexibility of LL distribution and improve its goodness-of-fit characteristics, thus offering broader applications. The OFr-G family is used to enlarge the capability of the baseline distribution's applicability in modeling different types of datasets. Moreover, utilizing the OFr-G framework enables us to investigate the tail properties of the distribution. We are motivated to introduce OFrLL distribution because

- It is more flexible and exhibits more complex shapes of density and hazard rate functions.
- It is capable of modeling constant or monotonically increasing/decreasing hazard rate.
- It can be viewed as a suitable distribution for fitting the skewed data. Also, the proposed model outclasses some well-established models in terms of two real life data sets.

The rest of the article is unfolded as follows: In section 2, we propose a new distribution, as obtained by incorporating OFr-G family by using LL distribution as the baseline distribution. In section 3, some statistical properties of the distribution are explored. We have shown the reliability measures of the new distribution in section 4. Some generating functions are discussed in section 5.Entropy measures and order statistics are debated in section 6 and section 7 respectively. The estimation of unknown parameters is given in section 8. To analyze the flexibility of estimation procedures a simulation study is given in section 9. Further to check the applicability of the model real data sets are used and are presented in section 10. Finally, conclusions is provided in section 11.

2. Odd-Fréchet Log-Logistic (OFrLL) Distribution

In this section we discuss the construction of the OFrLL distribution. Considering $G(x;\theta)$ to be the CDF of Log-Logistic distribution, then the CDF of OFrLL distribution can be obtained by inserting Eq. (2) in Eq. (3) and is given by

$$F(x,\alpha,\theta) = e^{-x^{-\alpha\theta}}, \quad x > 0,\, \theta > 0 \tag{5}$$

And the corresponding PDF is obtained by inserting Eq. (1) in Eq. (4)

$$f(x,\alpha,\theta) = \alpha\theta x^{-(\alpha\theta+1)} e^{-x^{-\alpha\theta}}, \quad x > 0,\, \theta > 0 \tag{6}$$

Where α and θ are parameters. The density plots for different parameter combinations are presented in Figure 1.

Interpretation: From these plots, it is clear that the proposed distribution has decreasing, constant, unimodal and increasing-decreasing density shapes. Each of these shapes represents a different pattern in how the data is distributed. These shapes of the plots suggest that the proposed distribution can take on multiple forms depending on the parameters being considered.

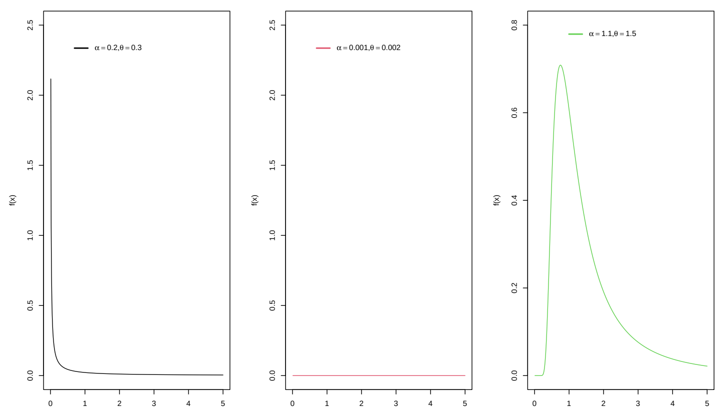

Figure 1. Probability density plots for different parameter combinations

3. Statistical Properties

In this section, some of the statistical properties of the OFrLL distribution are discussed.

3.1. **Quantile and Simulation.**

Quantiles are primarily important for quantile estimators and simulations. Also, the effect of the shape parameters on the skewness and kurtosis can be recognized using quantile measures. Using inverse CDF method quantile function of OFrLL distribution is given as:

Let $F(x) = u$.

Substituting value of $F(x)$,

$$e^{-x^{-\alpha\theta}} = u$$

$$x = \frac{1}{(-\log(u))^{\frac{1}{\alpha\theta}}} \tag{7}$$

Where u∼uniform (0,1). Using Eq. (7), the OFrLL distribution can easily be simulated. Also the p^{th} quantile of OFrLL distribution is given as

$$x_p = \frac{1}{(-\log(p))^{\frac{1}{\alpha\theta}}}$$

3.2. **Moments.** The r^{th} moment about the origin of the OFrLL distribution can be obtained as

$$\mu_r' = \int_0^\infty x^r f(x)\,dx = \int_0^\infty x^r \alpha\theta x^{-(\alpha\theta+1)} e^{-x^{-\alpha\theta}}\,dx$$

$$\mu_r' = \Gamma\left(1 - \frac{r}{\alpha\theta}\right) \tag{8}$$

3.2.1. *Mean and variance are given as:*

$$\text{Mean} = \mu_1' = \Gamma\left(1 - \frac{1}{\alpha\theta}\right)$$

$$\mu_2' = \Gamma\left(1 - \frac{2}{\alpha\theta}\right)$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\text{Variance} = \Gamma\left(1 - \frac{2}{\alpha\theta}\right) - \left(\Gamma\left(1 - \frac{1}{\alpha\theta}\right)\right)^2$$

3.3. **Incomplete moment.** The r^{th} incomplete moment about origin is defined as

$$I_r(t) = \int_0^t x^r g(x)\,dx$$

$$= \int_0^t x^r \alpha\theta x^{-(\alpha\theta+1)} e^{-x^{-\alpha\theta}}\,dx$$

After necessary substitutions and calculations,

$$I_r(t) = \Gamma\left(1 - \frac{r}{\alpha\theta}, t^{-\alpha\theta}\right) \tag{9}$$

For r=1, we get the first incomplete moment of OFrLL distribution.

$$I_1(t) = \Gamma\left(1 - \frac{1}{\alpha\theta}, t^{-\alpha\theta}\right) \tag{10}$$

4. Reliability Characteristics

In this section, the reliability measures of OFrLL distribution are discussed.

4.1. **Reliability function.** The reliability function of OFrLL distribution is given as

$$R(x) = 1 - e^{-x^{-\alpha\theta}}$$

4.2. **Hazard rate function.** The hazard rate function of OFrLL distribution is given as

$$h(x) = \frac{\alpha\theta x^{-(\alpha\theta+1)} e^{-x^{-\alpha\theta}}}{1 - e^{-x^{-\alpha\theta}}}$$

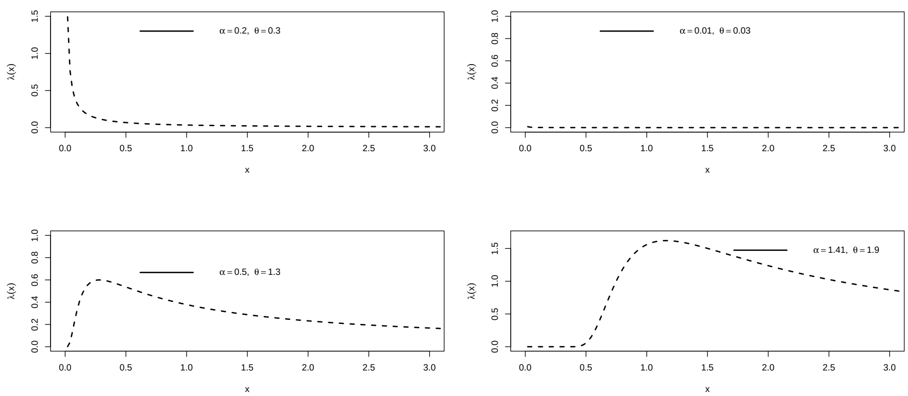

FIGURE 2. Fluctuations of hazard rate function along with parameters.

Interpretation: From Figure 2, it is evident that the model exhibits varying shapes based on different parameter values. This versatility indicates that the proposed model is well-suited for modeling datasets with varying failure rates.

4.3. **Reverse hazard rate function.** The reverse hazard rate function is given as

$$h_r(x) = \frac{\alpha\theta x^{-(\alpha\theta+1)} e^{-x^{-\alpha\theta}}}{e^{-x^{-\alpha\theta}}}$$

4.4. **Mean residual life.** In the context of reliability analysis, the mean residual life is the expected remaining lifetime of a component or system, given that it has already survived to a certain age and is given by

$$\mu(t) = \frac{1}{R(t)} \left(E(x) - \int_0^t x f(x; \alpha, \theta)\, dx \right) - t$$

Where

$$E(x) = \gamma \left(1 - \frac{r}{\alpha\theta} \right) \quad \& \quad I_1(t) = \int_0^t x\, f(x; \alpha, \theta)\, dx$$

Substituting values of $E(x)$, $I_1(t)$ and after necessary calculations we get,

$$\mu(t) = \frac{\Gamma\left(1 - \frac{1}{\alpha\theta}\right) - \Gamma\left(1 - \frac{1}{\alpha\theta}, t^{-\alpha\theta}\right)}{1 - e^{-t^{-\alpha\theta}}} - t$$

4.5. **Mean waiting time.** In reliability analysis, Mean waiting time is the average time elapsed between component failure. Mean waiting time is calculated as:

$$\bar{\mu}(t) = t - \frac{I_1(t)}{F(t)}$$

Substituting values of $E(x)$, $I_1(t)$ and after necessary calculations we get,

$$\mu(t) = \frac{te^{-t^{-\alpha\theta}} - \Gamma\left(1 - \frac{1}{\alpha\theta}, t^{-\alpha\theta}\right)}{e^{-t^{-\alpha\theta}}}$$

Mean waiting time is used in various reliability analysis applications, such as spare parts inventory management, predictive maintenance scheduling, system availability and uptime optimization by analyzing mean waiting time potential reliability issues. Optimal maintenance strategies and improved system uptime and availability and reduced maintenance costs can be achieved.

5. Generating Functions

This section explores the generating functions associated with the OFrLL distribution.

5.1. **Moment Generating Function.** The moment generating function is obtained as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\Gamma\left(1 - \frac{r}{\alpha\theta}\right)$$

5.2. **Characteristic Function.** The characteristic function is given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\mu_r'$$

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\Gamma\left(1 - \frac{r}{\alpha\theta}\right)$$

6. Entropy Measures

6.1. **Renyi Entropy.** The Renyi entropy is expressed as

$$I_\delta(x) = \frac{1}{(1-\delta)}\log\left(\int_{-\infty}^{\infty} f^\delta(x)\,dx\right) \quad \text{for } \delta > 0 \text{ and } \delta \neq 1$$

$$\frac{1}{(1-\delta)}\left((\delta-1)\log(\alpha\theta) + \log\left(\Gamma\left(\frac{\delta(\alpha\theta+1)-1}{\alpha\theta}\right)\right) - \frac{\delta(\alpha\theta+1)-1}{\alpha\theta}\log(\delta)\right)$$

6.2. **Harvda- Charvat Entropy.** Harvda-Charvat presented an idea of parametric entropy [10].The Harvda and Charvat entropy is given by

$$I_\delta(x) = \frac{1}{1-\delta}\left(1 - \int_0^{\infty} f^\delta(x)\,dx\right) \quad \delta > 0 \text{ and } \delta \neq 1$$

$$= \frac{1}{1-\delta}\left(1 - (\alpha\theta)^{\alpha-1}\frac{\Gamma\left(\frac{\delta(\alpha\theta+1)-1}{\alpha\theta}\right)}{\delta^{\frac{\delta(\alpha\theta+1)-1}{\alpha\theta}}}\right)$$

7. Order Statistics

Let $X_{(1)}, X_{(2)}, X_{(3)} \ldots X_{(n)}$ be the random sample of size n and let $X_{(r:n)}$ denote the r^{th} order statistic, then the PDF of $X_{(r:n)}$ is

$$f_{(r;n)}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \tag{11}$$

Using expansion

$$(1-x)^{n-r} = \sum_{l=0}^{\infty}\binom{n-r}{l}(-1)^l x^l$$

$$f_{(r;n)}(x) = \frac{1}{B(r,n+1-r)}\sum_{i=0}^{\infty}\binom{n-r}{i}(-1)^i\alpha\theta x^{-(\alpha\theta+1)}e^{-x^{-\alpha\theta(r+i)}}$$

The CDF is given by

$$F_{(r;n)}(x) = \sum_{j=r}^{n}\binom{n}{j}[F(x)]^j[1-F(x)]^{n-j} \tag{12}$$

$$F_{(r;n)}(x) = \frac{1}{B(j,n+1-j)}\sum_{j=r}^{n}\sum_{k=0}^{\infty}\binom{n-j}{k}(-1)^k\left(e^{-x^{-\alpha\theta}}\right)^{j+k}$$

8. Statistical Inference

This section deals with parameter estimation of OFrLL distribution using various estimation procedures.

8.1. **Maximum Likelihood Estimation Method.**

Let X_1, X_2, X_3, $\ldots$, X_n be a random sample of size n from OFrLL distribution then the log-likelihood function is given by

$$l = n\log(\alpha) + n\log(\theta) - \sum_{i=1}^{n}((\alpha\theta+1)\log(x_i)) - \sum_{i=1}^{n}x_i^{-\alpha\theta}$$

The estimate of parameters can be found by setting the following derivatives equal to 0 and solving the resulting equations,

For parameter θ,

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\alpha\log(x_i) - \alpha\sum_{i=1}^{n}x_i^{-\alpha\theta}\log(x_i)$$

$$\frac{n}{\theta} - \sum_{i=1}^{n}\alpha\log(x_i) - \alpha\sum_{i=1}^{n}x_i^{-\alpha\theta}\log(x_i) = 0$$

For parameter α,

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\theta\log(x_i) - \theta\sum_{i=1}^{n}x_i^{-\alpha\theta}\log(x_i)$$

$$\frac{n}{\alpha} - \sum_{i=1}^{n}\theta\log(x_i) - \theta\sum_{i=1}^{n}x_i^{-\alpha\theta}\log(x_i) = 0$$

The Newton–Raphson method, implemented using R software, is used to solve the above system of equations and estimate the unknown parameters of the OFrLL distribution.

8.2. **Aderson Darling (AD) Estimation.**

The Anderson and Darling estimate of the parameters is given by

$$AD(\alpha,\theta) = -n - \frac{1}{n}\sum_{k=1}^{n}(2k-1)\left(\ln F(x_{(k)}) + \ln \bar{F}(x_{(n+1-k)})\right)$$

Where $\bar{F}(x) = 1 - F(x)$ and for OFrLL is given by

$$AD(\alpha,\theta) = -n - \frac{1}{n}\sum_{k=1}^{n}(2k-1)\left(\ln \exp(-x_k^{-\alpha\theta}) + \ln\left(1 - \exp(-x_{(n+1-k)}^{-\alpha\theta})\right)\right)$$

8.3. **Cramer von Mises (CVM) estimation.** The estimation method was put forward by [11]. In this method parameters are estimated by minimizing the function $C(\alpha,\theta)$ given as

$$c(\alpha,\theta) = \frac{1}{12n} + \sum_{k=1}^{n}\left(F(x_{(k)}) - \frac{2k-1}{2n}\right)^2$$

And for OFrLL distribution is

$$c(\alpha,\theta) = \frac{1}{12n} + \sum_{k=1}^{n}\left(\exp(-x_{(k)}^{-\alpha\theta}) - \frac{2k-1}{2n}\right)^2$$

8.4. **Maximum product of spacing (MPS) estimation.** This method was put forward as an alternative to maximum likelihood estimation by [12]. The MPS estimators can be obtained by maximizing the following functions,

$$M(\alpha,\theta) = \frac{1}{n+1}\sum_{k=1}^{n+1}\log(D_k)$$

$$M(\alpha,\theta) = \frac{1}{n+1}\sum_{k=1}^{n+1}\log\left(e^{-x_{(k)}^{-\alpha\theta}} - e^{-x_{(k-1)}^{-\alpha\theta}}\right)$$

8.5. **Ordinary and Weighted Least Square Estimation.** The least square and weighted least square estimation methods to estimate the unknown parameters are due to [13]. The estimates are obtained by minimizing the below given function with respect to parameters,

$$S(\alpha,\theta) = \sum_{k=1}^{n} n_k\left(F(x_{(k)}) - \frac{k}{n+1}\right)^2$$

For OFrLL distribution

$$S(\alpha,\theta) = \sum_{k=1}^{n} n_k\left(e^{-x_{(k)}^{-\alpha\theta}} - \frac{k}{n+1}\right)^2$$

Substituting $n_k = 1$, the least square estimators can be obtained and similarly substituting $n_k = \frac{(n+1)^2(n+2)}{k(n-k+1)}$, the weighted least square estimators can be obtained.

9. Simulation Study

This section is dedicated to the numerical study using R software. In this section we performed a simulation study to find out the best method of estimation using the methods of section 8, in terms of bias and mean square error. We generated random samples of sizes (n) 20, 50, 125, 225, 300, 500 from OFrLL distribution with 1000 replications and with two different combinations of initial values of the parameters. For comparing effectiveness of different estimation methods we used ranks which are summarized in tables.

Results: From the Tables 1-4 we conclude that

- With increase in sample size (n) the estimated parameters get closer to their initial values.
- For all the estimation techniques, the value of mean square error of α and θ reduces as n increases, satisfying the consistency criteria. Bias of α and θ also decreases for all estimation methods as n increases.
- From Table 2 and Table 3,noticing sum of ranks for all the estimation methods we conclude that the MPSE has the smallest sum so it performs best in terms of bias and MSE as n increases for each parameter combinations taken. MLE is the second best.
- The inference drawn from the overall simulation results is that with increasing sample size bias and MSE for all parameters reduces and finally will approach zero. This shows the precision of estimation methods.

Table 1. For $\alpha = 0.11$, $\theta = 1.22$, Average estimate (AVE), Bias and MSE

N	Estimate	Parameter	MLE	ADE	CVME	MPSE	OLSE	WLSE
20	AVE	$\hat{\alpha}$	0.11543	0.10472	0.11240	0.09928	0.10618	0.11100
		$\hat{\theta}$	1.21084	1.21120	1.20219	1.21174	1.20325	1.19006
	$Bias$	$\hat{\alpha}$	0.01589[1]	0.01754[3]	0.02307[5]	0.01652[2]	0.02418[6]	0.02253[4]
		$\hat{\theta}$	0.01161[2]	0.01341[3]	0.01826[5]	0.01046[1]	0.01743[4]	0.03070[6]
	MSE	$\hat{\alpha}$	0.00047[2]	0.00048[3]	0.00121[5]	0.00040[1]	0.00110[4]	0.00135[6]
		$\hat{\theta}$	0.00040[2]	0.00069[3]	0.00183[5]	0.00032[1]	0.00147[4]	0.00489[6]
	$\sum$ *Ranks*		7[2]	12[3]	20[4]	5[1]	18[3]	22[5]
50	AVE	$\hat{\alpha}$	0.11291	0.10849	0.11042	0.10431	0.10738	0.11129
		$\hat{\theta}$	1.21366	1.21461	1.21424	1.21634	1.21527	1.20491
	$Bias$	$\hat{\alpha}$	0.01023[2]	0.01154[3]	0.01369[5]	0.01013[1]	0.01397[6]	0.01183[4]
		$\hat{\theta}$	0.00810[5]	0.00747[3]	0.00779[4]	0.00405[1]	0.00719[2]	0.01587[6]
	MSE	$\hat{\alpha}$	0.00019[2]	0.00021[3]	0.00031[5]	0.00015[1]	0.00033[6]	0.00028[4]
		$\hat{\theta}$	0.00013[2.5]	0.00013[2.5]	0.00029[4]	0.00009[1]	0.00032[5]	0.00076[6]
	$\sum$ *Ranks*		11.5[2.5]	11.5[2.5]	18[4]	4[1]	19[5]	20[6]
125	AVE	$\hat{\alpha}$	0.11180	0.10952	0.11051	0.10694	0.10939	0.11056
		$\hat{\theta}$	1.21534	1.21561	1.21657	1.21922	1.21635	1.21250
	$Bias$	$\hat{\alpha}$	0.00641[2]	0.00733[4]	0.00883[5]	0.00624[1]	0.00923[6]	0.00722[3]
		$\hat{\theta}$	0.00603[5]	0.00533[4]	0.00502[2]	0.00080[1]	0.00532[3]	0.00944[6]
	MSE	$\hat{\alpha}$	0.00007[2]	0.00009[3.5]	0.00014[5.5]	0.00006[1]	0.00014[5.5]	0.00009[3.5]
		$\hat{\theta}$	0.00006[2.5]	0.00006[2.5]	0.00014[5]	0.00005[1]	0.00013[4]	0.00019[6]
	$\sum$ *Ranks*		11.5[2]	14[3]	17.5[4]	4[1]	18.5[5.5]	18.5[5.5]
225	AVE	$\hat{\alpha}$	0.11047	0.10976	.11048	0.10784	0.10972	0.11049
		$\hat{\theta}$	1.21754	1.21625	1.21568	1.21961	1.21614	1.21548
	$Bias$	$\hat{\alpha}$	0.00444[1]	0.00525[3]	0.00694[6]	0.00474[2]	0.00683[5]	0.00557[4]
		$\hat{\theta}$	0.00439[2]	0.00457[3]	0.00494[5]	0.00040[1]	0.00466[4]	0.00738[6]
	MSE	$\hat{\alpha}$	0.00003[1]	0.00005[3.5]	0.00008[5.5]	0.00004[2]	0.00008[5.5]	0.00005[3.5]
		$\hat{\theta}$	0.00004[2]	0.00005[3]	0.00008[5]	0.00003[1]	0.00006[4]	0.00011[6]
	$\sum$ *Ranks*		6[1.5]	12.5[3]	21.5[6]	6[1.5]	18.5[4]	19.5[5]
300	AVE	$\hat{\alpha}$	0.11061	0.11006	0.11041	0.10846	0.10993	0.11005
		$\hat{\theta}$	1.21801	1.21668	1.21589	1.21968	1.21549	1.21727
	$Bias$	$\hat{\alpha}$	0.00385^{1}	0.00463^{3}	0.00540^{5}	0.00407^{2}	0.00574^{6}	0.00474^{4}
		$\hat{\theta}$	0.00392[3]	0.00439[4]	0.00461[5]	0.00033[1]	0.00353[2]	0.00660[6]
	MSE	$\hat{\alpha}$	0.00002[1.5]	0.00003[3]	0.00005[5.5]	0.00002[1.5]	0.00005[5.5]	0.00004[4]
		$\hat{\theta}$	0.00003[2]	0.00004[3]	0.00005[4.5]	0.00002[1]	0.00005[4.5]	0.00007[6]
	$\sum$ *Ranks*		7.5[2]	13[3]	20[4.5]	5.5[1]	18[6]	20[4.5]
500	AVE	$\hat{\alpha}$	0.11037	0.11006	0.11015	0.10906	0.10986	0.11015
		$\hat{\theta}$	1.21840	1.21747	1.21704	1.21977	1.21714	1.21896
	$Bias$	$\hat{\alpha}$	0.00286[1]	0.00354[3]	0.00446[6]	0.00301[2]	0.00438[5]	0.00372[4]
		$\hat{\theta}$	0.00245[2]	0.00390[5]	0.00343[4]	0.00024[1]	0.00324[3]	0.00517[6]
	MSE	$\hat{\alpha}$	0.00001[1.5]	0.00002[3.5]	0.00003[5.5]	0.00001[1.5]	0.00003[5.5]	0.00002[3.5]
		$\hat{\theta}$	0.00002[2.5]	0.00002[2.5]	0.00003[4.5]	0.00001[1]	0.00003[4.5]	0.00004[6]
	$\sum$ *Ranks*		7[1]	14[3]	20[6]	5.5[2]	18[4]	19.5[5]

Table 2. Partial and overall ranks of estimation procedures

Parameter	*N*	*MLE*	*ADE*	*CVME*	*MPSE*	*OLSE*	*WLSE*
$\alpha = 0.11, \theta = 1.22$	20	2	3	4	1	3	5
	50	2.5	2.5	4	1	5	6
	125	2	3	4	1	5.5	5.5
	225	1.5	3	6	1.5	4	5
	300	2	3	4.5	1	6	4.5
	500	1	3	6	2	4	5
$\sum$ *Ranks*		11[2]	17.5[3]	28.5[5]	7.5[1]	27.5[4]	40[6]

Table 3. : For $\alpha = 1.61$, $\theta = 0.71$, Average estimate (AVE), Bias and MSE

N	Estimate	Parameter	MLE	ADE	CVME	MPSE	OLSE	WLSE
20	*AVE*	$\hat{\alpha}$	1.61965	1.59010	1.60741	1.58869	1.59318	1.52133
		$\hat{\theta}$	0.72435	0.67745	0.71161	0.64809	0.67131	0.73117
	Bias	$\hat{\alpha}$	0.04277[2]	0.04535[3]	0.04592[4]	0.03503[1]	0.04781[5]	0.09307[6]
		$\hat{\theta}$	0.08025[1]	0.09871[3]	0.12036[4]	0.09358[2]	0.12043[5]	0.12078[6]
	MSE	$\hat{\alpha}$	0.00451[5]	0.00372[4]	0.00339[2]	0.00168[1]	0.00365[3]	0.01787[6]
		$\hat{\theta}$	0.01180[1]	0.01604[3]	0.02492[5]	0.01263[2]	0.02345[4]	0.03609[6]
	$\sum$ *Ranks*		9[2]	13[3]	15[4]	6[1]	17[5]	24[6]
50	*AVE*	$\hat{\alpha}$	1.60389	1.59617	1.60661	1.59802	1.60199	1.56667
		$\hat{\theta}$	0.72343	0.70227	0.71229	0.67812	0.69608	0.72816
	Bias	$\hat{\alpha}$	0.03558[5]	0.02970[4]	0.02520[2]	0.02343[1]	0.02654[3]	0.04861[6]
		$\hat{\theta}$	0.04915[1]	0.05848[3]	0.07862[5]	0.05789[2]	0.07721[6]	0.06728[4]
	MSE	$\hat{\alpha}$	0.00309[5]	0.00168[4]	0.00095[2]	0.00079[1]	0.00107[3]	0.00523[6]
		$\hat{\theta}$	0.00413[1]	0.00580[3]	0.01021[6]	0.00512[2]	0.00936[4]	0.00982[5]
	$\sum$ *Ranks*		12[2]	14[3]	15[4]	6[1]	16[5]	21[6]
125	*AVE*	$\hat{\alpha}$	1.60016	1.59407	1.60176	1.60391	1.60025	1.59097
		$\hat{\theta}$	0.72122	0.71316	0.71450	0.69561	0.70984	0.71701
	Bias	$\hat{\alpha}$	0.02104[4]	0.02558[5]	0.01683[2]	0.01509[1]	0.01851[3]	0.02927[6]
		$\hat{\theta}$	0.03316[1]	0.03467[2]	0.04920[5]	0.03480[3]	0.05194[6]	0.03638[4]
	MSE	$\hat{\alpha}$	0.00122[4]	0.00165[5]	0.00048[2]	0.00034[1]	0.00054[3]	0.00221[6]
		$\hat{\theta}$	0.00191[2]	0.00212[3]	0.00429[5]	0.00182[1]	0.00442[6]	0.00244[4]
	$\sum$ *Ranks*		11[2]	15[4]	14[3]	6[1]	18[5]	20[6]
225	*AVE*	$\hat{\alpha}$	1.60613	1.59785	1.60473	1.60464	1.60249	1.59962
		$\hat{\theta}$	0.71601	0.71316	0.71243	0.69767	0.70775	0.71459
	Bias	$\hat{\alpha}$	0.01148[2]	0.02558[6]	0.01628[3]	0.01101[1]	0.01734[4]	0.02219[5]
		$\hat{\theta}$	0.02465[1]	0.03467[4]	0.03556[5]	0.02512[2]	0.03639[6]	0.02643[3]
	MSE	$\hat{\alpha}$	0.00041[2]	0.00165[6]	0.00048[3]	0.00019[1]	0.00054[4]	0.00145[5]
		$\hat{\theta}$	0.00105[2]	0.00212[4]	0.00222[5]	0.00100[1]	0.00233[6]	0.00115[3]
	$\sum$ *Ranks*		7[2]	20[5.5]	16[3.5]	5[1]	20[5.5]	16[3.5]
300	*AVE*	$\hat{\alpha}$	1.60727	1.59836	1.60472	1.60628	1.60412	1.60169
		$\hat{\theta}$	0.71298	0.71090	0.71205	0.70150	0.71124	0.71208
	Bias	$\hat{\alpha}$	0.00883[1]	0.01756[5]	0.01516[3]	0.00929[2]	0.01572[4]	0.01842[6]
		$\hat{\theta}$	0.02089[1]	0.02414[4]	0.03043[5]	0.02114[2]	0.03097[6]	0.02231[3]
	MSE	$\hat{\alpha}$	0.00024[2]	0.00098[5]	0.00048[3]	0.00014[1]	0.00053[4]	0.00111[6]
		$\hat{\theta}$	0.00074[2]	0.00103[4]	0.00162[5]	0.00071[1]	0.00167[6]	0.00075[3]
	$\sum$ *Ranks*		6[1.5]	18[4.5]	16[3]	6[1.5]	20[6]	18[4.5]
500	*AVE*	$\hat{\alpha}$	1.60909	1.60536	1.60622	1.60740	1.60527	1.60614
		$\hat{\theta}$	0.71199	0.71023	0.71080	0.70408	0.70954	0.71188
	Bias	$\hat{\alpha}$	0.00601[1]	0.00990[3]	0.01217[5]	0.00710[2]	0.01262[6]	0.01153[4]
		$\hat{\theta}$	0.01657[2]	0.01915[4]	0.02254[6]	0.01610[1]	0.02232[5]	0.01843[3]
	MSE	$\hat{\alpha}$	0.00007[1]	0.00034[3]	0.00044[5]	0.00008[2]	0.00048[6]	0.00039[4]
		$\hat{\theta}$	0.00042[2]	0.00063[4]	0.00079[6]	0.00041[1]	0.00078[5]	0.00050[3]
	$\sum$ *Ranks*		6[1.5]	14[3.5]	22[5.5]	6[1.5]	22[5.5]	14[3.5]

Table 4. Partial and overall ranks of estimation procedures

Parameter	*N*	*MLE*	*ADE*	*CVME*	*MPSE*	*OLSE*	*WLSE*
$\alpha = 1.6, \theta = 0.71$	20	2	3	4	1	5	6
	50	2	3	4	1	5	6
	125	2	4	3	1	5	6
	225	2	5.5	3.5	1	5.5	3.5
	300	1.5	4.5	3	1.5	6	4.5
	500	1.5	3.5	5.5	1.5	5.5	3.5
$\sum$ *Ranks*		$11^{[2]}$	$23.5^{[4]}$	$23^{[3]}$	$7^{[1]}$	$32^{[6]}$	$29.5^{[5]}$

10. Applications

This section expounds the real life applications of OFrLL distribution. To illustrate the significance, flexibility and potentiality of OFrLL distribution, we introduce two real data sets.

Application 1: The data set 1 shows foreign direct investment, net inflows (% of GDP) in India from year 1995 to 2017 and is given below and was obtained from world bank [14] https://data.worldbank.org/indicator/BX.KLT.DINV.WD.GD.ZS (retrieved on August 8, 2024).

0.59498633, 0.617479352, 0.86020896, 0.625286204, 0.472644191, 0.765211694, 1.05638021, 1.011569465, 0.605887857, 0.765596855, 0.886098378, 2.130168425, 2.073394047, 3.620523235, 2.651590332, 1.635034094, 2.002063463, 1.31293453, 1.516276467, 1.69565959, 2.092115214, 1.937364122, 1.5073158338.

Application 2: The data set 2 shows CO2 emissions (metric tons per capita) in India from year 1994 to 2020 and is given below and was obtained from world bank [15] https://data.worldbank.org/indicator/ EN.ATM.CO2E.PC (retrieved on August 8, 2024).

0.725622135, 0.76518964, 0.787231756, 0.817360076, 0.818720963, 0.866242339, 0.885077949, 0.883746998, 0.897242649, 0.905456602, 0.955470157, 0.984261473, 1.036533922, 1.123599482, 1.180361247, 1.278873603, 1.338033835, 1.396878498, 1.498204123, 1.5276744, 1.642465277, 1.631323487, 1.639914019, 1.704926721, 1.795595299, 1.752534366, 1.576093232.

The potential of the proposed model is assessed by comparing its performance with other models, namely Alpha power log-logistic distribution (APLL) [4], Exponentiated Log-Logistic distribution (ELL) [16] and Log-Logistic distribution (LL) using the following goodness of fit measures(GoF) : Akaike Information Criterion(AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion(BIC) and Hannan Quinn Information Criterion (HQIC).Also, we present the fitness of the proposed model to the real data sets using the Kolmogorov-Smirnov (KS) statistics with its P-value. The MLEs of the parameters of the model and the GoF metrics are given in Table 5 and Table 6. The best match for the data set may be the model with the least numerical measures of the aforementioned GoF metrics. Additionally, the parameters of model are estimated using estimation methods discussed in section 8 and the estimates are presented in Table 7 and Table 8. Also the relative histograms for two data sets are presented in the Figure 3 and Figure 4.

Table 5. The MLEs and goodness of fit metrics for foreign direct investment data set

$Model$	$MLEs$	AIC	BIC	$CAIC$	$HQIC$	KS	$P-value$
$OFLL$	$\hat{\alpha}=0.99370, \hat{\theta}=1.94207$	52.46436	54.73535	53.06436	53.03551	0.13321	0.76078
ELL	$\hat{\alpha}=1.44610, \hat{\beta}=2.69357$	52.51056	54.78155	53.11055	53.08171	0.13950	0.71093
$APLL$	$\hat{\alpha}=3.02430, \hat{\beta}=2.91994$	52.90841	55.17940	53.50841	53.47956	0.14021	0.70514
LL	$\hat{\theta}=2.85479$	53.16846	54.80365	53.35863	53.45373	0.24166	0.11465

Table 6. The MLEs and goodness of fit metrics for CO_2 emission data set

$Model$	$MLEs$	AIC	BIC	$CAIC$	$HQIC$	KS	$P-value$
$OFLL$	$\hat{\alpha}=1.88178, \hat{\theta}=2.05546$	22.88301	25.47469	23.38301	42.53667	0.14443	0.57679
ELL	$\hat{\alpha}=1.60100, \hat{\beta}=4.82942$	24.93937	27.53104	25.43938	25.71000	0.15584	0.48088
$APLL$	$\hat{\alpha}=3.87650, \hat{\beta}=5.20646$	26.05034	28.64201	26.55034	26.82098	0.14888	0.53858
LL	$\hat{\theta}=5.08799$	27.97455	29.27039	28.13455	28.35987	0.22222	0.11871

Table 7. Estimates and GoF metrics for foreign direct investment data set

	$\hat{\alpha}$	$\hat{\theta}$	A^*	W^*	KS	$P-value$
MLE	0.99370	1.94207	0.59737	0.09880	0.13321	0.76078
ADE	1.39668	1.28582	0.58376	0.09671	0.13980	0.70846
$CVME$	1.39440	1.28909	0.58919	0.09673	0.13956	0.71041
$MPSE$	1.25198	1.37716	0.57669	0.09562	0.15080	0.61891
$OLSE$	1.36471	1.25451	0.57551	0.09544	0.15270	0.60344
$WLSE$	1.13609	1.62555	0.58886	0.09751	0.13226	0.76813

Table 8. Estimates and GoF metrics for CO_2emissions data set

	$\hat{\alpha}$	$\hat{\theta}$	A^*	W^*	KS	$P-value$
MLE	1.88178	2.05554	0.88467	0.14002	0.14443	0.57679
ADE	1.83484	1.82105	0.89162	0.14229	0.13190	0.68688
$CVME$	1.73971	1.82033	0.89425	0.14272	0.14501	0.57180
$MPSE$	1.54773	2.27261	0.88913	0.14187	0.12820	0.71923
$OLSE$	1.72408	1.76142	0.89631	0.14305	0.15553	0.48336
$WLSE$	3.45798	1.00401	0.88975	0.14198	0.12665	0.73268

Data application results: On the basis of results from Table 5 to Table 8, we found that the OFrLL distribution has the smallest AIC, BIC, CAIC and HQIC values than other fitted models. Also, it is shown that the OFrLL distribution outperforms other fitted models because it has larger p-value and the smallest K-S distance. Thus the OFrLL distribution fits both data sets better than other competing models. So, it is the most suitable model for the two datasets and emerges as a strong contender for modeling distributions.

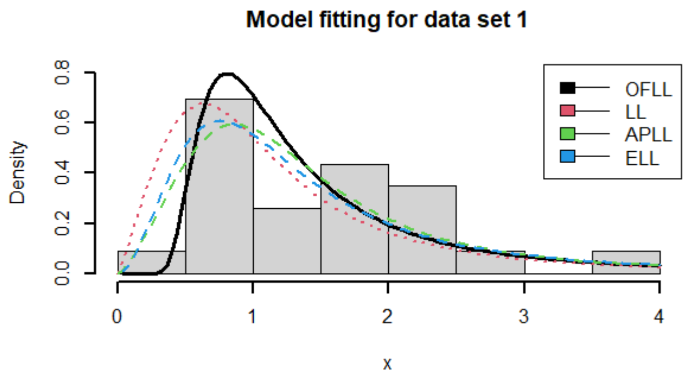

Figure 3. Relative histogram and density functions for data set 1

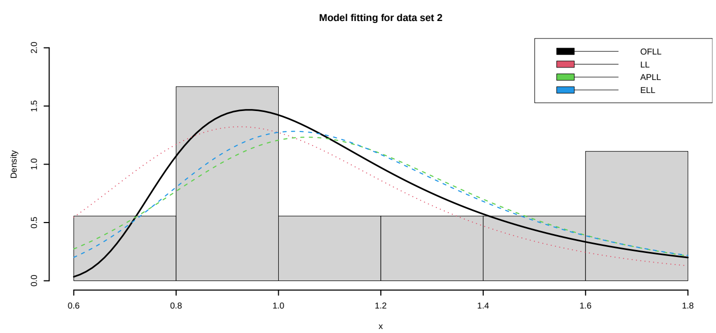

Figure 4. Relative histogram and density functions for data set 2

11. Conclusions

Generalization of distributions has proved useful in the statistical literature. In the present study, we propose a novel two parameter Log-logistic distribution namely Odd Fréchet Log-Logistic (OFrLL) distribution. The PDF, CDF and hazard function including their graphical illustration are provided. It's mathematical properties and reliability analysis have been comprehensively discussed. The unknown parameters of the proposed distribution are estimated using different estimation procedures. A simulation study with 1000 iterations is conducted using R-software for evaluating and analyzing performance of different estimation procedures and it is found that with increase in sample size, the estimated biases and MSEs of the parameters decreases under all the estimation procedures indicating precision of estimation methods. Also, it is observed that MPSE estimation procedure performs better than other estimation procedures signifying the effectiveness of the MPSE procedure. The superiority and efficiency of the proposed model over some competitive models is analysed using two real world data sets. From the comparison, we conclude that the proposed model performs better than competitive models. The proposed model exhibits variety of shapes of the PDF and hazard plots, which increases its flexibility in modeling different types of data. We hope OFrLL distribution draws attention towards wider applications in diverse fields.

Competing interests: The authors declare that there is no conflict of interest regarding the publication of this paper.

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