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Zero Modification of Hyperbolic Sine Distribution: Properties, Asymptotic Analysis and Applications

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Abstract. This paper introduces the zero-modified hyperbolic sine (ZMHS) distribution, a new two-parameter discrete probability model. The ZMHS distribution is a more general and flexible extension of the original hyperbolic sine distribution, which contains hyperbolic sine, zero-truncated hyperbolic sine, zero-inflated hyperbolic sine, and zero-deflated hyperbolic sine as special sub-models. Key probabilistic properties of the ZMHS are derived. The proposed distribution offers greater flexibility than the hyperbolic sine distribution, as evidenced by its index of dispersion. Maximum likelihood estimation is proposed for inferring the parameters of the ZMHS distribution, and the asymptotic normality of the resulting estimators is established. Monte Carlo simulations are conducted to evaluate the finite-sample performance of the estimators, and the results confirm that the maximum likelihood estimators are consistent, with average estimates approaching true parameter values as sample size increases. Applications to two real datasets, along with comparative analyses involving existing count data models, demonstrates the flexibility of the ZMHS distribution and its potential as an alternative modeling approach.

1. Introduction

Probability distributions are fundamental tools for modeling real-world phenomena that involve uncertainty. Examples include tracking the spread of diseases through case counts or analyzing the frequency of insurance claims among policyholders. Such events are often modeled using non-negative discrete probability distributions. When developing a new distribution, several characteristics must be considered, with flexibility being one of the most critical. In discrete distributions, flexibility refers to the ability to capture a variety of dispersal behaviors in count data, commonly categorized as underdispersion, equidispersion, and overdispersion. These behaviors are measured using the index of dispersion (IOD), defined as the ratio of variance to

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the mean: an IOD less than one indicates underdispersion, equal to one indicates equidispersion, and greater than one indicates overdispersion.

Classical discrete distributions such as the Poisson, Geometric, and Negative Binomial are limited in this regard, as each can model only a specific dispersal behavior. For example, the Poisson distribution generates only equidispersed data, where the variance equals the mean. Over the past two decades, considerable research has focused on developing flexible discrete distributions capable of modeling count data with diverse dispersion properties. Some attempts, such as in [1–7], however, lack sufficient flexibility. For example, the Poisson Epanechnikov-exponential distribution by [5] has an index of dispersion (IOD) greater than one and is therefore applicable only to overdispersed data. However, other studies have introduced discrete probability distributions that offer greater flexibility. For example, the study in [8] introduced the two-parameter Conway-Maxwell Poisson (CMP) distribution, a generalization of the Poisson distribution; [9] proposed another two-parameter flexible discrete distribution; [10] presented the two-parameter partial-geometric (PG) distribution, an extension of the geometric distribution; and [11] presented the three-parameter discrete hyperbolic mixture distribution. Despite their flexibility, the distributions in [8, 9] present computational challenges due to complex probability mass functions.

In addition to dispersion, real-world count data often exhibit excess zeros (zero-inflation) or fewer zeros than expected (zero-deflation), a feature not adequately addressed by classical distributions. Motivated by these limitations, this paper presents a novel two-parameter discrete distribution called the zero-modified hyperbolic sine distribution, designed to simultaneously accommodate zero-inflation and zero-deflation while providing flexible dispersion behavior. The zero-modified hyperbolic sine distribution is inspired by existing zero-modification models, such as zero-modified Poisson [12], zero-modified geometric [13], zero-modified binomial, and zero-modified negative binomial distributions [14]. It explicitly accounts for zero-modification, allowing the probabilities of zeros to be adjusted, as zero-inflated Poisson model handles excess zeros. By enhancing the flexibility of the original hyperbolic sine distribution in [11], the zero-modified hyperbolic sine distribution provides a mathematically tractable probability mass function that facilitates practical computation.

The primary objective of this study is to present the zero-modified hyperbolic sine distribution as a flexible alternative for modeling discrete count data, particularly in scenarios where both zero-modification and variable dispersion are presence and critical. This represents a key research gap and distinguishes the present study from the work of Kafi et al. [11], who introduced and employed the hyperbolic sine distribution as part of a mixture model to achieve flexibility in dispersion behavior. Using the hyperbolic sine distribution as its baseline, the present study offers less complex model with fewer parameters than the model proposed in [11]. This study also examines properties that were not discussed in [11], such as the mode of the hyperbolic sine distribution and the asymptotic normality of the parameter estimators, which provide both practical and theoretical advantages in statistics and probability.

2. Definition of zero-modified hyperbolic sine distribution

The idea of constructing the proposed distribution is influenced by the work of [12–14], where the main idea is to modify (inflate or deflate) the probability of zero. But first we need to remind the mass function of the hyperbolic sine distribution which is important in constructing the proposed model.

A random variable X has the hyperbolic sine distribution with parameter $\lambda > 0$, denoted by $\mathrm{HS}(\lambda)$, if its mass function is [11]

$$\mathbb{P}(X = k) = \frac{\lambda^{2k+1}}{(\sinh\lambda)(2k+1)!}, \ k = 0, 1, 2, 3, \dots$$

The mass function of the zero-truncated hyperbolic sine distribution with parameter $\lambda > 0$ is

$$\mathbb{P}(X = k|X > 0) = \frac{\mathbb{P}(X = k)}{1 - \mathbb{P}(X = 0)} = \frac{\lambda^{2k+1}}{(\sinh(\lambda) - \lambda)(2k+1)!}, \ k = 1, 2, 3, \dots$$

According to [14], a non-negative integer random variable Y follows discrete distributions with modified probability of zero, if its mass function is

$$\mathbb{P}(Y = k) = \begin{cases} p_0^M, & k = 0 \\ (1 - p_0^M)\, p_k^T, & k = 1, 2, 3, \dots \end{cases}$$

where p_0^M refers to a probability that the new distribution takes value zero, and p_k^T refers to mass function of zero-truncated distribution. Note that for extreme case $p_0^M = 1$, the corresponding distribution is degenerate, assigning probability 1 to zero. This is not interesting case from a probabilistic point of view and is therefore excluded from consideration.

Therefore, we define our new distribution as follows.

Definition 1. *A random variable Y has the zero-modified hyperbolic sine distribution with parameters $\lambda > 0$ and $0 \le \beta < 1$, denoted by $\mathrm{ZMHS}(\lambda, \beta)$, if its mass function is*

$$\mathbb{P}(Y = k) = \begin{cases} \beta, & k = 0 \\ \dfrac{(1 - \beta)\, \lambda^{2k+1}}{(\sinh(\lambda) - \lambda)\, (2k+1)!}, & k = 1, 2, 3, \dots \end{cases}$$

There are the following interesting sub-models of the zero-modified hyperbolic sine distribution:

1. The zero-truncated hyperbolic sine (ZTHS) distribution is the zero-modified hyperbolic sine distribution with the smallest possible $\beta = 0$.

2. The hyperbolic sine distribution is the zero-modified hyperbolic sine distribution with $\beta = \frac{\lambda}{\sinh\lambda}$.

3. For $\frac{\lambda}{\sinh\lambda} < \beta < 1$, we inflate the probability of zero; that is, $\mathbb{P}(Y=0)$ is larger than for the corresponding hyperbolic sine distribution.

4. For $0 \leq \beta < \frac{\lambda}{\sinh\lambda}$, we deflate the probability of zero; that is, $\mathbb{P}(Y=0)$ is smaller than for the corresponding hyperbolic sine distribution.

The graphs of mass function given of the zero-modified hyperbolic sine distribution are presented in Figure 1. As we can see from the plots, the mass function of the zero-modified hyperbolic sine distribution can be unimodal or bimodal. In certain cases the additional mode happens at zero.

The cumulative distribution function of Y is

$$F(y) = \mathbb{P}(Y \leq y) = \begin{cases} 0, & y < 0 \\ \beta, & 0 \leq y < 1 \\ \beta + \displaystyle\sum_{k=1}^{\lfloor y \rfloor} \frac{(1-\beta)\lambda^{2k+1}}{(\sinh(\lambda)-\lambda)(2k+1)!}, & y \geq 1 \end{cases},$$

where $\lfloor y \rfloor$ is the floor function (the greatest integer less than or equal to y).

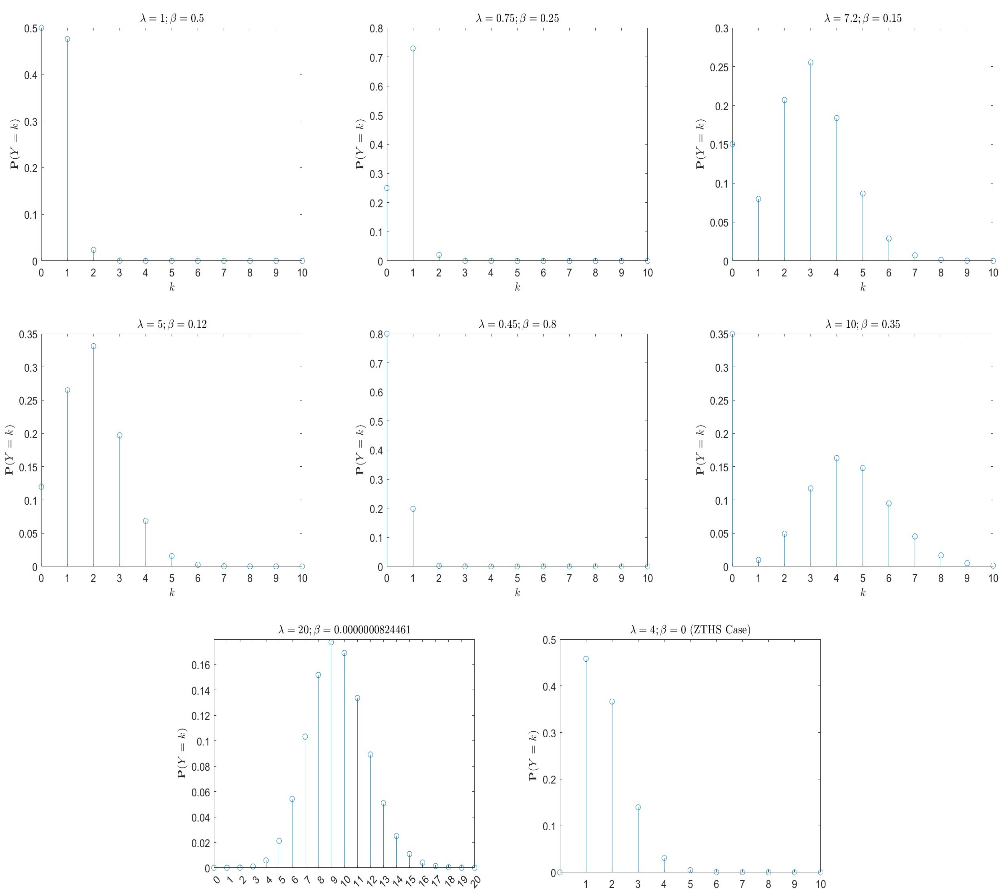

FIGURE 1. The mass function plots of the zero-modified hyperbolic sine distribution for different values of λ and β

3. Basic distributional properties of the zero-modified hyperbolic sine distribution

Some probabilistic properties of the zero-modified hyperbolic sine distributions, in particular the mean, variance, index of dispersion, quantiles, and moment generating, probability generating, and characteristic functions are provided in this section. In particular, index of dispersion can show the flexibility of the proposed distribution. Later in this section, the zero-modified hyperbolic sine distribution is shown to achieve greater flexibility than the original hyperbolic sine distribution. In this section, we assume that the random variable Y follows the zero-modified hyperbolic sine distribution; that is, $Y \sim \mathrm{ZMHS}(\lambda, \beta)$.

The generating functions are as follows.

Theorem 1. *The moment generating function is*

$$\psi(t) = \beta + \frac{(1-\beta)\, e^{-t/2}}{\sinh(\lambda) - \lambda} \left(\sinh\left(\lambda e^{t/2}\right) - \lambda e^{t/2}\right), \ t \in \mathbb{R}.$$

The probability generating function is

$$\tau(t) = \beta + \frac{1-\beta}{\sqrt{t}\,(\sinh(\lambda) - \lambda)} \left(\sinh(\lambda\sqrt{t}) - \lambda\sqrt{t}\right), \ t > 0.$$

The characteristic function is

$$\phi(t) = \beta + \frac{(1-\beta)\, e^{-it/2}}{\sinh(\lambda) - \lambda} \left(\sinh\left(\lambda e^{it/2}\right) - \lambda e^{it/2}\right), \ t \in \mathbb{R},$$

where $i = \sqrt{-1}$ is the imaginary unit.

Proof. The moment generating function of the zero-modified hyperbolic sine distribution can be obtained as follows:

$$\begin{aligned}
\psi(t) = \mathbb{E}(e^{tY}) &= \sum_{k=0}^{\infty} e^{tk}\mathbb{P}(Y=k) = e^0\mathbb{P}(Y=0) + \sum_{k=1}^{\infty} e^{tk}\mathbb{P}(Y=k) \\
&= \beta + \frac{1-\beta}{\sinh(\lambda)-\lambda}\sum_{k=1}^{\infty} e^{tk}\frac{\lambda^{2k+1}}{(2k+1)!} = \beta + \frac{(1-\beta)e^{-t/2}}{\sinh(\lambda)-\lambda}\sum_{k=1}^{\infty}\frac{(\lambda e^{t/2})^{2k+1}}{(2k+1)!} \\
&= \beta + \frac{(1-\beta)\, e^{-t/2}}{\sinh(\lambda)-\lambda}\left(\sinh\left(\lambda e^{t/2}\right) - \lambda e^{t/2}\right).
\end{aligned}$$

The probability generating function is

$$\begin{aligned}
\tau(t) = \mathbb{E}(t^{Y}) &= \sum_{k=0}^{\infty} t^{k}\mathbb{P}(Y=k) = t^0\mathbb{P}(Y=0) + \sum_{k=1}^{\infty} t^{k}\mathbb{P}(Y=k) \\
&= \beta + \frac{1-\beta}{\sinh(\lambda)-\lambda}\sum_{k=1}^{\infty}\frac{t^k\,\lambda^{2k+1}}{(2k+1)!} = \beta + \frac{1-\beta}{\sqrt{t}\,(\sinh(\lambda)-\lambda)}\sum_{k=1}^{\infty}\frac{\left(\lambda t^{1/2}\right)^{2k+1}}{(2k+1)!} \\
&= \beta + \frac{1-\beta}{\sqrt{t}\,(\sinh(\lambda)-\lambda)}\left(\sinh(\lambda\sqrt{t}) - \lambda\sqrt{t}\right).
\end{aligned}$$

The characteristic function can be derived in the similar way as the moment generating function, that is, $\phi(t) = \mathbb{E}\left(e^{itY}\right)$ where $i = \sqrt{-1}$ is the imaginary unit, and the result follows. □

According to [14], the n-th raw moment of a zero-modified random variable Y can be obtained as follows:

$$\mathbb{E}[Y^n] = (1 - \mathbb{P}(Y = 0))\,\mathbb{E}[X^n \mid X > 0], \tag{1}$$

where $\mathbb{E}[X^n \mid X > 0]$ is the n-th raw moment of the zero-truncation of a random variable X with a mass function $p_X(x)$ associated with Y. This finding can be used to verify the following theorem.

Theorem 2. *Let $Y \sim \mathrm{ZMHS}(\lambda, \beta)$ and $X \sim \mathrm{HS}(\lambda)$. The first two raw moments of Y are*

$$\mathbb{E}(Y) = (1-\beta)\,\frac{\lambda\cosh(\lambda) - \sinh(\lambda)}{2(\sinh(\lambda) - \lambda)},$$

$$\mathbb{E}(Y^2) = (1-\beta)\,\frac{(\lambda^2+1)\sinh(\lambda) - \lambda\cosh(\lambda)}{4(\sinh(\lambda) - \lambda)}.$$

The variance is

$$\mathrm{Var}(Y) = (1-\beta)\,\frac{(\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda}{4(\sinh\lambda - \lambda)} - \left((1-\beta)\,\frac{\lambda\cosh\lambda - \sinh\lambda}{2(\sinh\lambda - \lambda)}\right)^2.$$

Proof. Kafi et al. [11] established the first and second moments of the hyperbolic sine distributed random variable X, respectively, as follows:

$$\mathbb{E}(X) = \frac{\lambda}{2}\frac{\cosh\lambda}{\sinh\lambda} - \frac{1}{2},$$

$$\mathbb{E}(X^2) = -\frac{\lambda}{4}\frac{\cosh\lambda}{\sinh\lambda} + \frac{\lambda^2}{4} + \frac{1}{4}.$$

Hence, by using Equation (1), the expectation of the zero-modified hyperbolic sine distribution can be obtained as follows:

$$\begin{aligned}\mathbb{E}(Y) &= (1 - \mathbb{P}(Y = 0))\,\mathbb{E}(X \mid X > 0) = (1-\beta)\frac{\mathbb{E}(X)}{1 - \mathbb{P}(X = 0)}\\ &= (1-\beta)\frac{\frac{\lambda}{2}\frac{\cosh\lambda}{\sinh\lambda} - \frac{1}{2}}{1 - \frac{\lambda}{\sinh\lambda}} = (1-\beta)\,\frac{\lambda\cosh\lambda - \sinh\lambda}{2(\sinh\lambda - \lambda)}.\end{aligned}$$

The second raw moment can also be obtained using Equation (1) as follows:

$$\begin{aligned}\mathbb{E}(Y^2) &= (1 - \mathbb{P}(Y = 0))\,\mathbb{E}(X^2 \mid X > 0) = (1-\beta)\frac{\mathbb{E}(X^2)}{1 - \mathbb{P}(X = 0)}\\ &= (1-\beta)\frac{-\frac{\lambda}{4}\frac{\cosh\lambda}{\sinh\lambda} + \frac{\lambda^2}{4} + \frac{1}{4}}{1 - \frac{\lambda}{\sinh\lambda}} = (1-\beta)\,\frac{(\lambda^2+1)\sinh(\lambda) - \lambda\cosh(\lambda)}{4(\sinh(\lambda) - \lambda)}.\end{aligned}$$

For the variance, we calculate

$$\begin{aligned}\operatorname{Var}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}^2(Y) \\ &= (1-\beta)\,\frac{(\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda}{4(\sinh\lambda-\lambda)} - \left((1-\beta)\,\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}\right)^2.\end{aligned}$$

□

Theorem 3. *The index of dispersion (IOD) is*

$$\mathrm{IOD} = \frac{(\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda}{2(\lambda\cosh\lambda-\sinh\lambda)} - (1-\beta)\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}.$$

The dispersion behavior of the zero-modified hyperbolic sine distribution, as measured by its index of dispersion (IOD), is characterized as follows:

(1) *The distribution is equidispersed (that is,* $\mathrm{IOD}=1$*) if and only if*

$$\beta = 1 - \frac{(\sinh\lambda-\lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda-\sinh\lambda)^2}.$$

(2) *The distribution is underdispersed (that is,* $\mathrm{IOD}<1$*) if and only if*

$$0 \le \beta < 1 - \frac{(\sinh\lambda-\lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda-\sinh\lambda)^2}.$$

(3) *The distribution is overdispersed (that is,* $\mathrm{IOD}>1$*) if and only if*

$$1 - \frac{(\sinh\lambda-\lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda-\sinh\lambda)^2} < \beta < 1.$$

Proof. The IOD of the zero-modified hyperbolic sine distribution can be obtained as follows:

$$\begin{aligned}\mathrm{IOD} &= \frac{\operatorname{Var}(Y)}{\mathbb{E}(Y)} \\ &= \frac{(1-\beta)\,\frac{(\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda}{4(\sinh\lambda-\lambda)} - \left((1-\beta)\,\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}\right)^2}{(1-\beta)\,\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}} \\ &= \frac{(\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda}{2(\lambda\cosh\lambda-\sinh\lambda)} - (1-\beta)\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}.\end{aligned}$$

Case 1: Equidispersion occurs when $\mathrm{IOD}=1$; that is, if and only if

$$\frac{(1-\beta)\,\frac{(\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda}{4(\sinh\lambda-\lambda)} - \left((1-\beta)\,\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}\right)^2}{(1-\beta)\,\frac{\lambda\cosh\lambda-\sinh\lambda}{2(\sinh\lambda-\lambda)}} = 1.$$

Let $A = \lambda\cosh\lambda - \sinh\lambda$, $B = \sinh\lambda - \lambda$, and $C = (\lambda^2+1)\sinh\lambda - \lambda\cosh\lambda$. Then, the last expression can be written as follows.

$$\frac{C}{2A} - (1-\beta)\frac{A}{2B} = 1$$

$$
\begin{aligned}
\iff \quad & -(1-\beta)\frac{A}{2B} = 1 - \frac{C}{2A} \\
\iff \quad & -(1-\beta) = \left(1 - \frac{C}{2A}\right)\frac{2B}{A} \\
\iff \quad & \beta - 1 = \frac{2B}{A} - \frac{BC}{A^2} \\
\iff \quad & \beta = 1 - \frac{B(C-2A)}{A^2} \\
& = 1 - \frac{(\sinh\lambda - \lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda - \sinh\lambda)^2}
\end{aligned}
$$

Case 2: Suppose that IOD < 1. Then this holds if and only if

$$
\begin{aligned}
& \frac{C}{2A} - (1-\beta)\frac{A}{2B} < 1 \\
\iff \quad & -(1-\beta)\frac{A}{2B} < 1 - \frac{C}{2A} \\
\iff \quad & -(1-\beta) < \left(1 - \frac{C}{2A}\right)\frac{2B}{A} \\
\iff \quad & \beta - 1 < \frac{2B}{A} - \frac{BC}{A^2} \\
\iff \quad & \beta < 1 - \frac{B(C-2A)}{A^2} \\
& = 1 - \frac{(\sinh\lambda - \lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda - \sinh\lambda)^2}
\end{aligned}
$$

However, since $0 \leq \beta < 1$, we have

$$
0 \leq \beta < 1 - \frac{(\sinh\lambda - \lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda - \sinh\lambda)^2}.
$$

Case 3: Suppose that IOD > 1. Then this holds if and only if

$$
\begin{aligned}
& \frac{C}{2A} - (1-\beta)\frac{A}{2B} > 1 \\
\iff \quad & -(1-\beta)\frac{A}{2B} > 1 - \frac{C}{2A} \\
\iff \quad & -(1-\beta) > \left(1 - \frac{C}{2A}\right)\frac{2B}{A} \\
\iff \quad & \beta - 1 > \frac{2B}{A} - \frac{BC}{A^2} \\
\iff \quad & \beta > 1 - \frac{B(C-2A)}{A^2}
\end{aligned}
$$

$$= 1 - \frac{(\sinh\lambda - \lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda - \sinh\lambda)^2}$$

However, since $0 \leq \beta < 1$, we have

$$1 - \frac{(\sinh\lambda - \lambda)\big((\lambda^2+3)\sinh\lambda - 3\lambda\cosh\lambda\big)}{(\lambda\cosh\lambda - \sinh\lambda)^2} < \beta < 1.$$

$\square$

Theorem 3 implies that the zero-modified hyperbolic sine distribution can be a suitable option for modeling the overdispersed, underdispersed, and equidispersed count data. Therefore, by adding an extra parameter β, the zero-modified hyperbolic sine distribution is proven to have more flexibility than the hyperbolic sine distribution.

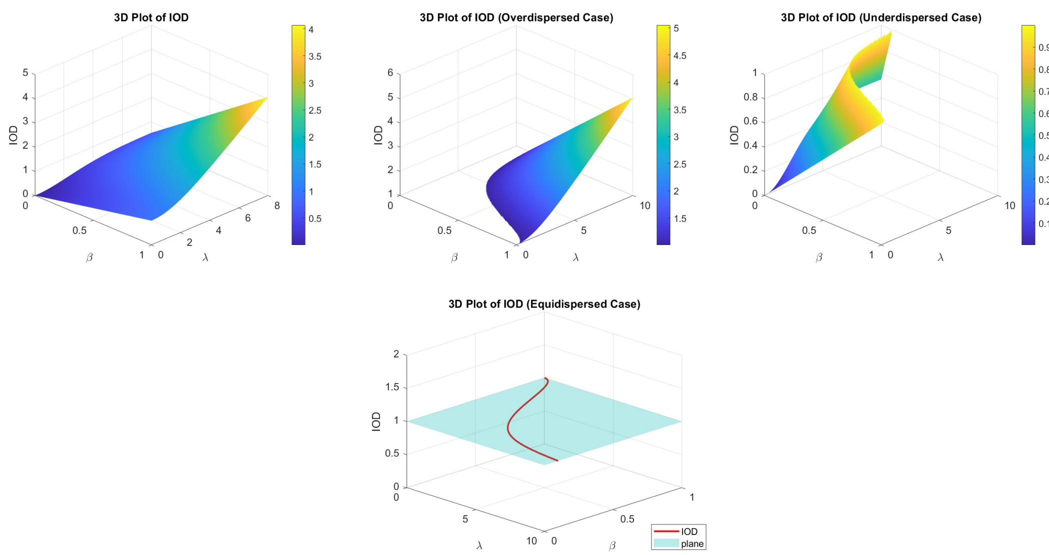

FIGURE 2. 3D plots of index of dispersion of the zero-modified hyperbolic sine distribution

In support of Theorem 3, the IOD 3D plots of the zero-modified hyperbolic sine for selected values of λ and β are presented in Figure 2, illustrating all possible dispersion behaviors of the zero-modified hyperbolic sine distribution.

Remark 4. The IOD of the hyperbolic sine distribution exceeds one, classifying it as overdispersed [11]. Introducing the parameter β allows the zero-modified hyperbolic sine distribution to accommodate underdispersion and equidispersion as well.

There is no closed-form for the quantiles of the zero-modified hyperbolic sine distribution, but we can establish the following result.

Theorem 5. *Let $0 \leq r < 1$ and $Y \sim \text{ZMHS}(\lambda, \beta)$. The r-quantile (or $100r$-th percentile) of Y is the non-negative integer y_r given as follows:*

$$y_r = \begin{cases} 0, & \text{if } r \leq \beta \\ y_r \in \mathbb{Z}^+ \ni c \leq \sum_{k=1}^{y_r} \frac{\lambda^{2k+1}}{(2k+1)!} \leq \frac{\lambda^{2y_r+1}}{(2y_r+1)!} + c & \text{if } r > \beta \end{cases},$$

where $c = (r-\beta)\,\frac{\sinh\lambda - \lambda}{1-\beta}$.

Proof. The r-quantiles of an integer-valued discrete distribution are the non-negative integers y_r satisfying

$$\mathbb{P}(Y < y_r) \leq r \leq \mathbb{P}(Y \leq y_r).$$

The case $0 \leq r \leq \beta$ is obvious that $y_r = 0$ is the only r-quantile satisfying this condition.

$$\mathbb{P}(Y < 0) = 0 \leq r \leq \beta = \mathbb{P}(Y \leq 0).$$

For the case $\beta < r < 1$, we note that $\mathbb{P}(Y \leq y_r) \geq r$ can be written as

$$\beta + \frac{1-\beta}{\sinh\lambda - \lambda} \sum_{k=1}^{y_r} \frac{\lambda^{2k+1}}{(2k+1)!} \geq r \iff \sum_{k=1}^{y_r} \frac{\lambda^{2k+1}}{(2k+1)!} \geq (r-\beta)\,\frac{\sinh\lambda - \lambda}{1-\beta}.$$

Also, $\mathbb{P}(Y < y_r) \leq r$ can be written as

$$\beta + \frac{1-\beta}{\sinh\lambda - \lambda} \sum_{k=1}^{y_r-1} \frac{\lambda^{2k+1}}{(2k+1)!} \leq r \iff \sum_{k=1}^{y_r-1} \frac{\lambda^{2k+1}}{(2k+1)!} \leq (r-\beta)\,\frac{\sinh\lambda - \lambda}{1-\beta}.$$

Therefore, in this case, we obtain that y_r is any non-negative integer satisfying

$$(r-\beta)\,\frac{\sinh\lambda - \lambda}{1-\beta} \leq \sum_{k=1}^{y_r} \frac{\lambda^{2k+1}}{(2k+1)!} \leq \frac{\lambda^{2y_r+1}}{(2y_r+1)!} + (r-\beta)\,\frac{\sinh\lambda - \lambda}{1-\beta}.$$

□

The zero-modified hyperbolic sine distribution can be unimodal and bimodal, as it is established in the following theorem.

Theorem 6. *The mode of the zero-modified hyperbolic sine distribution is:*

$$\text{mode} = \begin{cases} 0, & \text{if } \beta > \frac{1-\beta}{\sinh\lambda - \lambda} \max_{k\geq 1} \left\{ \frac{\lambda^{2k+1}}{(2k+1)!} \right\} \text{ and } \lambda^2 < 20 \\ \left\lfloor \frac{-1+\sqrt{4\lambda^2+1}}{4} \right\rfloor, & \text{if } \beta = \frac{\lambda}{\sinh\lambda} \\ \left\{ 0, \left\lfloor \frac{-1+\sqrt{4\lambda^2+1}}{4} \right\rfloor \right\}, & \text{if } \beta > \frac{(1-\beta)\lambda^3}{6(\sinh\lambda - \lambda)} \text{ and } \lambda^2 \geq 20 \end{cases},$$

where $\lfloor\cdot\rfloor$ denotes the floor function.

Proof. Remind that a mode of discrete probability distibution is any local maxima for a mass function. The proof is given case by case as follows:

Case 1. Given $\beta > \frac{1-\beta}{\sinh\lambda-\lambda}\max_{k\geq 1}\left\{\frac{\lambda^{2k+1}}{(2k+1)!}\right\}$, we obtain $\mathbb{P}(Y=0) > \mathbb{P}(Y=k)$ for all $k \geq 1$ and hence zero is clearly a mode. Note that the ratio

$$\frac{\mathbb{P}(Y=k+1)}{\mathbb{P}(Y=k)} = \frac{\lambda^2}{(2k+3)(2k+2)} \leq \frac{\lambda^2}{5\cdot 4} = \frac{\lambda^2}{20}, \qquad k \geq 1.$$

Hence, if $\lambda^2 < 20$, then the ratio $\frac{\mathbb{P}(Y=k+1)}{\mathbb{P}(Y=k)} < 1$ for all $k \geq 1$; that is, $\mathbb{P}(Y=k)$ is decreasing over $k \geq 1$. Therefore, in this case, the ZMHS distribution is unimodal with the mode at zero.

Case 2. If $\beta = \frac{\lambda}{\sinh\lambda}$, we have the case of the hyperbolic sine distribution. Consider the ratio $\frac{\mathbb{P}(Y=k+1)}{\mathbb{P}(Y=k)}$ for $k \geq 0$, that is,

$$\frac{\mathbb{P}(Y=k+1)}{\mathbb{P}(Y=k)} = \frac{\lambda^2}{(2k+3)(2k+2)}.$$

Note that the mass function is increasing as long as

$$\frac{\lambda^2}{(2k+3)(2k+2)} > 1,$$

that is, $\lambda^2 > (2k+3)(2k+2)$. Since the mode must be a non-negative integer, the mode k^* is the integer satisfying

$$(2k^*)(2k^*+1) \leq \lambda^2 \leq (2k^*+3)(2k^*+2).$$

This guarantees that

$$\mathbb{P}(Y=k^*) \geq \mathbb{P}(Y=k^*-1) \quad \text{and} \quad \mathbb{P}(Y=k^*) \geq \mathbb{P}(Y=k^*+1).$$

The largest k for which the mass function is still increasing is the solution of

$$(2k^*)(2k^*+1) = \lambda^2.$$

Solving this quadratic equation yields

$$k_{1,2} = \frac{-2 \pm \sqrt{4-4(4)(-\lambda^2)}}{2(4)} = \frac{-1 \pm \sqrt{4\lambda^2+1}}{4}.$$

Since k must be a non-negative integer, we take the positive root, and hence

$$k^* = \left\lfloor \frac{-1+\sqrt{4\lambda^2+1}}{4} \right\rfloor.$$

Case 3. Given $\beta > \frac{(1-\beta)\lambda^3}{6(\sinh\lambda-\lambda)}$, we obtain $\mathbb{P}(Y=0) > \mathbb{P}(Y=1)$, that is, zero is a mode. For the mode to equal one, we must have

$$\frac{\mathbb{P}(Y=1)}{\mathbb{P}(Y=2)} = \frac{20}{\lambda^2} > 1,$$

which is impossible since $\lambda^2 \geq 20$. Therefore, mode $\neq 1$. For $k \geq 2$, the regions of increase and decrease of the mass function $\mathbb{P}(Y=k)$ are the same as those of the hyperbolic sine distribution (Case 2), and we consequently obtain the remaining mode. □

4. Parameter estimation

Let $\boldsymbol{Y} = \{Y_1, Y_2, \dots, Y_n\}$ be an independent and identically distributed random sample of size n from the zero-modified hyperbolic sine distribution ZMHS(λ, β). Consequently, each Y_i can take the value $k \in \mathbb{Z}_{\geq 0}$. We consider the method of maximum likelihood.

4.1. **Maximum likelihood estimation.** For every $k = 0, 1, 2, \dots$, let random variable N_k be the number of observations from the random sample $\boldsymbol{Y} = \{Y_1, Y_2, \dots, Y_n\}$ that are equal to k. Note that $\sum_{k=0}^{\infty} N_k = n$ and $\sum_{k=1}^{\infty} kN_k = n\overline{Y}$, where $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. As we will see from the following statistic N_0 that represent the number of zero observations plays the main role. This random variable has a binomial distribution with parameters n and β; that is, $\mathbb{E}(N_0) = n\beta$.

The likelihood function can be written as

$$\begin{aligned} L(\lambda, \beta; \boldsymbol{Y}) &= \prod_{k=0}^{\infty} \mathbb{P}(Y = k)^{N_k} = \mathbb{P}(Y = 0)^{N_0} \prod_{k=1}^{\infty} \mathbb{P}(Y = k)^{N_k} \\ &= \beta^{N_0} \prod_{k=1}^{\infty} \left[\frac{1-\beta}{\sinh\lambda - \lambda} \, \frac{\lambda^{2k+1}}{(2k+1)!} \right]^{N_k}. \end{aligned}$$

The corresponding log-likelihood function is:

$$\begin{aligned} \ell(\lambda, \beta; \boldsymbol{Y}) = N_0 \ln\beta + (n - N_0) \ln\!\left(\frac{1-\beta}{\sinh\lambda - \lambda} \right) \\ + (2n\overline{Y} + n - N_0) \ln\lambda - \sum_{k=1}^{\infty} N_k \ln\!\big((2k+1)!\big). \end{aligned}$$

The maximum likelihood estimators for λ and β, denoted by $\hat{\lambda}$ and $\hat{\beta}$, are the values that maximize the log-likelihood function. They can be obtained by solving the score equations

$$\frac{\partial \ell}{\partial \lambda} = -(n - N_0) \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} + \frac{2n\overline{Y} + n - N_0}{\lambda} = 0,$$

$$\frac{\partial \ell}{\partial \beta} = \frac{N_0}{\beta} - \frac{n - N_0}{1-\beta} = 0.$$

By solving the second score equation, we obtain

$$\frac{N_0}{\beta} = \frac{n - N_0}{1-\beta},$$

or equivalently,

$$\hat{\beta} = \frac{N_0}{n}.$$

Note that the first score equation can be simplified as

$$\begin{aligned} (n - N_0) \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} &= \frac{2n\overline{Y} + n - N_0}{\lambda} \\ \iff \quad \lambda \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} &= \frac{2n\overline{Y}}{n - N_0} + 1. \end{aligned}$$

There is no closed-form solution for $\hat{\lambda}$; it must be obtained numerically, for instance by using the Newton–Raphson method. Moreover, we can show the uniqueness of the MLE $\hat{\lambda}$ as follows.

Recall the score equation for λ:

$$\lambda \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} = \frac{2n\overline{Y}}{n - N_0} + 1.$$

Define

$$a = \frac{2n\overline{Y}}{n - N_0} + 1 = 2\,\frac{\sum_{k=1}^{\infty} kN_k}{\sum_{k=1}^{\infty} N_k} + 1,$$

and define the function

$$g(\lambda) = \lambda \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda}, \qquad \lambda > 0.$$

We must show that the equation $g(\lambda) = a$ has a unique solution.

Let us analyze

$$a = 2\,\frac{\sum_{k=1}^{\infty} kN_k}{\sum_{k=1}^{\infty} N_k} + 1.$$

For all $k \geq 1$, we obtain

$$a \geq 2\,\frac{\sum_{k=1}^{\infty} N_k}{\sum_{k=1}^{\infty} N_k} + 1 = 3.$$

Thus, $a \geq 3$, with equality only if $N_k = 0$ for all $k > 1$.

It is clear that $g(\lambda)$ is continuous and strictly increasing on $(0, \infty)$. Moreover,

$$\begin{aligned}
\lim_{\lambda \to 0^+} g(\lambda) &= \lim_{\lambda \to 0^+} \left[\lambda \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda}\right] \\
&= \lim_{\lambda \to 0^+} \frac{\cosh\lambda - 1 + \lambda\sinh\lambda}{\cosh\lambda - 1} \\
&= \lim_{\lambda \to 0^+} \frac{\sinh\lambda + (\sinh\lambda + \lambda\cosh\lambda)}{\sinh\lambda} \\
&= 2 + \lim_{\lambda \to 0^+} \frac{\lambda\cosh\lambda}{\sinh\lambda} = 2 + 1 = 3.
\end{aligned}$$

Furthermore,

$$\lim_{\lambda \to \infty} g(\lambda) = \lim_{\lambda \to \infty} \lambda \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} = \infty.$$

Graphically, consider the horizontal line $y_1 = a$ and the curve $y_2 = g(\lambda)$. Since $g(\lambda)$ is strictly increasing on $(0, \infty)$ and bounded below by 3, the curve $y = g(\lambda)$ can intersect the horizontal line $y = a$ at most once whenever $a \geq 3$. Hence, there is exactly one intersection between the line $y_1 = a$ and the curve $y_2 = g(\lambda)$, and therefore $\hat{\lambda}$ has a unique solution (see Figure 3). For instance, the intersection point between $y_1 = 4$ (blue dashed line) and $y_2 = g(\lambda)$ (black solid curve), denoted by (λ, y), is $(3.212,\ 4)$.

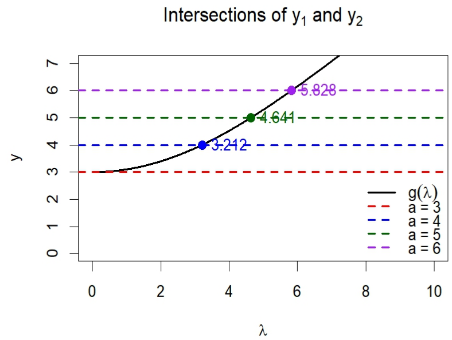

FIGURE 3. Intersections between the line $y_1 = a$ and the curve $y_2 = g(\lambda)$ for different values of a

Since the maximum likelihood estimator $\hat{\lambda}$ does not admit a closed-form expression, it is fully rigorous to define it as follows:

$$\hat{\lambda} = \arg\max_{\lambda>0} \ell(\lambda, \hat{\beta}),$$

where $\hat{\beta} = \frac{N_0}{n}$, or equivalently,

$$\hat{\lambda} \text{ is the unique solution of } U_\lambda(\lambda) = 0,$$

where

$$U_\lambda(\lambda) = -(n - N_0)\frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} + \frac{2n\overline{Y} + n - N_0}{\lambda}, \quad \lambda > 0$$

is a score function with respect to λ.

Lemma 1 (Existence and smoothness of the score function)**.** *For every $\lambda > 0$, the score function*

$$U_\lambda(\lambda) = -(n - N_0)\frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} + (2n\overline{Y} + n - N_0)\frac{1}{\lambda}$$

is well defined and continuously differentiable on $(0, \infty)$.

Proof. We can consider $U_\lambda(\lambda)$, being a linear combination of functions

$$f_1(\lambda) = \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda} \quad \text{and} \quad f_2(\lambda) = \frac{1}{\lambda}.$$

The hyperbolic functions $\cosh\lambda$ and $\sinh\lambda$ are entire and therefore infinitely differentiable on $\mathbb{R}$. Define

$$h(\lambda) = \sinh\lambda - \lambda.$$

For $\lambda > 0$,

$$h'(\lambda) = \cosh\lambda - 1 > 0 \quad \text{and} \quad h(0) = 0,$$

which implies $h(\lambda) > 0$ for all $\lambda > 0$. Hence the function $f_1(\lambda)$ is well defined on $(0, \infty)$. Since both numerator and denominator are differentiable and the denominator does not vanish on $(0, \infty)$, this function is continuously differentiable on $(0, \infty)$.

Moreover, the function $f_2(\lambda) = \frac{1}{\lambda}$ is continuously differentiable on $(0, \infty)$. It follows that $U_\lambda(\lambda)$, being a linear combination of continuously differentiable functions on $(0, \infty)$, belongs to the set $C^1((0, \infty))$, i.e., the set of all continuous functions on $(0, \infty)$ whose first derivative is also continuous on $(0, \infty)$.

An explicit expression for the derivative is obtained via the quotient rule:

$$U'_\lambda(\lambda) = -(n - N_0)\,\frac{\sinh\lambda(\sinh\lambda - \lambda) - (\cosh\lambda - 1)^2}{(\sinh\lambda - \lambda)^2} - \frac{2n\overline{Y} + n - N_0}{\lambda^2}.$$

□

Remark 7. The differentiability of $U_\lambda(\lambda)$ on $(0, \infty)$ ensures that Newton–Raphson iterations are well defined for any initial value $\lambda^{(0)} > 0$, providing a valid numerical procedure for computing the maximum likelihood estimator $\hat{\lambda}$.

4.2. **On the asymptotic normality of the method of maximum likelihood estimators for the parameters of the ZMHS distribution.** The asymptotic normality of the method of maximum likelihood estimators $\hat{\lambda}$ and $\hat{\beta}$ is given by the following theorem.

Theorem 8. *The maximum likelihood estimators* $\hat{\lambda} = \arg\max_{\lambda>0} \ell(\lambda, \hat{\beta})$ *and* $\hat{\beta} = \frac{N_0}{n}$ *are jointly asymptotically normal with mean* $(\lambda, \beta)^T$ *and covariance matrix*

$$\Lambda = \begin{pmatrix} \frac{1}{(1-\beta)\left[w'(\lambda) + \frac{1}{\lambda} w(\lambda)\right]} & 0 \\ 0 & \beta(1-\beta) \end{pmatrix},$$

where $w(\lambda) = \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda}$*; that is,*

$$\sqrt{n}\begin{pmatrix} \hat{\lambda} - \lambda \\ \hat{\beta} - \beta \end{pmatrix} \text{ converges in distribution as } n \to \infty \text{ to } \mathcal{N}(\mathbf{0}, \Lambda).$$

The statistics $\overline{Y}$ *and* N_0 *are the minimal sufficient statistics for the zero-modified hyperbolic sine (ZMHS) distribution.*

Proof. The joint asymptotic normality follows from the general result on the maximum likelihood estimators. We need to establish the parameters of the asymptotic normality only. Recall that the first partial derivatives (score equations) are

$$\frac{\partial \ell}{\partial \lambda} = -(n - N_0)\, w(\lambda) + \frac{2n\overline{Y} + n - N_0}{\lambda}, \qquad \frac{\partial \ell}{\partial \beta} = \frac{N_0}{\beta} - \frac{n - N_0}{1 - \beta},$$

where

$$w(\lambda) = \frac{\cosh\lambda - 1}{\sinh\lambda - \lambda}.$$

The second partial derivatives (entries of the Hessian matrix) are

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -(n - N_0)\, w'(\lambda) - \frac{2n\overline{Y} + n - N_0}{\lambda^2},$$
$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{N_0}{\beta^2} - \frac{n - N_0}{(1 - \beta)^2},$$

$$\frac{\partial^2 \ell}{\partial\lambda\partial\beta} = \frac{\partial^2 \ell}{\partial\beta\,\partial\lambda} = 0.$$

The expected values are

$$\begin{aligned}
\mathbb{E}\left(\frac{\partial^2 \ell}{\partial\lambda^2}\right) &= -(n - n\beta)\, w'(\lambda) - \frac{2n\,\mathbb{E}(\overline{Y}) + n - n\beta}{\lambda^2} \\
&= -n\left[(1-\beta)\, w'(\lambda) + \frac{2\left((1-\beta)\,\dfrac{\lambda\cosh\lambda - \sinh\lambda}{2(\sinh\lambda - \lambda)}\right) + 1 - \beta}{\lambda^2}\right] \\
&= -n(1-\beta)\left[w'(\lambda) + \frac{\lambda(\cosh\lambda - 1)}{\lambda^2(\sinh\lambda - \lambda)}\right] \\
&= -n(1-\beta)\left[w'(\lambda) + \frac{1}{\lambda}\, w(\lambda)\right].
\end{aligned}$$

$$\mathbb{E}\left(\frac{\partial^2 \ell}{\partial\beta^2}\right) = -\frac{n\beta}{\beta^2} - \frac{n - n\beta}{(1-\beta)^2} = -\frac{n}{\beta(1-\beta)}.$$

$$\mathbb{E}\left(\frac{\partial^2 \ell}{\partial\lambda\,\partial\beta}\right) = \mathbb{E}\left(\frac{\partial^2 \ell}{\partial\beta\,\partial\lambda}\right) = 0$$

and the Fisher information matrix is

$$I(\lambda,\beta) = -\mathbb{E}\begin{pmatrix} \frac{\partial^2 \ell}{\partial\lambda^2} & \frac{\partial^2 \ell}{\partial\lambda\partial\beta} \\ \frac{\partial^2 \ell}{\partial\beta\partial\lambda} & \frac{\partial^2 \ell}{\partial\beta^2} \end{pmatrix} = n\begin{pmatrix} (1-\beta)\left[w'(\lambda) + \frac{1}{\lambda}\, w(\lambda)\right] & 0 \\ 0 & \frac{1}{\beta(1-\beta)} \end{pmatrix}.$$

Therefore, since the maximum likelihood estimator $\hat{\lambda}$ is an implicit function of the sample mean, which is asymptotically normal, it follows that for large n,

$$\sqrt{n}\begin{pmatrix} \hat{\lambda} - \lambda \\ \hat{\beta} - \beta \end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Lambda\right),$$

where

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the zero mean vector, and Λ denotes the covariance matrix of the maximum likelihood estimators $\hat{\lambda}$ and $\hat{\beta}$, that is,

$$\Lambda = I^{-1}(\lambda,\beta) = \begin{pmatrix} \frac{1}{(1-\beta)\left[w'(\lambda) + \frac{1}{\lambda}\, w(\lambda)\right]} & 0 \\ 0 & \beta(1-\beta) \end{pmatrix}.$$

□

Remark 9. Theorem 8 is an asymptotic result. Applying it, we can make the following conclusion:

- MLE estimators $\hat{\lambda}$ and $\hat{\beta}$ are consistent and asymptotically unbiased.

- The variance of the estimators:

$$\operatorname{Var}(\hat{\lambda}) = \frac{1}{n} \frac{1}{(1-\beta)\left[w'(\lambda) + \frac{1}{\lambda} w(\lambda)\right]} + O(n^{-2}),$$
$$\operatorname{Var}(\hat{\beta}) = \frac{1}{n}\beta(1-\beta) + O(n^{-2}).$$

- MLE estimators $\hat{\lambda}$ and $\hat{\beta}$ are asymptotically uncorrelated, that is

$$\operatorname{Cov}(\hat{\lambda}, \hat{\beta}) = O(n^{-2}).$$

- For a sufficiently large sample size n, a $(1-\alpha)100\%$ confidence interval for λ and β, respectively, are

$$\hat{\lambda} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{n} \frac{1}{(1-\beta)\left[w'(\lambda) + \frac{1}{\lambda} w(\lambda)\right]}},$$
$$\hat{\beta} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{n}\beta(1-\beta)},$$

where $z_{\frac{\alpha}{2}}$ is the $(1-\frac{\alpha}{2})$ quantile of the standard normal distribution.

5. Numerical simulations

5.1. **On a zero-modified hyperbolic sine random variate.** The zero-modified hyperbolic sine distribution ZMHS(λ, β) can be considered as zero-inflation of the zero-truncated hyperbolic sine distribution. Because of that, we can suggest the following algorithm to generate a zero-modified hyperbolic sine random variate.

Step 0. Fix the values of $\lambda > 0$ and $0 \leq \beta < 1$.

Step 1. Generate a random variate u following the uniform distribution on the interval $[0, 1]$; that is, $u \sim U(0, 1)$.

Step 2. If $u < \beta$, then set $x = 0$. If $u \geq \beta$, generate random variate x that follows the zero-truncated hyperbolic sine distribution with parameter λ; that is, $x \sim$ ZTHS(λ).

Step 3. Return x as a zero-modified hyperbolic sine random variate.

Remark 10. Basically, a random variate can be obtained using the r-quantile (or $100r$-th percentile) provided in Theorem 5. However, the above algorithm provides a simple and efficient procedure for generating random variates from the zero-modified hyperbolic sine distribution. It is structured as a two-component combining a point mass at zero with a zero-truncated hyperbolic sine distribution; makes it particularly well suited for Monte Carlo simulation.

5.2. **Simulation results.** As discussed in Section 4, the method of maximum likelihood estimation (MLE) estimator for parameter λ of the zero-modified hyperbolic sine distribution does not have an exact (closed-form) solution. This section aims to examine the performance of the proposed estimators, especially an MLE $\hat{\lambda}$. We use two metrics to assess overall performance (measures both accuracy and precision), using the bias and the mean squared error (MSE).

Monte Carlo experiment is repeated $N = 10,000$ times for each scenario. The formula of the bias and the mean squared error (MSE) are then given as follows:

$$\text{Bias} = \theta - \frac{\sum_{j=1}^{N} \widehat{\theta}_j}{N}; \quad \text{MSE} = \frac{\sum_{j=1}^{N} (\widehat{\theta}_j - \theta)^2}{N},$$

where $\widehat{\theta}_j$ is the estimated value for the considered parameter θ (λ or β) on the j-th simulation and θ is the true value of the considered parameter.

Table 1. Bias and MSE for different parameter values and sample sizes

Parameter	Bias			MSE		
	$n = 100$	$n = 500$	$n = 1000$	$n = 100$	$n = 500$	$n = 1000$
$\lambda = 1$	-0.0768	-0.0095	-0.0022	0.1542	0.0212	0.0103
$\beta = 0.5$	-0.0008	-0.0003	-0.0001	0.0025	0.0005	0.0002
$\lambda = 0.3$	-0.1658	-0.0591	-0.0258	0.0993	0.0358	0.0154
$\beta = 0.5$	-0.0008	-0.0002	0.0001	0.0025	0.0005	0.0003
$\lambda = 0.75$	-0.0726	-0.0099	-0.0055	0.1073	0.0138	0.0067
$\beta = 0.25$	0.0001	0.0000	0.0000	0.0019	0.0004	0.0002
$\lambda = 5$	-0.0041	-0.0006	-0.0009	0.1255	0.0245	0.0122
$\beta = 0.5$	0.0007	0.0003	0.0002	0.0025	0.0005	0.0003
$\lambda = 0.65$	-0.3735	-0.1365	-0.0599	0.4862	0.1813	0.0821
$\beta = 0.9$	-0.0001	-0.0001	0.0001	0.0009	0.0002	0.0001

Table 1 presents the simulation results and indicates that the estimates of the parameters are more or less close to their true values, with the bias never exceeding 0.4. As expected, the bias decreases as the sample size increases. This is also reflected in the MSE, which decreases with increasing sample size. Therefore, it is sufficient to conclude that MLE estimators are suitable for the point estimation of each parameter.

6. Application to real datasets

This section presents the application of the zero-modified hyperbolic sine distribution to real datasets, highlighting its usefulness and comparing its performance with the hyperbolic sine, Conway-Maxwell-Poisson (CMP), and zero-modified geometric (ZMG; also known as partial-geometric) distributions. Remind that the CMP and ZMG distributions include the Poisson and geometric distributions, respectively, as special cases. Moreover, both are also classified as flexible distributions with respect to dispersion index ([8], [10]).

We consider the dataset that represents the number of sperms in an egg with respect to fertilization in a sea-urchin's egg [15]; this dataset is underdispersion with mean = 0.7625 and variance = 0.4366. The second dataset is the number of dicentric chromosomes observed after exposure to radiation at doses of 0.6 [5]; this dataset is overdispersion with mean = 0.3233 and variance = 0.3937. The fit of two datasets to all competing distributions is assessed using appropriate metrics, including the maximum log-likelihood, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and the Chi-squared goodness-of-fit statistic along with its p-value, as summarized in Tables 2 and 3.

Table 2. Fitting result for dataset 1

Number of Sperms	Observed	Expected Frequency			
		ZMHS	CMP	ZMG	HS
0	28	28	28.2859	28	33.2478
1	44	43.6997	42.9323	44.3279	34.37312
2	7	7.6336	8.2906	6.5402	10.66097
3	1	0.6350	0.4793	0.9649	1.574547
Fitted parameters		$\widehat{\lambda} = 1.8691$	$\widehat{\lambda} = 1.5178$	$\widehat{p} = 0.8525$	$\widehat{\lambda} = 2.4906$
		$\widehat{\beta} = 0.35$	$\widehat{\nu} = 2.9745$	$\widehat{\alpha} = 3.7556$	
χ^2 statistic		0.2645	0.7960	0.036	4.99133
p-value		0.6071	0.3723	0.8495	0.0824
Log-likelihood		-77.28373	-77.4821	-77.31934	-79.79031
AIC		158.5675	158.9642	158.6387	161.5806
BIC		163.3315	163.7282	163.4027	163.9626

Table 3. Fitting result for dataset 2

Number of Chromsm	Observed	Expected Frequency			
		ZMHS	CMP	ZMG	HS
0	473	473	471.7168	473	447.1872
1	119	117.7934	123.7125	122.3726	164.6415
2	34	34.8676	28.20241	27.59381	18.1849
3	3	4.91477	5.923231	6.222134	0.95645
4	2	0.40411	1.173741	1.40303	0.02935
Fitted parameters		$\widehat{\lambda} = 2.4331$	$\widehat{\lambda} = 0.2623$	$\widehat{p} = 0.7745$	$\widehat{\lambda} = 1.4863$
		$\widehat{\beta} = 0.7496$	$\widehat{\nu} = 0.2022$	$\widehat{\alpha} = 0.8601$	
χ^2 statistic		0.0531	1.9944	2.4840	44.2428
p-value		0.8178	0.1579	0.115	0.0000
Log-likelihood		-463.7798	-463.7487	-463.9968	-482.7638
AIC		931.5597	931.4973	931.9936	967.5276
BIC		940.4543	940.392	940.8883	971.9749

As shown in Tables 2 and 3, the p-values obtained from the chi-squared goodness-of-fit test for the zero-modified hyperbolic sine distribution exceed the 5% significance level, indicating that the model provides an adequate fit to both datasets. Furthermore, among the candidate models considered, the zero-modified hyperbolic sine distribution demonstrates the best fit distribution, as evidenced by its lowest AIC and BIC values.

Remark 11. For all fitted models in Table 2, the expected frequency in the highest count category is below 5, so the regularity conditions for the chi-square goodness-of-fit approximation are not fully satisfied. A common correction in such situations is to pool tail categories; however, in the present setting this is not feasible. Pooling the upper count categories would reduce the number of classes to three, and after accounting for parameter estimation, the resulting chi-square test would have zero degrees of freedom. Consequently, the chi-square goodness-of-fit statistics are reported for descriptive purposes only, while formal model comparison is based on likelihood-based criteria such as the log-likelihood, AIC, and BIC.

7. Conclusion

This study proposes a new distribution, referred to as the zero-modified hyperbolic sine distribution, developed by adding an extra parameter to hyperbolic sine distribution. An extra parameter, which represents the probability that the zero-modified hyperbolic sine random variable takes value zero, is able to inflate or deflate the occuring of zeros. The flexibility of the zero-modified hyperbolic sine distribution is demonstrated through its index of dispersion (IOD), which indicates its ability to accommodate underdispersed, equidispersed, and overdispersed count data. Additionally, the probability mass function of the zero-modified hyperbolic sine distribution may be either unimodal or bimodal.

Parameter estimation for the zero-modified hyperbolic sine distribution is conducted using the method of maximum likelihood estimation (MLE). The asymptotic normality of the MLE estimators are proven. However, the effectiveness of the proposed estimation methods are assessed via Monte Carlo simulation under a range of parameter configurations and sample sizes. The results demonstrate that the MLE performs well, yielding the relatively small mean squared error for all sample sizes.

Moreover, the zero-modified hyperbolic sine can be a promising alternative distribution to capture count data. According to Section 6, real datasets provided are suitable to the zero-modified hyperbolic sine distribution achieving the smallest Akaike Information Criterion and Bayesian Information Criterion.

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