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The Unit Quasi-Lindley Distribution: Properties, Inference, and Applications to Real Data

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Abstract. This article presents the Unit Quasi-Lindley distribution, a two-parameter model for continuous data defined on the unit interval, derived via an exponential transformation of the Quasi-Lindley distribution. Fundamental properties are established, including moments, the generating function, and the quantile function, along with an analysis of the shapes of the density and hazard rate functions, which encompass monotonic and bathtub-shaped patterns. The model parameters are estimated using maximum likelihood, and the efficiency of the estimators is evaluated through Monte Carlo simulation studies. Two applications to real-world datasets demonstrate that the proposed distribution provides a superior fit compared to existing unit-interval distributions in the literature.

1. Introduction

Researchers frequently use continuous distributions in $(0, 1)$ to model percentages, rates, and bounded data in various scientific fields. These models are crucial for analyzing common datasets, including economic market shares, medical recovery rates, test scores, and engineering reliability.

Modeling bounded data requires distributions restricted to the unit interval to ensure valid, meaningful, and statistically sound interpretations. The beta distribution is the standard, premier choice for this purpose, offering unparalleled flexibility to model various shapes, including skewed, symmetric, and J-shaped densities, on a confined range.

The foundational work of [2] enabled the modeling of bounded response variables by reparameterizing beta distributions in terms of mean and precision, facilitating covariate analysis

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for rates and proportions.

Beyond the commonly used beta distribution, alternatives such as the Kumaraswamy distribution have been introduced to leverage computational benefits, specifically a closed-form cumulative distribution function (cdf) [3]. In a parallel trend, the Lindley distribution [5] has emerged as a robust foundation for new lifetime models.

Researchers often prefer the Lindley distribution and its flexible two-parameter extension, the Quasi Lindley (QL) distribution [10], over the exponential distribution for modeling lifetime data. This effectiveness is often paired with the productive technique of mapping positive real-valued distributions to the unit interval to create new models. By applying an exponential transformation to the Lindley distribution, [7] introduced the unit-Lindley distribution, a one-parameter model designed for bounded data. Despite ample literature on both subjects, the transformation of the QL distribution to the unit interval is missing. This approach aims to bridge this gap by utilizing the extra parameter in the QL distribution to provide a more flexible alternative for unit-interval modeling.

Taking the log of a QL random variable builds the unit QL (UQL) distribution, which is a powerful tool for analyzing data that always falls between zero and one. This flexible distribution easily adapts to various data shapes and hazard rates, from steady increases to bathtub curves. Its built-in mathematical formulas for moments and quantiles make generating random samples and analyzing the data much easier.

This paper proceeds as follows: Section 2 defines the UQL distribution and examines the shapes of its basis functions. Section 3 details the maximum likelihood estimation approach and presents simulation results. Section 4 applies the distribution to empirical data, highlighting its superiority over rival models. The paper ends with conclusions in Section 5.

2. The UQL Distribution

The probability density function (pdf) of the QL distribution with parameters $\alpha, \theta > 0$ is given by

$$f(x;\alpha,\theta) = \frac{\theta\,(\alpha+\theta\,x)\,\mathrm{e}^{-\theta\,x}}{\alpha+1}\;\; x>0. \tag{1}$$

The QL distribution generalizes the Lindley distribution ($\alpha=\theta$) and the gamma $(2,\theta)$ distribution ($\alpha=0$). Consequently, it can be represented as a combination of them.

Setting $Y=\exp(-X)$ in (1), we can define the pdf of the UQL distribution as

$$f(y) = \frac{\theta\,(\alpha-\theta\log y)\,y^{\theta-1}}{\alpha+1},\;\; 0<y<1. \tag{2}$$

Its cdf becomes

$$F(y) = \frac{y^{\theta}\,(1+\alpha-\theta\log y)}{\alpha+1},$$

and its hazard rate function (hrf) is

$$h(y) = \frac{\theta\,(\alpha - \theta \log y)\, y^{\theta-1}}{\alpha + 1 - y^{\theta}\,\{1 + \alpha - \theta \log y\}}\,.$$

Figure 1 displays the density and hrf of the UQL distribution, emphasizing its adaptability for unit interval data. Because the distribution supports shapes ranging from decreasing to skewed unimodal, and hrfs featuring both monotonic and bathtub patterns, it serves as a highly robust model for complex, bounded data in survival and reliability analysis.

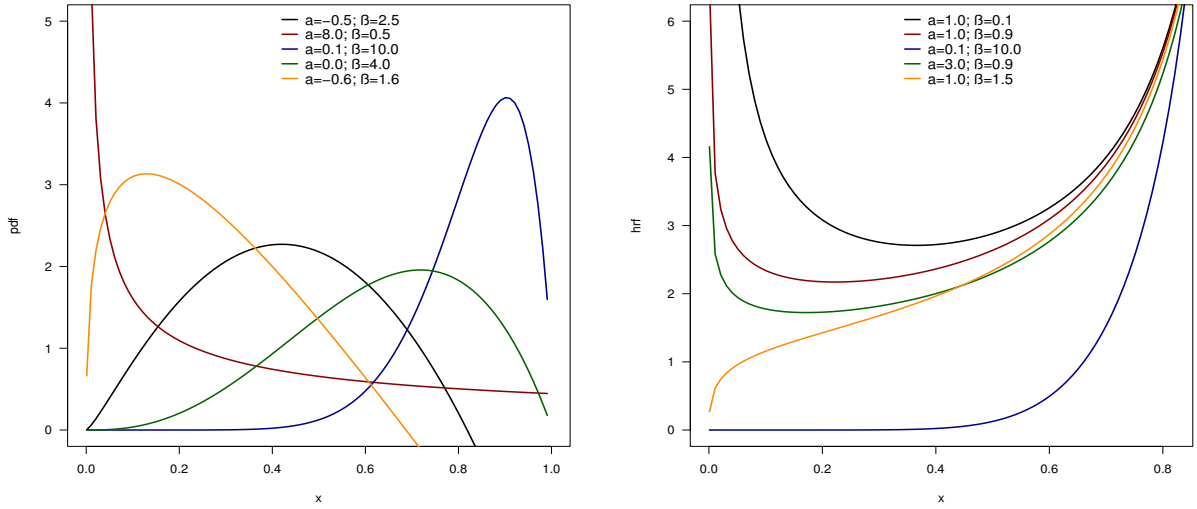


FIGURE 1. Density and hrf of the UQL distribution.

Proposition 1. Let $Y \sim \mathrm{UQL}(\alpha, \theta)$. Consequently, the UQL is expressible via a convex mixture.

$$f_Y(y) = \frac{\alpha}{\alpha+1} f_1(y) + \frac{1}{\alpha+1} f_2(y), \qquad 0 < y < 1,$$

where

$$f_1(y) = \theta y^{\theta-1} \quad \text{and} \quad f_2(y) = \theta^2(-\log y) y^{\theta-1}$$

are valid pdfs on $(0, 1)$.

Proof. The UQL density is

$$f(y) = \frac{\theta(\alpha - \theta \log y)}{\alpha+1} y^{\theta-1} = \frac{\alpha\,\theta}{\alpha+1} y^{\theta-1} + \frac{\theta^2}{\alpha+1}(-\log y)\, y^{\theta-1}\,.$$

Given that $\int_0^1 f_1(y)\,dy = \int_0^1 f_2(y)\,dy = 1$ and the weights sum to one, the result follows. □

The above density $f_1(y)$ corresponds to the distribution of $Y_1 = \exp(-X_1)$ with $X_1 \sim$ Exponential(θ), while $f_2(y)$ refers to $Y_2 = \exp(-X)$ with $X_2 \sim \mathrm{Gamma}(2, \theta)$.

Mapping the QL distribution to the unit interval allows the UQL distribution to retain its mixture structure.

2.1. **Main properties.** This section examines the foundational properties of the UQL, specifically focusing on moments, measures of central tendency, and dispersion. Evaluating these descriptive statistics is vital for accurately modeling distribution behavior and facilitating robust statistical inferences.

The rth ordinary moment of Y, say $\mu'_r = \mathbb{E}(Y^r) = \int_0^1 y^r\, f(y)\, \mathrm{d}y$, can be derived using the linear combination in Proposition 1:

$$\begin{aligned}\mu'_r =& \frac{\alpha\,\theta}{\alpha+1}\int_0^1 y^r\, y^{\theta-1}\mathrm{d}y + \frac{\theta^2}{\alpha+1}\int_0^1 y^r\,(-\log y)\, y^{\theta-1}\mathrm{d}y \\ =& \frac{\alpha\,\theta}{(1+\alpha)(r+\theta)} + \frac{\theta^2}{(1+\alpha)(r+\theta)^2}.\end{aligned}$$

This structural property significantly simplifies the analytical derivation of not only the mean and variance, but also the generating function. Thus,

$$\mu'_r = \frac{\theta\,[\theta+\alpha(r+\theta)]}{(\alpha+1)(\theta+r)^2}, \quad r = 1, 2, \ldots \tag{3}$$

This explicit formulation permits the straightforward computation of all positive integer moments, thereby establishing the mathematical tractability of the UQL. The expected value of Y is

$$\mu'_1 = \mathbb{E}(Y) = \frac{\theta(\theta+\alpha+\alpha\,\theta)}{(\alpha+1)(\theta+1)^2}\,.$$

The mean represents the central tendency and relies on the α and θ shape parameters to provide flexible location adjustment. The variance is calculated by

$$\mathrm{Var}(Y) = \frac{\theta}{(\alpha+1)}\left[\frac{\theta+\alpha(\theta+2)}{(\theta+2)^2} - \frac{\theta(\theta+2+\alpha\,\theta)^2}{(\alpha+1)(\theta+1)^2}\right].$$

The availability of closed-form expressions for the mean and variance facilitates method-of-moments estimation and permits the analytical derivation of asymptotic properties for statistical inference.

The generating function (gf) of Y, $M_Y(t) = \mathbb{E}(e^{tY})$, admits the representation starting from proposition 1 as (for $t < 0$)

$$M_Y(t) = \frac{1}{(1+\alpha)}\left[\alpha\,\theta\,(-t)^{-\theta}\,[\Gamma(\theta) - \Gamma(\theta,-t)] + {}_2F_2(\theta,\theta;1+\theta,1+\theta;t)\right],$$

where $\Gamma(\theta,-t)$ is the upper incomplete gamma function, and ${}_2F_2(\theta,\theta;1+\theta,1+\theta;t)$ denotes the generalized hypergeometric function.

The presence of these special functions reflects the mathematical structure shaped by the UQL density's logarithmic component. Despite its complexity, this expression permits numerical evaluation and full distributional characterization, allowing moments to be derived through successive differentiation at $t = 0$.

Following the derivation of the ordinary moments, the rth incomplete moment, $m_r(z) = \int_0^z y^r f(y)\mathrm{d}y$, is given by

$$m_r(z) = \frac{z^{r+\theta}\,\theta\,[\theta + \alpha\,(r+\theta) - \theta\,(r+\theta)\log z]}{(1+\alpha)\,(r+\theta)^2}\,.$$

For a given probability $\nu \in (0,1)$, the Bonferroni and Lorenz curves of Y are defined as $B(\nu) = m_1(q)/(\nu\mu_1')$ and $L(\nu) = m_1(q)/\mu_1'$, respectively, where $q = Q(\nu)$ denotes the quantile function (qf) evaluated at ν, and $m_1(q)$ is the first incomplete moment. Illustrative plots of these curves for some values of α and θ are shown in Figure 2, highlighting the UQL model's flexibility in characterizing inequality and income-related measures.

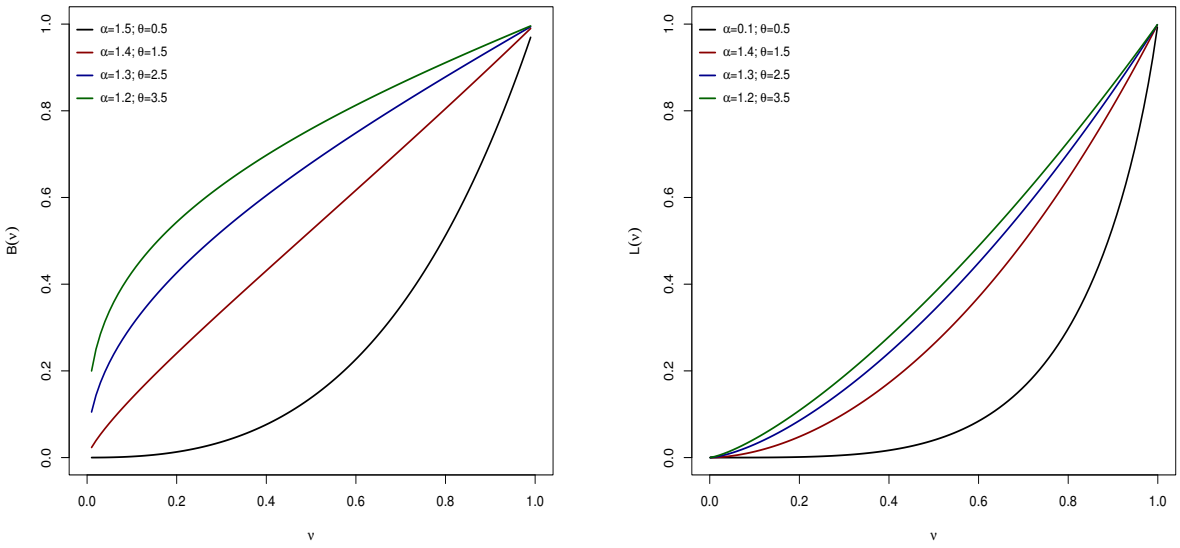


FIGURE 2. Bonferroni and Lorenz curves of the UQL distribution.

2.2. **Quantile function.** One of the key advantages of the UQL is the availability of a closed-form expression for its qf, say $Q(u) = F^{-1}(u)$. Thus, for $u \in (0,1)$, the qf of Y has the form

$$Q(u) = \exp\left\{\frac{\alpha+1}{\theta} + \frac{1}{\theta}W_{-1}\left(\frac{-u(\alpha+1)}{\mathrm{e}^{(\alpha+1)}}\right)\right\}\,, \tag{4}$$

where $W_{-1}(\cdot)$ denotes the lower branch of the Lambert W function. Because its quantile function is available in explicit form, the UQL avoids the numerical root-finding required by many other unit-interval distributions. Thus, random observations from the UQL can be efficiently generated from (4). Further, the Bowley skewness [4] and Moors kurtosis [8] of Y are

$$\mathcal{B} = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}\,,$$

and

$$\mathcal{M} = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)},$$

respectively. Plots of these measures for the UQL distribution as functions of θ reported in Figure 3 show that both measures decrease when θ increases.

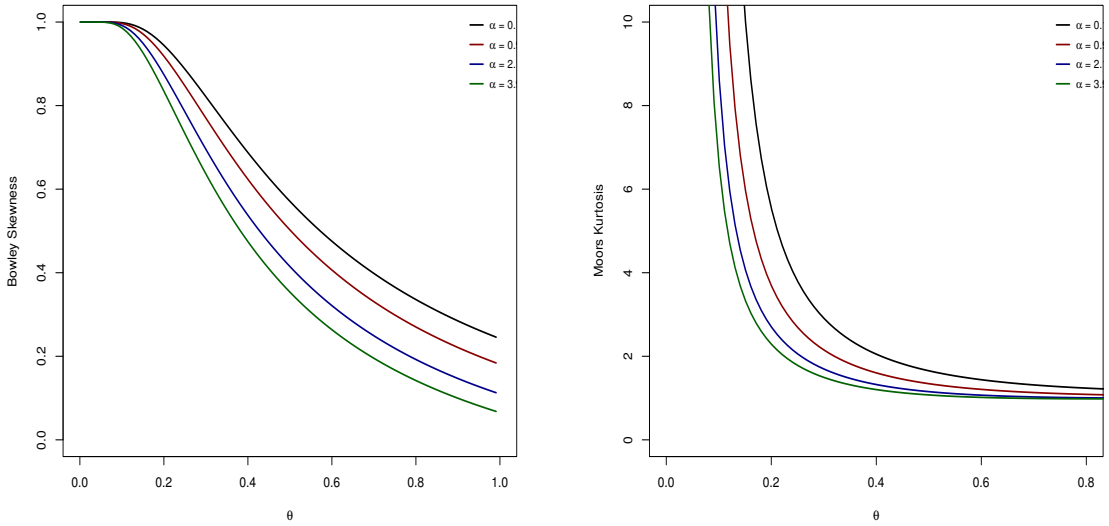


FIGURE 3. Bowley skewness and Moors kurtosis of the UQL distribution.

Figure 4 illustrates the qf of Y for two simulation scenarios. The estimated pdf and cdf of Y show excellent agreement with the empirical histogram and cdf, confirming the adequacy of the function.

Theorem 1. The pdf and cdf of the UQL(α, θ) distribution is log-concave and log-convex on $(0, 1)$ for all $\alpha > 0$ and $\theta \geq 1$.

Proof. The log-density is

$$\log f(y) = \log \theta - \log(\alpha + 1) + \log(\alpha - \theta \log y) + (\theta - 1) \log y.$$

Differentiate with respect to y

$$\frac{d \log f(y)}{dy} = \frac{-\theta}{y(\alpha - \theta \log y)} + \frac{\theta - 1}{y}.$$

Differentiate one more with respect to y

$$\frac{d^2 \log f(y)}{dy^2} = -\frac{\theta}{y^2(\alpha - \theta \log y)} - \frac{\theta^2}{y^2(\alpha - \theta \log y)^2} - \frac{\theta - 1}{y^2}.$$

For $\alpha > 0$ and $\theta \geq 1$, all three terms are non-positive for $0 < y < 1$, which implies

$$\frac{d^2 \log f(y)}{dy^2} \leq 0.$$

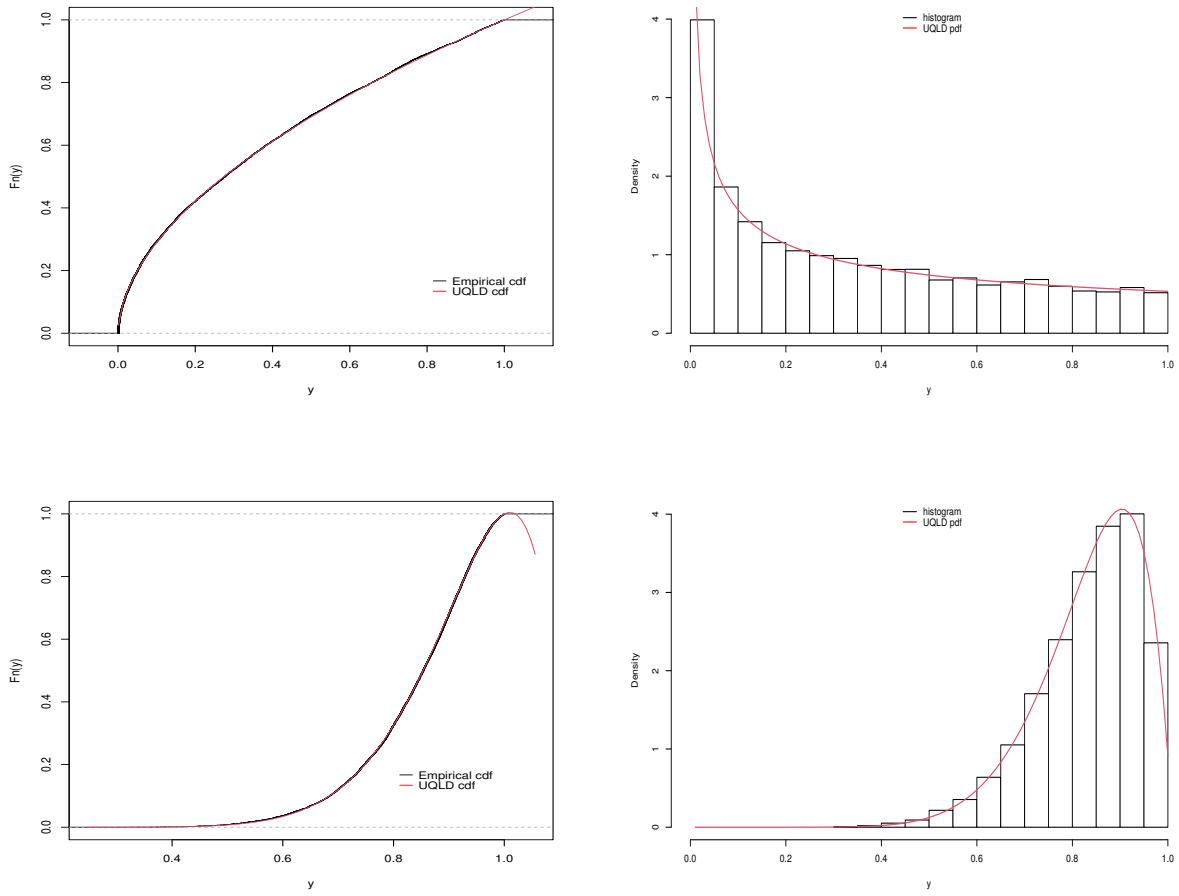


FIGURE 4. Histogram and empirical cdf of the generated samples, along with the estimated pdf and cdf for the scenarios $(\alpha, \theta) = (0.4, 0.9)$ and $(\alpha, \theta) = (0.1, 10)$.

Hence, the UQL density is log-concave on $(0, 1)$. Let $S(y) = \bar{F}(y)$. Log-convexity is equivalent to

$$\frac{d^2 \log S(y)}{dy^2} \geq 0.$$

A direct calculation gives

$$\frac{d^2 \log S(y)}{dy^2} = \frac{S''(y)}{S(y)} - \left(\frac{S'(y)}{S(y)}\right)^2.$$

For $0 < \theta \leq 1$, the derivative $S'(y)$ is decreasing and $S''(y)$ is non-negative for all $0 < y < 1$, which leads to

$$\frac{d^2 \log S(y)}{dy^2} \geq 0.$$

Hence the survival function is log-convex. □

2.3. **Stochastic ordering.** Let X and Y be the two random variables with cdfs $F_X(\cdot)$ and $F_Y(\cdot)$ respectively. Then, X is said to be smaller than Y in the

(1) Stochastic order $(X \prec_s Y), if F_X(t) \geq F_Y(t), \forall\, t.$
(2) Convex order $(X \prec_{cx} Y)$ if for all convex functions ϕ and provided expectations exist, $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$.
(3) Hazard rate order $(X \prec_{hr} Y), if\ h_X(t) \geq h_Y(t), \forall\, t.$
(4) Likelihood Ratio order $(X \prec_{lr} Y), if\ \frac{f_X(t)}{f_Y(t)}$ is decreasing in t.

Likelihood ratio order $\Rightarrow$ *Hazard rate order* $\Rightarrow$ *Stochastic order if* $\mathbb{E}(X) = \mathbb{E}(Y)$, *then Convex order* $\iff$ *Stochastic order.*

Theorem 2. Let $Y_i \sim \text{UQL}(\alpha_i, \theta)$ with $\alpha_i \leq \alpha_{i+1}$ and fixed $\theta > 0$. Then Y_2 is smaller than Y_1 in the likelihood ratio order, that is,

$$Y_1 \prec_{lr} Y_2, Y_1 \prec_{hr} Y_2, Y_1 \prec_s Y_2 \text{ and } Y_1 \prec_{cx} Y_2$$

Proof. The UQL density is

$$f(y; \alpha, \theta) = \frac{\theta(\alpha - \theta \log y)}{\alpha + 1} y^{\theta - 1}, \quad 0 < y < 1.$$

Consider the likelihood ratio

$$R(y) = \frac{f(y; \alpha_2, \theta)}{f(y; \alpha_1, \theta)} = \frac{\alpha_2 + 1}{\alpha_1 + 1} \frac{\alpha_2 - \theta \log y}{\alpha_1 - \theta \log y}.$$

Differentiate with respect to y

$$\frac{dR(y)}{dy} = -\frac{\theta(\alpha_2 - \alpha_1)}{y(\alpha_1 - \theta \log y)^2} < 0, \quad 0 < y < 1,$$

since $\alpha_2 \geq \alpha_1$ and $\theta > 0$. Hence, $R(y)$ is decreasing in y. Further, $Y_1 \prec_{lr} Y_2, Y_1 \prec_{hr} Y_2, Y_1 \prec_s Y_2$ and $Y_1 \prec_{cx} Y_2$ holds. □

3. Estimation and Simulations

Let $(y_1, \cdots, y_n)$ denote an independent and identically distributed (i.i.d.) random sample drawn from the UQL distribution with parameter vector $\boldsymbol{\zeta} = (\alpha, \theta)^\top$. The log-likelihood function has then form

$$\ell(\boldsymbol{\zeta}) = n \log \theta + (\theta - 1) \sum_{i=1}^{n} \log y_i + \sum_{i=1}^{n} \log[\alpha - \theta \log y_i] - n \log(\alpha + 1)\,.$$

The corresponding score vector $\mathbf{U}(\boldsymbol{\zeta}) = (\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \theta})^\top$ has components

$$\frac{\partial \ell(\boldsymbol{\zeta})}{\partial \alpha} = \sum_{i=1}^{n} \frac{1}{\alpha - \theta \log y_i} - \frac{n}{\alpha + 1}\,,$$

and

$$\frac{\partial \ell(\boldsymbol{\zeta})}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log y_i + \sum_{i=1}^{n} \frac{-\log y_i}{\alpha - \theta \log y_i}\,.$$

The second-order partial derivatives (Hessian matrix elements) are given by

$$\frac{\partial^2 \ell(\boldsymbol{\zeta})}{\partial \alpha^2} = -\sum_{i=1}^{n} \frac{1}{(\alpha - \theta \log y_i)^2} + \frac{n}{(\alpha+1)^2},$$

$$\frac{\partial^2 \ell(\boldsymbol{\zeta})}{\partial \alpha\, \partial \theta} = \frac{\partial^2 \ell(\boldsymbol{\zeta})}{\partial \theta\, \partial \alpha} = \sum_{i=1}^{n} \frac{\log y_i}{(\alpha - \theta \log y_i)^2},$$

$$\frac{\partial^2 \ell(\boldsymbol{\zeta})}{\partial \theta^2} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} \frac{(\log y_i)^2}{(\alpha - \theta \log y_i)^2}.$$

The maximum likelihood estimator (MLE) $\hat{\boldsymbol{\zeta}} = (\hat{\alpha}, \hat{\theta})^\top$ is obtained by solving the system of equations $\mathbf{U}(\boldsymbol{\zeta}) = \mathbf{0}$. These equations are nonlinear, making it necessary to use numerical optimization methods like the Newton-Raphson or quasi-Newton algorithms. Therefore, for any $\alpha > 0$, we can express the expected Fisher information matrix as:

$$\mathcal{I}(\boldsymbol{\zeta}) = \begin{pmatrix} \mathcal{I}_{\alpha\alpha} & \mathcal{I}_{\alpha\theta} \\ \mathcal{I}_{\theta\alpha} & \mathcal{I}_{\theta\theta} \end{pmatrix},$$

which leads to

$$\mathcal{I}(\boldsymbol{\zeta}) = \begin{pmatrix} \frac{n}{\alpha+1}\left[\mathrm{e}^{\alpha} E_1(\alpha) - \frac{1}{(\alpha+1)}\right] & \frac{n}{(\alpha+1)\theta}\left[1 - \alpha\, \mathrm{e}^{\alpha} E_1(\alpha)\right] \\ \frac{n}{(\alpha+1)\theta}\left[1 - \alpha\, \mathrm{e}^{\alpha} E_1(\alpha)\right] & \frac{n}{\theta^2} + \frac{n}{(\alpha+1)\,\theta^2}\left[-\alpha + \alpha^2 \mathrm{e}^{\alpha} E_1(\alpha) + 1\right] \end{pmatrix}.$$

Here, $\mathcal{I}(\boldsymbol{\zeta})$ denotes the expected Fisher information matrix and $E_1(\cdot)$ denotes the exponential integral function.

This analysis assesses MLE performance for the UQL distribution parameters across four scenarios. For each sample size and parameter configuration (α, θ), 1,000 random samples are generated using the quantile function in (4) with a fixed seed (2013) for reproducibility. The log-likelihood function is optimized via the BFGS algorithm using true parameter values as initial inputs. Estimator performance is evaluated using average estimates (AEs), bias, and mean squared errors (MSEs).

This analysis investigates the performance of MLEs for the parameters of the UQL distribution in four scenarios. For each sample size configuration and parameter values (α, θ), we generate 1,000 random samples using the qf presented in (4), with a fixed random seed (2013) to ensure reproducibility. The log-likelihood function is optimized using the BFGS algorithm, with the true values as initials, and the resulting estimates are evaluated under the average estimate (AEs), biases, and mean squared error (MSE).

Overall, MLEs for the UQL distribution are dependable. Researchers need at least 200 data points for good accuracy, and should use 400 or more when dealing with tricky parameter settings.

TABLE 1. Monte Carlo simulation results for the UQL distribution.

		(0.1, 0.5)			(0.5, 0.8)		
n	ζ	AE	Bias	MSE	AE	Bias	MSE
50	α	0.1530	0.0530	0.0663	0.7282	0.2282	1.2015
	θ	0.5016	0.0016	0.0038	0.8158	0.0158	0.0175
100	α	0.1168	0.0168	0.0167	0.6079	0.1079	0.3344
	θ	0.5018	0.0018	0.0018	0.8072	0.0072	0.0088
200	α	0.1052	0.0052	0.0068	0.5420	0.0420	0.0953
	θ	0.5021	0.0021	0.0010	0.8041	0.0041	0.0042
400	α	0.1023	0.0023	0.0037	0.5227	0.0227	0.0369
	θ	0.5014	0.0014	0.0005	0.8021	0.0021	0.0021

		(1.0, 0.2)			(0.1, 10)		
n	ζ	AE	Bias	MSE	AE	Bias	MSE
50	α	1.5448	0.5448	6.2061	0.1531	0.0531	0.0662
	θ	0.2095	0.0095	0.0015	9.9700	-0.0300	1.1043
100	α	1.6042	0.6042	5.9580	0.1167	0.0167	0.0167
	θ	0.2022	0.0022	0.0009	10.0128	0.0128	0.6076
200	α	1.2503	0.2503	1.5527	0.1053	0.0053	0.0068
	θ	0.2008	0.0008	0.0004	10.0307	0.0307	0.3363
400	α	1.1239	0.1239	0.3715	0.1022	0.0022	0.0037
	θ	0.2001	0.0001	0.0002	10.0271	0.0271	0.1832

		(2.0, 1.0)			(3.0, 2.0)		
n	ζ	AE	Bias	MSE	AE	Bias	MSE
50	α	1.5624	-0.4376	2.4324	1.8108	-1.1892	5.3841
	θ	1.1350	0.1350	0.0610	2.3959	0.3959	0.3441
100	α	1.8612	-0.1388	2.4789	2.2192	-0.7808	4.5807
	θ	1.0741	0.0741	0.0294	2.2400	0.2400	0.1673
200	α	2.3952	0.3952	5.1470	3.0318	0.0318	8.6492
	θ	1.0297	0.0297	0.0148	2.1253	0.1253	0.0738
400	α	2.3470	0.3470	4.9363	3.0303	0.0303	4.5735
	θ	1.0127	0.0127	0.0087	2.0708	0.0708	0.0383

4. Applications

4.1. **Três Marias Reservoir data.** The UQL distribution's practical value is validated using real-world data from the Três Marias Reservoir in Minas Gerais, Brazil. Spanning 108 months (January 2011 to December 2019), this useful volume (UV) series provides a benchmark against which the modified Kumaraswamy distribution [9] was previously compared.

We evaluated the goodness-of-fit of each model using the Cramér-von Mises (W^*) and Anderson-Darling (A^*) metrics [1], as well as the Akaike Information Criterion (AIC), the Consistent AIC (CAIC), the Bayesian IC (BIC), the Hannan-Quinn IC (HQIC), and the Kolmogorov-Smirnov (KS) statistic (accompanied by its p-value). Lower values for these metrics signify a superior model fit.

Analysis of the UV data demonstrates a mean of 0.4329 (a standard deviation of 0.2594). The distribution displays a mild positive skew (0.4980) and a platykurtic profile (kurtosis of -0.7822), indicating a flatter peak and lighter tails relative to a normal distribution. The minimum and maximum values of 0.0290 and 0.9900 confirm that the dataset is bounded on the standard unit interval, validating its use with unit distributions.

We evaluate the UQL distribution against the Kumaraswamy (KW) and Beta (BT) distributions, which are the standard baselines for modeling variables on the unit interval. Table 2 presents the maximum likelihood estimates (MLEs) and standard errors (SEs) for all fitted models, confirming that every estimate shows high precision.

Table 3 highlights that the UQL distribution is the best-fitting model, as it yields the lowest values of W^*, A^*, AIC, CAIC, BIC, and HQIC among the competing models, in addition to the smallest KS statistic (0.083) and largest p-value (0.434). Graphical evidence in Figure 5 confirms that the UQL distribution fits the empirical histogram and cdf better than the KW and BT alternatives. Furthermore, the GLR test [11] indicates a superior fit for the UQL model, with strong test statistics of 20.789 (vs. KW) and 19.500 (vs. BT).

Table 2. MLEs and SEs for the Três Marias Reservoir data.

Distribution	MLEs (SEs)	
UQL(α, θ)	0.2966	1.6557
	(0.1845)	(0.1530)
KW(a, b)	1.1734	1.4624
	(0.1311)	(0.1929)
BT(β, α)	1.2050	1.4734
	(0.1511)	(0.1910)

4.2. **Barra Bonita Reservoir data.** Secondly, the UQL distribution is applied to analyze monthly UV data from the Barra Bonita Reservoir. Situated on the Tietê River in São Paulo,

TABLE 3. Adequacy measures for the Três Marias Reservoir data.

Distribution	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
UQL	0.111	0.600	-9.378	-9.264	-4.014	-7.203	0.083	0.434
KW	0.192	1.046	-4.275	-4.161	1.089	-2.100	0.101	0.218
BT	0.188	1.027	-4.507	-4.393	0.857	-2.332	0.101	0.213

(a) (b)

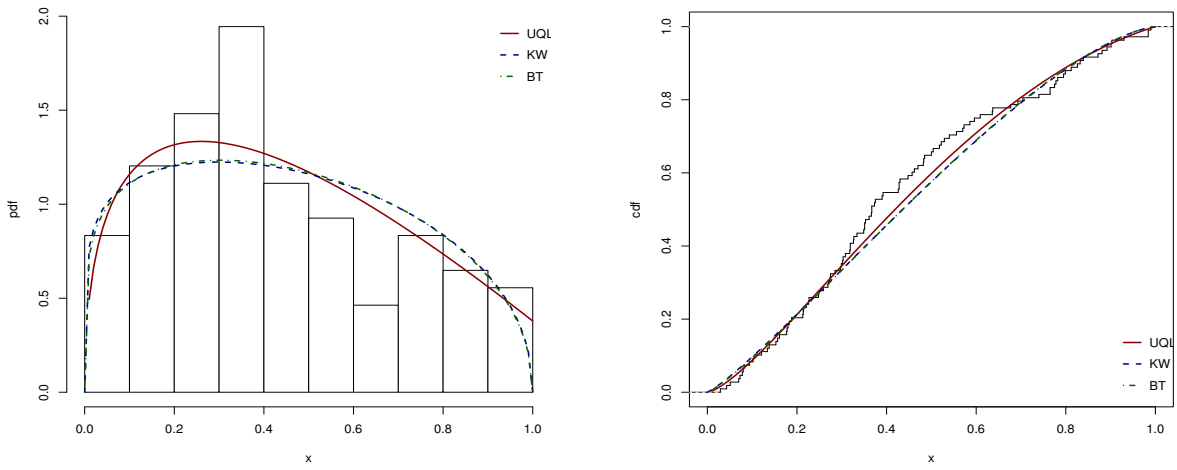


FIGURE 5. Fitted pdf and cdf for the Três Marias Reservoir data.

Brazil, this reservoir is vital to the national hydroelectric grid, making accurate water level modeling essential for resource management.

Descriptive statistics show an average UV of 0.5645 with a standard deviation of 0.3186. The observed range (0.0219 to 0.9900) confirms that the data are bound to the unit interval, making them ideal for unit distribution modeling.

We compare the UQL distribution against the KW and BT distributions. Table 5 presents the MLEs and corresponding SEs for each fitted model, all of which demonstrate satisfactory precision.

As shown in Table 4, the adequacy measures consistently favor the UQL model. It achieves the lowest values for W^*, A^*, AIC, CAIC, and BIC among the competing models, as well as the smallest KS statistic (0.065) and the highest p-value (0.747), providing strong evidence for its effectiveness in modeling the observed data. Furthermore, the GLR test confirms the UQL model's superior fit at conventional significance levels, yielding statistics of 24.1063 (UQL vs. KW) and 22.8356 (UQL vs. BT). Figure 6 supports this conclusion, illustrating that the UQL density and cdf plots fit the empirical data better than the KW and BT alternatives.

(a) (b)

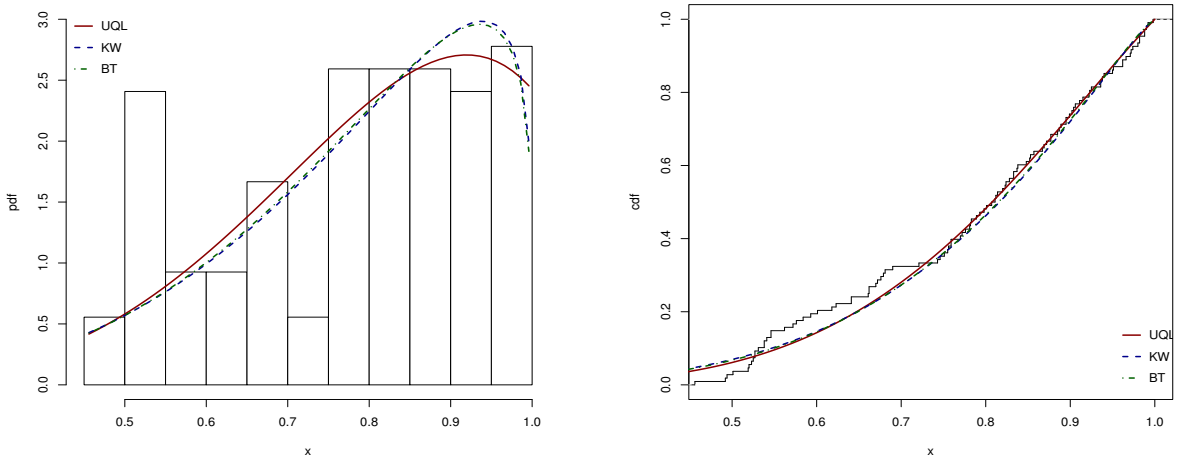


FIGURE 6. Fitted pdf and cdf for the Barra Bonita Reservoir data

All previous results are obtained using the `AdequacyModel` [6] package in `R`, with the numerical method Broyden-Fletcher-Goldfarb-Shanno (BFGS).

TABLE 4. MLEs and SEs for the Barra Bonita Reservoir data.

Distribution	MLEs (SEs)	
UQL(α, θ)	0.7269	5.7595
	(0.4484)	(0.6821)
KW(a, b)	4.1310	1.2240
	(0.4860)	(0.1573)
BT(β, α)	4.3888	1.2325
	(0.6186)	(0.1509)

TABLE 5. Adequacy measures for the Barra Bonita Reservoir data.

Distribution	W^*	A^*	AIC	CAIC	BIC	HQIC	KS	p-value
UQL	0.065	0.617	-123.616	-123.502	-118.252	-121.441	0.065	0.747
KW	0.078	0.723	-121.113	-120.999	-115.749	-118.938	0.070	0.655
BT	0.076	0.707	-121.483	-121.369	-116.119	-119.308	0.070	0.654

5. Conclusions

This paper introduced the Unit Quasi-Lindley distribution (UQL), a two-parameter model for bounded continuous data derived from the Quasi-Lindley distribution. Closed-form properties were established, including moments, generating and quantile functions. The hazard rate showed high flexibility, capturing monotonic, non-monotonic, and bathtub shapes. Parameters were estimated by maximum likelihood, with simulations confirming estimator consistency. Applications to reservoir volume data (Três Marias and Barra Bonita) demonstrated superior performance of UQL compared to Kumaraswamy and beta distributions across adequacy measures, including the generalized likelihood ratio test. Overall, UQL proved a robust and versatile tool for modeling bounded hydrological data.

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