Eur. J. Stat. 2 (2022) 10
doi: 10.28924/ada/stat.2.10

Inverse Two-Parameter Lindley Distribution and Its Applications

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Abstract. This paper proposes an Inverse two-parameter Lindley distribution (ITPLD). This is originated from Lindley distribution and two-parameter Lindley distribution. Its mathematical and statistical properties which includes its survival function, hazard rate function, shape characteristics of the density, stochastic ordering, entropy measure, and stress-strength reliability were discussed. The estimation of parameters was carried out using the method of maximum likelihood. Also, in the application of the model, HQIC, BIC, CAIC, AIC, and K.S are used to test for the goodness of fit of the model which was applied to two real data sets. The Inverse two-parameter Lindley distribution was compared with Inverse Lindley, Inverse Akash, and Inverse Exponential distributions in order to determine its superiority.

1. Introduction

As proposed by [1], the probability density function (PDF) and cumulative distribution function (CDF) of a one parameter lifetime distribution is respectively given by the following;

$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; \quad x > 0, \theta > 0 \tag{1.1}$$

$$F(x;\theta) = 1 - \frac{(\theta+1)+\theta x}{\theta+1}e^{-\theta x}; \quad x > 0, \theta > 0 \tag{1.2}$$

Detailed about its mathematical and statistical properties including hazard rate and application from lifetime data have been discussed by [1]. The Inverse Lindley distribution

Received: 22 Jan 2022.

Key words and phrases. inverse distribution; stress-strength reliability; maximum likelihood estimator; goodness of fit.

has been proposed by [2] which is an Inverse of Lindley distribution (ILD). Its important statistical properties, stress and strength reliability measure including its stress and strength parameter estimation method on real lifetime data has been shown in their paper. Its PDF and CDF are given respectively by

$$f(x;\theta)=\frac{\theta^2}{(\theta+1)x^3}(1+x)e^{-\theta/x};\quad x>0,\theta>0 \tag{1.3}$$

$$F(x;\theta)=1+\frac{\theta}{(\theta+1)x}e^{-\theta/x};\quad x>0,\theta>0 \tag{1.4}$$

In another study, [3] introduced a two-parameter Lindley distribution (TPLD) and it has been studied and discussed as a lifetime model. Its PDF and CDF are given respectively by

$$f(x;\theta,\alpha)=\frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x};\quad x>0,\theta>0,\alpha>0 \tag{1.5}$$

$$F(x;\theta,\alpha)=1-\frac{(\theta+\alpha)+\alpha\theta x}{\theta+\alpha}e^{-\theta x};\quad x>0,\theta>0,\alpha>0 \tag{1.6}$$

The main motivation of this paper is to introduce a new life time distribution:

(i) It is observed that Inverse Lindley distribution is a more flexible distribution than Lindley and exponential distribution especially for biological data.

(ii)Likewise, it is envisaged that that the inverse of the two-parameter Lindley distribution would also be more flexible and gives good fit over the two-parameter Lindley distribution as well as Inverse Lindley distribution.

The study has been divided into Nine sections, introduction of proposed study is discussed in the first section. Inverse two-parameter distribution and the shape characteristics of the density has been defined in the second section. Survival and hazards rate function are discussed in third section. Stochastic ordering has been discussed in the fourth section. In the fifth section, entropy measure has been discussed. Stress-Strength reliability measure has been derived in the sixth section. Maximum likelihood estimation method has been derived for estimation parameter of the proposed distribution in seventh section. In the eighth section, application of the proposed distribution on real lifetime data has been presented. Conclusions have been given in the last section.

2. Inverse Two-Parameter Lindley Distribution

If a random variable Y has a two-parameter Lindley distribution TPLD(θ), then the random variable X = ($^1/_Y$) is said to be follow the Inverse two-parameter Lindley distribution having a scale parameter θ and shape parameter α with its probability density function (PDF), is defined by

$$f(x;\theta,\alpha) = \frac{\theta^2}{(\theta+\alpha)x^3}(x+\alpha)e^{-\theta/x}; \quad x>0, \theta>0, \alpha>0 \tag{2.1}$$

It is denoted by ITPLD(θ). The cumulative distribution function (CDF) of Inverse two-parameter Lindley distribution is given by

$$F(x;\theta,\alpha) = \left[1+\frac{\theta\alpha}{(\theta+\alpha)x}\right]e^{-\theta/x}; \quad x>0, \theta>0, \alpha>0 \tag{2.2}$$

Since this continuous distribution has the nice closed form expressions for the CDF, hazard function as well as stress-strength reliability, its relevance for survival analysis can never be denied in the literate.

2.1 Shape Characteristics of the Density

The first derivation of (2.1) is given by

$$\frac{d}{dx}f(x) = -\left(\frac{\theta^2}{\theta+\alpha}\right)\frac{e^{-\frac{\theta}{x}}}{x^5}\left(2x^2-(\theta-3\alpha)x-\theta\alpha\right) \tag{2.3}$$

And $\frac{d}{dx}f(x)\Big|_{x=M_0} = M_0$ is the mode of an Inverse two-parameter Lindley random variable and it is given by

$$M_0 = \frac{-(\theta-3\alpha)+\sqrt{(\theta-3\alpha)^2+8\theta\alpha}}{4} \tag{2.4}$$

Thus, the graphs of the PDF and CDF are given in Figure 1 and Figure 2 respectively.

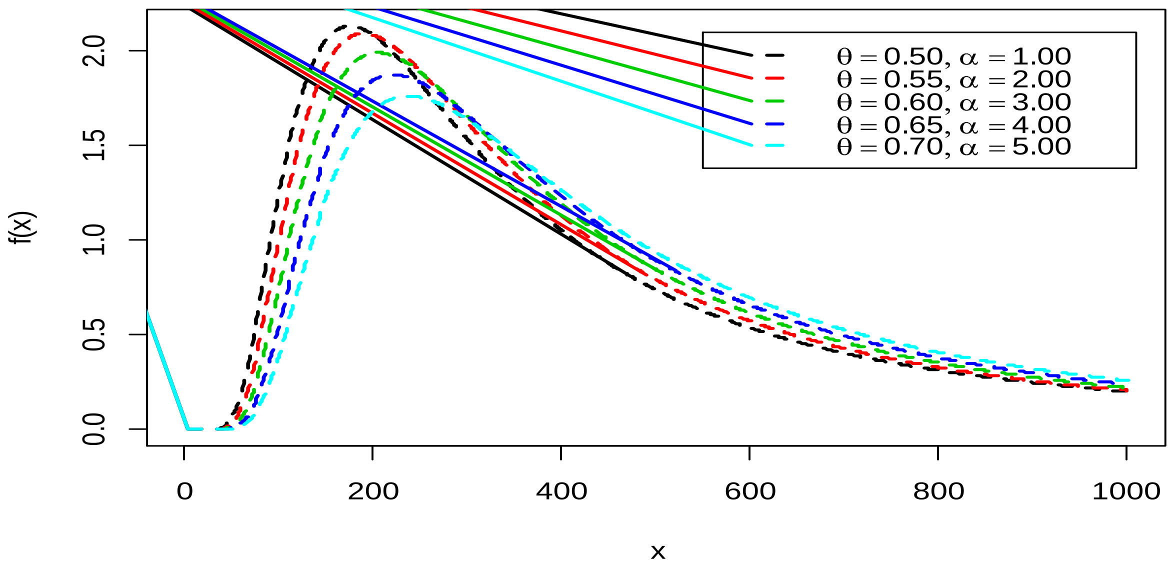

Figure 1. PDF plot of the ITPLD for varying values of the parameters θ and α.

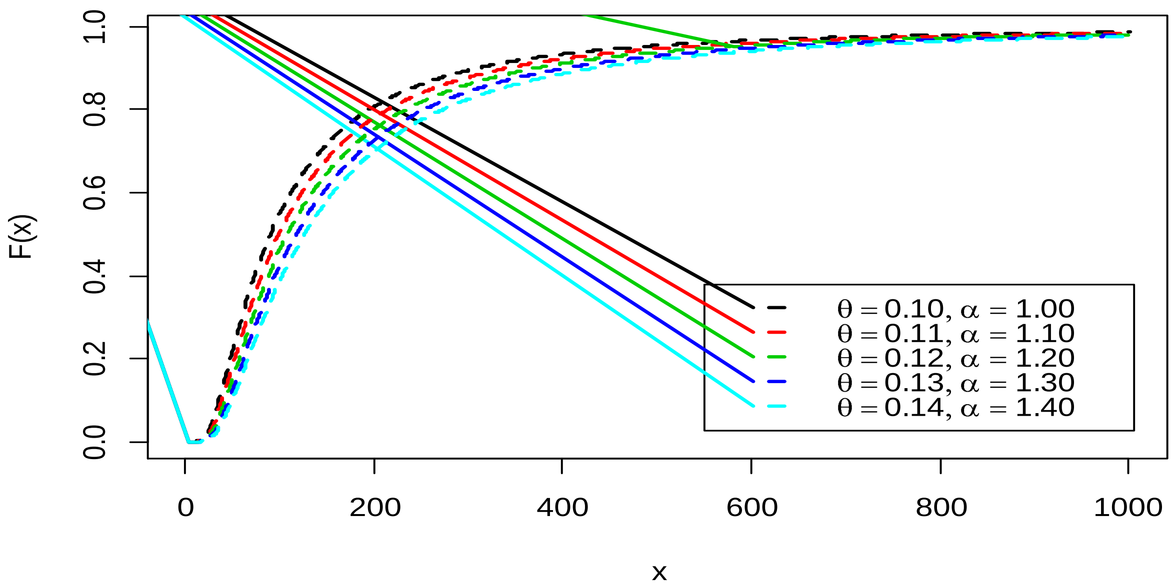

Figure 2. CDF plot of the ITPLD for varying values of the parameters θ and α

3. SURVIVAL FUNCTION AND HAZARD RATE FUNCTION

Survival function $S(x;\theta,\alpha)$ of the Inverse two-parameter Lindley distribution (ITPLD) can be defined as

$$S(x;\theta,\alpha) = 1 - F(x;\theta,\alpha) \tag{3.1}$$

$$S(x;\theta,\alpha) = 1 - \left[1 + \frac{\theta\alpha}{(\theta+\alpha)x}\right] e^{-\theta/x} \tag{3.2}$$

And the hazard rate function $h(x;\theta,\alpha)$ of the Inverse two-parameter Lindley distribution (ITPLD) can be defined as

$$h(x;\theta,\alpha) = \frac{f(x;\theta,\alpha)}{1-F(x;\theta,\alpha)} = \frac{f(x;\theta,\alpha)}{S(x;\theta,\alpha)} \tag{3.3}$$

$$h(x;\theta,\alpha) = \frac{\theta^2(x+\alpha)}{x^2\left[x(\theta+\alpha)\left(e^{\theta/x}-1\right)-\theta\alpha\right]} \tag{3.4}$$

Thus, the graph of the survival function and hazard rate function are given in Figure 3 and Figure 4 respectively.

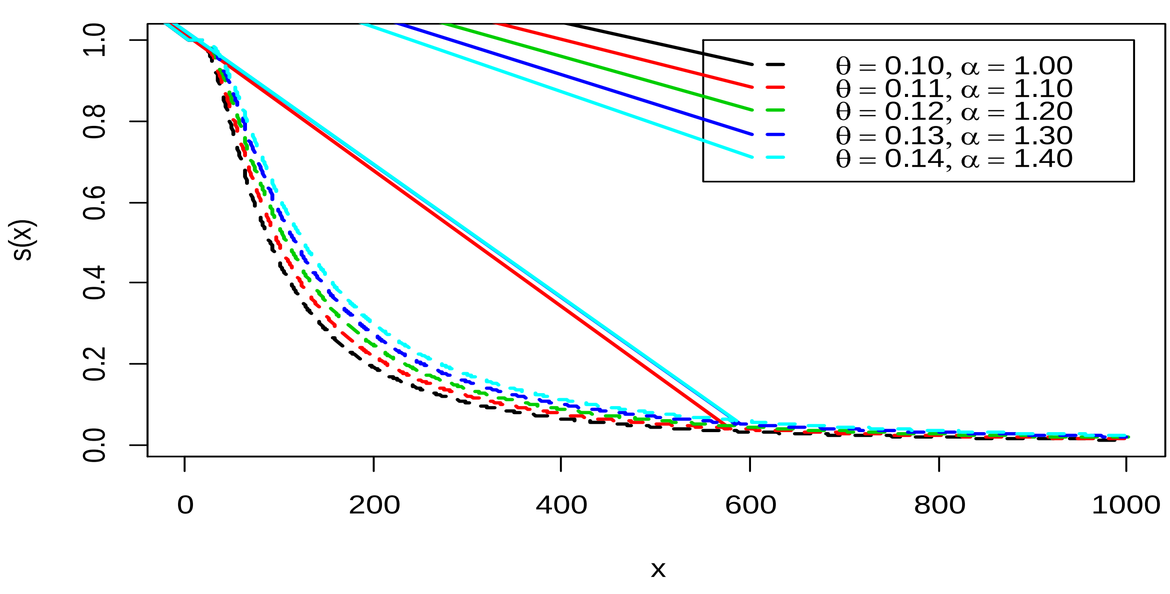

Figure 3. Survival function plot of the ITPLD for varying values of the parameters θ and α.

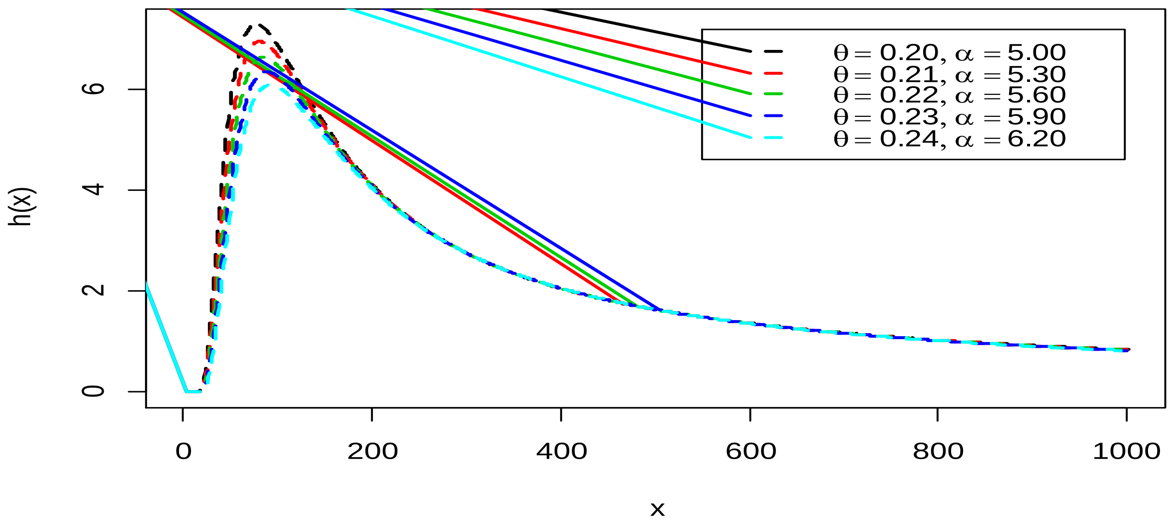

Figure 4. Hazard rate function plot of the ITPLD for varying values of the parameters θ and α.

4. Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) Stochastic order $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x.

(ii) Hazard rate function $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all x.

(iii) Mean residual life function $X \leq_{mrl} Y$ if $m_X(x) \geq m_Y(x)$ for all x.

(iv) Likelihood ratio order $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x

The following results due to [4] are well known for establishing stochastic ordering of distributions.

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y)$$
$$\Downarrow$$
$$(X \leq_{st} Y)$$

The Inverse two-parameter Lindley distribution (ITPLD) is ordered with respect to the strongest likelihood ratio ordering as shown in the following theorem.

Theorem. Let X be Inverse two-parameter Lindley distribution (α_1, θ_1) and Y be Inverse two-parameter Lindley distribution (α_2, θ_2). If $\alpha_1 = \alpha_2$ and $\theta_1 \geq \theta_2$ (or if $\theta_1 = \theta_2$ and $\alpha_1 \geq \alpha_2$), then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof. We have

$$\frac{f(x;\theta_1,\alpha_1)}{f(x;\theta_2,\alpha_2)} = \frac{\theta_1^2(\theta_2+\alpha_2)}{\theta_2^2(\theta_1+\alpha_1)}\left(\frac{x+\alpha_1}{x+\alpha_2}\right)e^{-(\theta_1-\theta_2)/x}; \quad x > 0$$

Now

$$log\frac{f(x;\theta_1,\alpha_1)}{f(x;\theta_2,\alpha_2)} = log\left[\frac{\theta_1^2(\theta_2+\alpha_2)}{\theta_2^2(\theta_1+\alpha_1)}\right] + log\left(\frac{x+\alpha_1}{x+\alpha_2}\right) - (\theta_1-\theta_2)/x$$

This gives

$$\frac{d}{dx}log\frac{f(x;\theta_1,\alpha_1)}{f(x;\theta_2,\alpha_2)} = \frac{(\alpha_2-\alpha_1)}{(x+\alpha_1)(x+\alpha_2)} + (\theta_1-\theta_2)/x^2$$

Thus, for ($\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$) or ($\alpha_1 > \alpha_2$ and $\theta_1 = \theta_2$)

$$\frac{d}{dx}log\frac{f(x;\theta_1,\alpha_1)}{f(x;\theta_2,\alpha_2)} > 0$$

This implies that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

5. Entropy Measure

Entropy of a random variable X is a measure of variation of uncertainty. A popular entropy measure is Renyi entropy [5]. If X is a continuous random variable having probability density function$f(.)$, then Renyi entropy is defined as

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\{\int f^{\gamma}(x)\,dx\} \qquad (5.1)$$

Where $\gamma > 0$ and $\gamma \neq 1$.

For the Inverse two-parameter Lindley distribution, the Renyi entropy measure is defined by;

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty \frac{\theta^{2\gamma}}{(\theta+\alpha)^\gamma}\left[\frac{(\alpha+x)^\gamma}{x^{3\gamma}}\right]e^{-\frac{\theta\gamma}{x}}dx$$

We know that $(1+z)^j = \sum_{j=0}^{\infty}\binom{\gamma}{j}z^j$ and $\int_0^\infty e^{-\frac{b}{x}}x^{-a-1}dx = \frac{\Gamma(a)}{b^a}$

$$= \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}}{(\theta+\alpha)^\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}(\alpha)^j\int_0^\infty \frac{e^{-\frac{\theta\gamma}{x}}}{x^{3\gamma-j}}\mathrm{dx}\right]$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\frac{\theta^{2\gamma}\alpha^j}{(\theta+\alpha)^\gamma}\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\Gamma(3\gamma-j-1)}{(\theta\gamma)^{3\gamma-j-1}}\right] \tag{5.2}$$

6. Stress-Strength Reliability

The stress- strength reliability describes the life of a component which has random strength X that is subjected to a random stress Y. When the stress applied to it exceeds the strength, the component fails instantly and the component will function satisfactorily till X >Y. Therefore, $R = P(Y < X)$ is a measure of component reliability and in statistical literature it is known as stress-strength parameter.

Let Y and X be independent stress and strength random variables that follow Inverse two-parameter Lindley distribution with parameter θ_1, α_1 *and* θ_2, α_2 respectively. Then, the stress-strength reliability R is defined as

$$\mathrm{R} = \mathrm{P[Y<X]} = \int_0^\infty \mathrm{P}[\mathrm{Y} < X \mid \mathrm{X} = x]\, \mathrm{f_x}(x)\mathrm{d}x$$

$$= \int_0^\infty f(x,\theta_1,\alpha_1)\ F(x,\theta_2,\alpha_2)dx$$

$$= \int_0^\infty \frac{\theta_1^2}{\theta_1+\alpha_1}\left(\frac{x+\alpha_1}{x^3}\right)e^{\theta_1/x} \times \left[1+\frac{\theta_2\alpha_2}{(\theta_2+\alpha_2)x}\right]e^{\theta_2/x}dx$$

Using inverse gamma function, it can be written as

$$R = \theta_1^2\left[\frac{2\theta_2\alpha_1\alpha_2 + \theta_2\alpha_2(\theta_1+\theta_2) + \alpha_1(\theta_2+\alpha_2)(\theta_1+\theta_2) + (\theta_2+\alpha_2)(\theta_1+\theta_2)^2}{(\theta_1+\alpha_1)(\theta_2+\alpha_2)(\theta_1+\theta_2)^3}\right]$$

7. Maximum Likelihood Estimation Method

Let $L(x_1, x_2, \ldots, x_n;\ \theta, \alpha)$ be a random sample from the Inverse two-parameter Lindley distribution ITPLD (2.1). The likelihood function, L of ITPLD (2.1) is given by

$$L(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n}[f(x_i; \theta, \alpha)]$$

That is, $L = \left(\frac{\theta^2}{\theta+\alpha}\right)^n \prod_{i=1}^{n} \frac{(x_i+\alpha)}{x_i^3} e^{\sum \theta/x_i}$

The log likelihood function is thus obtained as

$$logL = nlog\left(\frac{\theta^2}{\theta+\alpha}\right) + \sum_{i=1}^{n} log(x_i+\alpha) - \sum_{i=1}^{n} log\left(x_i^3\right) - \theta\sum_{i=1}^{n}(1/x_i)$$

the maximum likelihood estimates $\hat{\theta}$ and $\hat{\alpha}$ of parameter θ and α are the solutions of the log-likelihood equation $\frac{\partial logL}{\partial\theta} = 0$ and $\frac{\partial logL}{\partial\alpha} = 0$. It is obvious that $\frac{\partial logL}{\partial\theta} = 0$ and $\frac{\partial logL}{\partial\alpha} = 0$ will not be in closed forms and hence some numerical optimization technique can be used in the equation for θ and α. In this paper the nonlinear method available in R software has been used to find the MLE of the parameters θ and α.

8. Data Application

The Inverse two-parameter Lindley distribution (ITPLD) was applied to two real life data sets in order to assess its statistical superiority over other models; the Inverse Lindley distribution (ILD), Inverse Akash distribution (IAD), and Inverse Exponential distribution (IED), to demonstrate that the theoretical results in the previous sections can be used in practice. The first data set represents the data on endurance of deep grove ball bearing measured in millions of revolutions before failure and the second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli. The data sets are as follows;

Data set 1:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

Data set 2:

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115,116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230, 231, 240, 245, 251, 253, 254, 254, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555

First, we checked the validity of the Inverse two-parameter Lindley distribution for the given data sets by using Akaike information criterion (AIC), Bayesian information criterion (BIC), Negative Log-Likelihood Function (MLE), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), Standard Error Estimate of the Parameter (SE), and The Estimate of the Parameter. We compared the applicability of Inverse two-parameter Lindley distribution with competing Inverse distributions; Inverse Lindley distribution (ILD), Inverse Akash distribution (IAD), and Inverse Exponential distribution (IED), based on real data sets.

For Data set 1: The performance of the ITPLD with respect to the ILD, IAD and IED using the observations in Data set 1 is shown in Table 1.

Table 1: Performance Ratings of Inverse two-parameter Lindley distribution Using Data Set 1.

Model	MLE	Estimates	S.E	HQIC	BIC	CAIC	AIC
ITPLD	117.1435	$\hat{\theta} = 102.999$ $\hat{\alpha} = 693.27$	16.02870 584.8700	238.8582	240.5580	238.8870	238.2870
ILD	121.7273	$\hat{\theta} = 56.0400$	11.48721	245.7401	246.5900	245.6450	245.4545
IAD	121.7345	$\hat{\theta} = 55.1471$	11.47636	245.7545	246.6045	245.6594	245.4690
IED	121.7296	$\hat{\theta} = 55.0747$	11.48386	245.7447	246.5946	245.6496	245.4591

From Table 1, the ITPLD has the lowest MLE value of 117.1435, the lowest AIC value of 238.2870, the lowest BIC value of 240.5580, the lowest HQIC value of 238.8582, and the lowest CAIC value of 238.8870 therefore, the ITPLD provides a better fit than the ILD, IAD, and IED.

For Data set 2: The performance of the ITPLD with respect to the ILD, IAD and IED using the observations in Data set 2 is shown in Table 2.

Table 2: Performance Ratings of Inverse two-parameter Lindley distribution Using Data Set 2.

Model	MLE	Estimates	S.E	HQIC	BIC	CAIC	AIC
ITPLD	446.3421	$\hat{\theta} = 207.414$ $\hat{\alpha} = 984.66$	16.02870 584.8700	898.4969	901.2375	896.8581	896.6842
ILD	451.8770	$\hat{\theta} = 114.572$	13.38758	906.6604	908.0307	905.8112	905.7540
IAD	451.8707	$\hat{\theta} = 113.624$	13.38453	906.6478	908.0181	905.7986	905.7414
IED	451.8751	$\hat{\theta} = 113.589$	13.38661	906.6564	908.0268	905.8072	905.7501

From Table 2, the ITPLD has the lowest MLE value of 446.3421, the lowest AIC value of 896.6842, the lowest BIC value of 901.2375, the lowest HQIC value of 898.4969, and the lowest CAIC value of 896.8581 therefore, the ITPLD provides a better fit than the ILD, IAD, and IED.

9. Conclusion

A two-parameter lifetime distribution named, "Inverse two-parameter Lindley distribution" has been proposed. Its statistical properties including shape characteristics of density, survival function, hazard rate function, stochastic ordering has been discussed. Further, expressions for entropy measure and, Stress-Strength Reliability of the proposed distribution have been derived. The method of maximum likelihood estimation has also been discussed for estimating its parameter. Finally, the goodness of fit test using MLE, AIC, BIC, HQIC, and CAIC based on two real lifetime data sets, the applicability and superiority over Inverse Lindley, Inverse Akash, and Inverse Exponential distributions while modeling certain lifetime data have been established.

Acknowledgements

The authors are grateful to God Almighty and to the editor- in - chief and to the anonymous reviewers whose comments led to the improved quality of this paper.

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